

DISTANCE BETWEEN TWO KEPLERIAN ORBITS

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ABSTRACT. In this paper, constrained minimization for the point of closest approach of two conic sections is developed. For this development, we considered the nine cases of possible conics, namely, (elliptic–elliptic), (elliptic–parabolic), (elliptic–hyperbolic), (parabolic–elliptic), (parabolic–parabolic), (parabolic–hyperbolic), (hyperbolic–elliptic), (hyperbolic–parabolic), and (hyperbolic–hyperbolic). The developments are considered from two points of view, namely, analytical and computational. For the analytical developments, the literal expression of the minimum distance equation (S) and the constraint equation (G), including the first and second derivatives for each case, are established. For the computational developments, we construct an efficient algorithm for calculating the minimum distance by using the Lagrange multiplier method under the constraint on time. Finally, we compute the closest distance S between two conics for some orbits. The accuracy of the solutions was checked under the conditions that $L|_{solution} \leq \varepsilon_1$; $G|_{solution} \leq \varepsilon_2$, where $\varepsilon_{1,2} < 10^{-10}$. For the cases of (parabolic–parabolic), (parabolic–hyperbolic), and (hyperbolic–hyperbolic), we studied thousands of comets, but the condition of the closest approach was not met.

Keywords: distance function, Lagrange multiplier, objective function, Hessian matrix, minimum distance

1. INTRODUCTION

The problem of determining the point of closest approach of two orbits has important applications. In planetary theory, the point of closest approach is interesting for visual observations of planetary surfaces. Moreover, the closest approach is essential for determining the critical distance at which a warning is activated. On the other hand, the point of closest approach is crucial for rendezvous considerations. Specifically, for space probes, photographic limits or on–off camera times require the determination of the closest approach for predicting when to activate the camera or transmission equipment (Liu et al., 2004). The prefilter is required to cancel the objects that are not in conjunction with the target spacecraft before further close approach analysis. The most common prefilter methods are the apogee–perigee filter and altitude difference filter methods (Zheng and Wu, 2004). The close approach analysis can be applied on the orbital elements and also for geometrical analysis of the orbital



elements. In these cases, the information of the closest approach events could be obtained by applying the differential method. On the other hand, the numerical methods are based on the orbital ephemeris of objects either at certain time steps during a certain interval or the position and velocity information obtained from the orbital model. The relative position and closest approach information are obtained by numerical processing methods, such as difference, interpolation, fitting, and polynomial root-finding to the orbital ephemerides or positions. Kholshchevnikov and Vassiliev (1999) developed the critical points of the distance function between two confocal Keplerian elliptic orbits, which was reduced to the eighth-order degree (Zheng and Wu, 2004). In nondegenerate cases, a polynomial of a lower degree with such properties does not exist. Baluyev and Kholshchevnikov (2005) extended the concept to all possible cases of orbital pairs required in two-body problems. Hoots et al. (1984) completed the calculation of the number of times of future close approaches between pairs of satellites, which was formulated using analytical techniques. The resulting analytical equations are solved using numerical iterative techniques similar to solving Kepler's equation. A solution is obtained in a very efficient manner by using a series of prefilters that eliminate many cases from further consideration. The method is valid for all values of eccentricities <1 and all relative geometries between the two orbits. This approach produces results in a very efficient and reliable manner. Sharaf and Sharaf (1997) extended the closest approach concept by using the general formulations of the closest approach (universal closest approach) at all orbits. Universal formulations of closest approach problems are established and solved by two methods. The first method uses the technique of unconstrained minimization and needs the solution of the universal Kepler's equation twice, while for the second method, a constraint minimization technique is developed, which needs the solution of two nonlinear simultaneous equations (Sharaf and Sharaf, 1997). Denenberg and Gurfil (2016) described the probability of collision using the closest approach (between space debris and spacecraft) and calculated the time of closest Approach (TCA) between the spacecraft and the space debris by three methods:

The first is a surrogate-based optimization (SBO) algorithm, using the Alfano (1994) Close Approach Software (ANCAS) as the model, allowing a compromise between calculation speed and accuracy. The second is a generalization of ANCAS over initial conditions as well as time. The third uses ANCAS generalization as a model for SBO (Denenberg and Gurfil, 2016). In the present paper, we established a constraint minimization technique to determine the point of closest approach of any two conic orbits whose orbital elements are known. We studied three different cases of orbits through numerical analysis; furthermore, the determinant of the global minimum of the coasting function was derived. The objective function will be as follows: (elliptic–elliptic), (elliptic–parabolic), and (elliptic–hyperbolic) (under (G) constraint).

2. DISTANCE FUNCTION

In this section, the constrained minimization problem is developed for the determination of the point of closest approach of any two conic sections.

2.1. The plane coordinates

Let $(\xi^{(1)}, \eta^{(1)})$, $(\xi^{(2)}, \eta^{(2)})$ and $(\xi^{(3)}, \eta^{(3)})$ be the plane coordinates for elliptic, parabolic and hyperbolic orbits, respectively (Escobal, 1965). Figure 1 shows the coordinate system used. The set of axes (ξ, η) is introduced with the origin at the focus; the positive ξ pointing along the minimum pericenter and the η axis are advanced by a right angle to ξ .

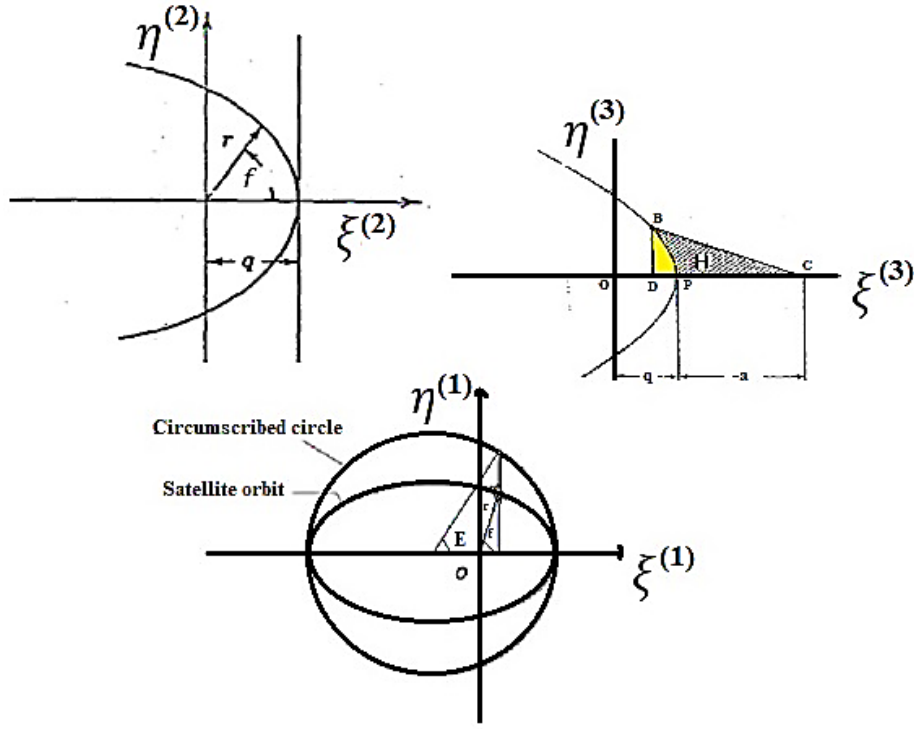


Figure 1. Axes (ξ, η) : eccentric and true anomalies (Brown, 2004; Moulton, 1960)

The rectangular coordinates of the two bodies $i = 1$ or 2 in their orbits are as follows:

$$\xi_i^{(1)} = a_i (\cos E_i - e_i), \eta_i^{(1)} = a_i \sqrt{1 - e_i^2} \sin E_i \quad (1)$$

$$\xi_i^{(2)} = q_i \left(1 - \tan^2 \frac{f_i}{2}\right), \eta_i^{(2)} = 2q_i \tan \frac{f_i}{2} \quad (2)$$

$$\xi_i^{(3)} = a_i (\cosh H_i - e_i), \eta_i^{(3)} = -a_i \sqrt{e_i^2 - 1} \sinh H_i \quad (3)$$

where a , e , and q are the semimajor axis, eccentricity, and the pericenter distance (perihelion distance), respectively, and f , E , and H are the true, elliptic eccentric, and the hyperbolic eccentric anomalies, respectively.

2.2. Transformation to the fundamental plane

The transformation of the orbital plane coordinates to the fundamental plane (ecliptic or equator) coordinates (x, y, z) are obtained by the following well-known vector mappings:

$$\vec{r}_i^{(k)} = \xi_i^{(k)} \vec{p}_i + \eta_i^{(k)} \vec{Q}_i, \quad i = 1, 2, \quad k = 1, 2 \text{ or } 3 \quad (4)$$

where the components of the unit vectors $\vec{p}_i = (p_{xi}, p_{yi}, p_{zi})$ and $\vec{Q}_i = (Q_{xi}, Q_{yi}, Q_{zi})$ are given in terms of the orbit orientation angles I , ω , and Ω by the following equations:

$$\vec{p}_{xi} = \cos \omega_i \cos \Omega_i - \sin \omega_i \sin \Omega_i \cos I_i \quad (5)$$

$$\vec{p}_{yi} = \cos \omega_i \sin \Omega_i + \sin \omega_i \cos \Omega_i \cos I_i \quad (6)$$

$$\vec{p}_{zi} = \sin \omega_i \sin I_i \quad (7)$$

$$\vec{Q}_{xi} = p_{xi}(I, \omega + 90^\circ, \Omega) = -\sin \omega_i \cos \Omega_i + \cos \omega_i \sin \Omega_i \cos I_i \quad (8)$$

$$\vec{Q}_{yi} = p_{yi}(I, \omega + 90^\circ, \Omega) = -\sin \omega_i \sin \Omega_i + \cos \omega_i \cos \Omega_i \cos I_i \quad (9)$$

$$\vec{Q}_{zi} = p_{zi}(I, \omega + 90^\circ, \Omega) = \cos \omega_i \sin I_i \quad (10)$$

where I, ω and Ω are the inclination, argument of pericenter, and longitude of the ascending node, respectively (Vallado, 2013).

2.3. Distance and constraint functions

The problem is to find the minimum of the difference between $\vec{r}_1^{(k_1)}$ and $\vec{r}_2^{(k_2)}$, where $k_{1,2} = 1, 2$ or 3 and k_1 may or may not be equal to k_2 . Thus, the objective is to minimize the equivalent expression:

$$(\Delta r)^2 = (\vec{r}_1^{(k_1)} - \vec{r}_2^{(k_2)}) \cdot (\vec{r}_1^{(k_1)} - \vec{r}_2^{(k_2)}) \quad (11)$$

where $(\vec{a}) \cdot (\vec{b})$ is used to denote the scalar product of the two vectors \vec{a} and \vec{b} . The above equation [Eq. (11)] could be rewritten as follows:

$$(\vec{p}_n \cdot \vec{p}_n) = 1, (\vec{Q}_n \cdot \vec{Q}_n) = 1, (\vec{p}_n \cdot \vec{Q}_n) = 0, n = 1, 2 \quad (12)$$

$$S \equiv (\Delta r)^2 = (\xi_1^{(k_1)})^2 + (\eta_1^{(k_1)})^2 + (\xi_2^{(k_2)})^2 + (\eta_2^{(k_2)})^2 + \alpha \xi_1^{(k_1)} \xi_2^{(k_2)} + \beta \xi_1^{(k_1)} \eta_2^{(k_2)} + \gamma \xi_2^{(k_2)} \eta_1^{(k_1)} + \zeta \eta_1^{(k_1)} \eta_2^{(k_2)} \quad (13)$$

$$\alpha = -2(\vec{P}_1 \cdot \vec{P}_2), \beta = -2(\vec{P}_1 \cdot \vec{Q}_2), \gamma = -2(\vec{P}_2 \cdot \vec{Q}_1), \zeta = -2(\vec{Q}_1 \cdot \vec{Q}_2) \quad (14)$$

where S is the objective function to be minimized (Figure 2).

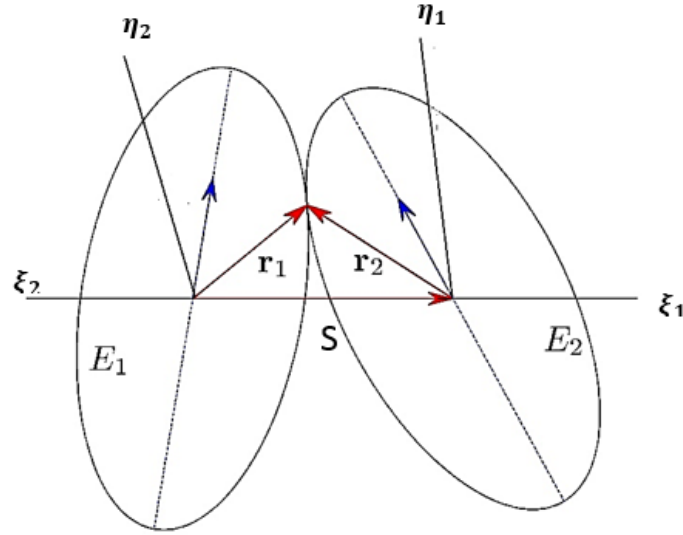


Figure 2. Distance S between two elliptic orbits

The relations between position and time for the different conic sections are given as follows (Escobal, 1965):

$$\begin{aligned} n_i^{(1)}(t - \tau_i^{(1)}) &= M_i^{(1)} = E_i - e \sin E_i \\ n_i^{(2)}(t - \tau_i^{(2)}) &= M_i^{(2)} = \sqrt{2} q_i^{3/2} \left\{ \tan \frac{f_i}{2} + \frac{1}{3} \tan^3 \frac{f_i}{2} \right\} \quad i = 1, 2 \\ n_i^{(3)}(t - \tau_i^{(3)}) &= M_i^{(3)} = e_i \operatorname{sh} H_i - H_i \end{aligned} \quad (15)$$

where n, τ , and M are, respectively, the mean motion, the time of pericenter passage, and mean anomaly. Equation (13) is the objective function, which is to be minimized. Since the

problem involves the dynamics of the two bodies, it must also be for any universal time t and its corresponding position and time:

$$\frac{M_1^{(k_1)}}{n_1^{(k_1)}} + \tau_1^{(k_1)} = t = \frac{M_2^{(k_2)}}{n_2^{(k_2)}} + \tau_2^{(k_2)} \quad (16)$$

The constraint of the problem G can be stated as follows:

$$G = \frac{M_1^{(k_1)}}{n_1^{(k_1)}} - \frac{M_2^{(k_2)}}{n_2^{(k_2)}} + \tau_1^{(k_1)} - \tau_2^{(k_2)} = 0 \quad (17)$$

3. PROBLEM FORMULATION

In this section, we shall establish computational algorithms for the point of closest approach of the two possible cases of approaching conic sections. Each case is characterized by two integers indicating the types of conic sections of the case $k_1 = 1$ and $k_2 = 3$, which means that the first conic section is elliptic, while the second is hyperbolic. It should be noted that the case with $k_1 = 1$ and $k_2 = 3$ is not the case with $k_1 = 3$ and $k_2 = 1$; therefore, they should be treated separately. Furthermore, if k_1 equals k_2 , the two approaching conics are of the same type. Each case is then developed analytically and computationally. For the analytical developments, the literal forms of the distance equation (S) and the constrained equation (G) are applied. We turn now to the study of minimization with constraints. It is convenient to use the Lagrangian associated with the constrained problem, defined as follows:

$$L(x, y, \lambda) = f(x, y) + \lambda G(x, y) \quad (18)$$

The local minimum and maximum points are the roots of the gradient $\mathbf{L} = 0$.

According to the constraint minimization techniques, the new function to be minimized is as follows:

$$L(\lambda, x_1, x_2) = S(x_1, x_2) + \lambda G(x_1, x_2) \quad (19)$$

Here, λ is known as the Lagrange multiplier, (S) is the objective function, and (G) is the subject of the constraint; x_1, x_2 represent the different anomalies. The extreme point of any system of nonlinear equations can be obtained by setting $\nabla L = 0$.

$$\nabla L = \left(\frac{\partial L}{\partial \lambda}, \frac{\partial L}{\partial x_1^{(k_1)}}, \frac{\partial L}{\partial x_2^{(k_2)}} \right) = 0 \quad (1)$$

The approach of using the Lagrange construction and setting its gradient to zero is known as the method of Lagrange multipliers (Junkins, 2004).

$$\frac{\partial L}{\partial \lambda} = 0 \& G = 0 \quad (2)$$

$$\frac{\partial L}{\partial x_1^{(k_1)}} = \frac{\partial S}{\partial x_1^{(k_1)}} + \lambda \frac{\partial G}{\partial x_1^{(k_1)}} = 0 \quad (3)$$

$$\frac{\partial L}{\partial x_2^{(k_2)}} = \frac{\partial S}{\partial x_2^{(k_2)}} + \lambda \frac{\partial G}{\partial x_2^{(k_2)}} = 0 \quad (4)$$

$$\lambda = - \frac{\frac{\partial S}{\partial x_1^{(k_1)}}}{\frac{\partial G}{\partial x_1^{(k_1)}}} = - \frac{\frac{\partial S}{\partial x_2^{(k_2)}}}{\frac{\partial G}{\partial x_2^{(k_2)}}} \quad (5)$$

From Eqs. (21), (22), (23), and (24), we can write

$$\frac{\partial S}{\partial x_2^{(k_2)}} \frac{\partial G}{\partial x_1^{(k_1)}} - \frac{\partial S}{\partial x_1^{(k_1)}} \frac{\partial G}{\partial x_2^{(k_2)}} = 0 \quad (6)$$

A sufficient condition for a minimum is that the Hessian matrix is positive definite (a symmetric matrix is positive definite if and only if all eigenvalues are positive). The minimum or maximum solutions are checked by the Hessian matrix, as follows:

$$H = \begin{bmatrix} 0 & \frac{\partial G}{\partial x_1^{(k_1)}} & \frac{\partial G}{\partial x_2^{(k_2)}} \\ \frac{\partial G}{\partial x_1^{(k_1)}} & \frac{\partial^2 S}{\partial x_1^{(k_1)^2}} + \lambda \frac{\partial^2 G}{\partial x_1^{(k_1)^2}} & \frac{\partial^2 S}{\partial x_1^{(k_1)} \partial x_2^{(k_2)}} + \lambda \frac{\partial^2 G}{\partial x_1^{(k_1)} \partial x_2^{(k_2)}} \\ \frac{\partial G}{\partial x_2^{(k_2)}} & \frac{\partial^2 S}{\partial x_1^{(k_1)} \partial x_2^{(k_2)}} + \lambda \frac{\partial^2 G}{\partial x_1^{(k_1)} \partial x_2^{(k_2)}} & \frac{\partial^2 S}{\partial x_2^{(k_2)^2}} + \lambda \frac{\partial^2 G}{\partial x_2^{(k_2)^2}} \end{bmatrix} \quad (26)$$

$$k_{1,2}=1,2,3 \text{ and } x_i^{(k)} = \begin{cases} E_i & \text{if } k = 1 \\ f_i & \text{if } k = 2 \\ H_i & \text{if } k = 3 \end{cases} \quad (27)$$

3.1. The sufficient condition for the closest approach

During a space mission, all types of conic motion appear. Therefore, we need formulae for determining the closest approach for any possible two conic orbits. Nine cases have been deduced and discussed. Substituting from Eqs. (1), (2), and (3) into Eqs. (13) and (14), we write the formulas for every two orbits separately.

3.1.1. The point of closest approach between two elliptic conic sections ($k_1 = 1, k_2 = 1$)

$$\begin{aligned} S(E_1, E_2) = & a_1 a_2 \left(-e_1 (\alpha \cos E_2 - \alpha e_2 + \beta \sqrt{1 - e_2^2} \sin E_2 + \cos E_2) (\alpha \cos E_1 \right. \\ & + \gamma \sqrt{1 - e_1^2} \sin E_1) - e_2 (\alpha \cos E_1 + \gamma \sqrt{1 - e_1^2} \sin E_1) \\ & + \sqrt{1 - e_2^2} \sin E_2 (\beta \cos E_2 + \sqrt{1 - e_1^2} \zeta \sin E_1) \left. \right) + a_1^2 (e_1 \cos E_1 - 1)^2 \\ & + a_2^2 (e_2 \cos E_2 - 1)^2 \end{aligned} \quad (28)$$

$$G(E_1, E_2) = \sqrt{\left(\frac{a_1^3}{\mu}\right)} (E_1 - e_1 \sin(E_1)) - \sqrt{\left(\frac{a_2^3}{\mu}\right)} (E_2 - e_2 \sin(E_2)) + \tau_1 - \tau_2 = 0 \quad (29)$$

Eqs. (28) and (29) represent the objective function (S) and constraint (G) of two elliptic orbits. The basic equations of our special elliptical cases are as follows:

$$\begin{aligned} \frac{\partial S}{\partial E_1} = & -2a_1^2 e_1^2 \sin E_1 \cos E_1 + a_1 (2a_1 e_1 - \alpha \xi_2 - \beta \eta_2) \sin E_1 \\ & + a_1 \sqrt{(1 - e_1^2)} (\gamma \xi_2 + \zeta \eta_2) \cos(E_1) \end{aligned} \quad (30)$$

$$\begin{aligned} \frac{\partial S}{\partial E_2} = & -2a_2^2 e_2^2 \sin E_2 \cos E_2 + a_2 (2a_2 e_2 - \alpha \xi_1 - \gamma \eta_1) \sin E_2 \\ & + a_2 \sqrt{(1 - e_2^2)} (\beta \xi_1 + \zeta \eta_1) \cos(E_2) \end{aligned} \quad (31)$$

$$\frac{\partial G}{\partial E_1} = \sqrt{(a_1^3/\mu)} (1 - e_1 \cos E_1) \quad (32)$$

$$\frac{\partial G}{\partial E_2} = -\sqrt{(a_2^3/\mu)}(1 - e_2 \cos E_2) \quad (33)$$

We solve the system of Eqs. (30), (31), (32), and (33) using Eqs. (21), (22), (23), and (25); thus, the expressions are written as follows:

$$L(E_1, E_2) = \frac{\partial S}{\partial E_2} \frac{\partial G}{\partial E_1} - \frac{\partial S}{\partial E_1} \frac{\partial G}{\partial E_2} = 0 \quad (34)$$

$$G(E_1, E_2) = 0 \quad (35)$$

3.1.2. *The point of closest approach between elliptic and parabolic conic sections ($k_1 = 1$, $k_2 = 2$)*

$$\begin{aligned} S(E_1, f_2) = & \frac{2a_1 q_2}{1 + \cos f_2} (-e_1(\alpha + \beta \sin f_2) + e_1^2(-\sin E_1)(\gamma + \zeta \sin f_2)) \\ & + \cos E_1(\alpha + \beta \sin f_2) + \sin E_1(\gamma + \zeta \sin f_2) \\ & + a_1^2(e_1^4 \sin^2 E_1 + e_1^2 \cos 2E_1 - 2e_1 \cos E_1 + 1) \\ & - \frac{1}{2}q_2^2(\cos 2f_2 - 3)\sec^4 \frac{1}{2}f_2 \end{aligned} \quad (36)$$

$$G(E_1, f_2) = \sqrt{\left(\frac{a_1^3}{\mu}\right)}(E_1 - e_1 \sin E_1) - \sqrt{\left(\frac{2q_2^3}{\mu}\right)}\left(\tan \frac{f_2}{2} + \frac{1}{3}\tan^3 \frac{f_2}{2}\right) + \tau_1 - \tau_2 = 0 \quad (37)$$

Here, Eqs. (34) and (36) represent the objective function (S) and constraint (G), respectively, of the elliptic and parabolic orbits. The basic equations of our special elliptic and parabolic cases are as follows:

$$\begin{aligned} \frac{\partial S}{\partial E_1} = & -2a_1^2 e_1^2 \sin E_1 \cos E_1 + a_1(2a_1 e_1 - \alpha \xi_2 - \beta \eta_2) \sin E_1 \\ & + a_1 \sqrt{(1 - e_1^2)}(\gamma \xi_2 + \zeta \eta_2) \cos(E_1) \end{aligned} \quad (38)$$

$$\frac{\partial S}{\partial f_2} = q_2 \sec^2 \left(\frac{f_2}{2}\right) \left\{ 2q_2 \tan^3 \left(\frac{f_2}{2}\right) + \beta \xi_1 + \zeta \eta_1 - (\alpha \xi_1 + \gamma \eta_1) \tan \left(\frac{f_2}{2}\right) \right\} \quad (39)$$

$$\frac{\partial G}{\partial E_1} = \sqrt{\left(\frac{a_1^3}{\mu}\right)}(1 - e_1 \cos E_1) \quad (40)$$

$$\frac{\partial G}{\partial f_2} = -\sqrt{\frac{q_2^3}{2\mu}} \sec^4 \left(\frac{f_2}{2}\right) \quad (41)$$

The system of Eqs. (38), (39), (40), and (41) is solved using Eqs. (21), (22), (23), and (24); thus, we write as follows:

$$L(E_1, f_2) = \frac{\partial S}{\partial f_2} \frac{\partial G}{\partial E_1} - \frac{\partial S}{\partial E_1} \frac{\partial G}{\partial f_2} = 0 \quad (42)$$

$$G(E_1, f_2) = 0 \quad (43)$$

3.1.3. The point of closest approach between elliptic and hyperbolic conic sections ($k_1 = 1$, $k_2 = 3$)

$$\begin{aligned}
S(E_1, H_2) = & a_1 a_2 \{ -e_1 (\alpha \cos E_1 + \gamma \sqrt{1 - e_1^2} \sin E_1) \\
& + e_1 (\alpha e_2 + \beta \sqrt{e_2^2 - 1} \sinh H_2 - \alpha \cosh H_2) \\
& + \cosh H_2 (\alpha \cos E_1 + \gamma \sqrt{1 - e_1^2} \sin E_1) \\
& - \sqrt{e_2^2 - 1} \sinh H_2 (\beta \cos E_1 + \sqrt{1 - e_1^2} \sin E_1) \} \\
& + a_2^2 (e_2 \cosh H_2 - 1)^2 + a_2^2 (e_1 \cos E_1 - 1)^2
\end{aligned} \tag{44}$$

$$\begin{aligned}
G(E_1, H_2) = & \sqrt{\left(\frac{a_1^3}{\mu}\right)} (E_1 - e_1 \sin(E_1)) - \sqrt{\left(-\frac{a_2^3}{\mu}\right)} (e_2 \sinh(H_2) - H_2) + \tau_1 - \tau_2 \\
= & 0
\end{aligned} \tag{45}$$

Here, Eqs. (44) and (45) represent the objective function (S) and constraint (G), respectively, of elliptic and hyperbolic orbits. The basic equations of our special elliptic and hyperbolic cases are as follows:

$$\begin{aligned}
\frac{\partial S}{\partial E_1} = & -2a_1^2 e_1^2 \sin E_1 \cos E_1 + a_1 (2a_1 e_1 - \alpha \xi_2 - \beta \eta_2) \sin E_1 \\
& + a_1 \sqrt{(1 - e_1^2)} (\gamma \xi_2 + \zeta \eta_2) \cos(E_1)
\end{aligned} \tag{46}$$

$$\begin{aligned}
\frac{\partial S}{\partial H_2} = & 2a_2^2 e_2^2 \sinh H_2 \cosh H_2 + a_2 (-2a_2 e_2 + \alpha \xi_1 + \gamma \eta_1) \sinh H_2 \\
& - a_2 \sqrt{e_2^2 - 1} (\beta \xi_1 + \zeta \eta_1) \cosh H_2
\end{aligned} \tag{47}$$

$$\frac{\partial G}{\partial E_1} = \sqrt{\left(\frac{a_1^3}{\mu}\right)} (1 - e_1 \cos E_1) \tag{48}$$

$$\frac{\partial G}{\partial H_2} = \sqrt{-\frac{a_2^3}{\mu}} (1 - e_2 \cosh H_2) \tag{49}$$

The system of Eqs. (46), (47), (48), and (49) are solved using Eqs. (21), (22), (23), and (24); thus, we write as follows:

$$L(E_1, H_2) = \frac{\partial S}{\partial H_1} \frac{\partial G}{\partial E_1} - \frac{\partial S}{\partial E_1} \frac{\partial G}{\partial H_1} = 0 \tag{50}$$

$$G(E_1, H_2) = 0 \tag{51}$$

3.1.4. The point of closest approach between parabolic and elliptic conic sections ($k_1 = 2$, $k_2 = 1$)

$$S(f_1, E_2) = a_2 q_1 \sec^2 \frac{f_1}{2} \left(-e_2 (\alpha + \gamma \sin f_1) + \sqrt{1 - e_2^2} \sin E_2 (\beta + \zeta \sin f_1) \cos E_2 (\alpha + \gamma \sin f_1) \right) + a_2^2 (e_2 \cos E_2 - 1)^2 - \frac{1}{2} q_1^2 (\cos 2f_1 - 3) \sec^4 \frac{f_1}{2} \quad (52)$$

$$G(f_1, E_2) = \sqrt{\left(\frac{2q_1^3}{\mu}\right)} \left(\tan \frac{f_1}{2} + \frac{1}{3} \tan^3 \frac{f_1}{2} \right) - \sqrt{\left(\frac{a_2^3}{\mu}\right)} (E_2 - e_2 \sin(E_2)) + \tau_1 - \tau_2 = 0 \quad (53)$$

Here, Eqs. (52) and (53) represent the objective function (S) and constraint (G), respectively, of parabolic and elliptic orbits. The basic equations of our special parabolic and elliptic orbit systems are as follows:

$$\frac{\partial S}{\partial f_1} = q_1 \sec^2 \left(\frac{f_1}{2} \right) \left\{ 2q_1 \tan^3 \left(\frac{f_1}{2} \right) + \gamma \xi_2 + \zeta \eta_2 - (\alpha \xi_2 + \beta \eta_2) \tan \left(\frac{f_1}{2} \right) \right\} \quad (54)$$

$$\frac{\partial S}{\partial E_2} = -2a_2^2 e_2^2 \sin E_2 \cos E_2 + a_2 (2a_2 e_2 - \alpha \xi_1 - \gamma \eta_1) \sin E_2 + a_2 \sqrt{(1 - e_2^2)} (\beta \xi_1 + \zeta \eta_1) \cos(E_2) \quad (55)$$

$$\frac{\partial G}{\partial f_1} = \sqrt{\frac{q_1^3}{2\mu}} \sec^4 \left(\frac{f_1}{2} \right) \quad (56)$$

$$\frac{\partial G}{\partial E_2} = -\sqrt{\left(\frac{a_2^3}{\mu}\right)} (1 - e_2 \cos E_2) \quad (57)$$

The system of Eqs. (54), (55), (56), and (57) are solved using Eqs. (21), (22), (23), and (24); thus, we obtain the following:

$$L(f_1, E_2) = \frac{\partial S}{\partial f_1} \frac{\partial G}{\partial E_2} - \frac{\partial S}{\partial E_2} \frac{\partial G}{\partial f_1} = 0 \quad (58)$$

$$G(f_1, E_2) = 0 \quad (59)$$

3.1.5. The point of closest approach between two parabolic conic sections ($k_1 = 2, k_2 = 2$)

$$S(f_1, f_2) = q_1^2 \sec^4 \frac{f_1}{2} + q_1 q_2 \sec^2 \frac{f_1}{2} \sec^2 \frac{f_2}{2} + (\alpha + \beta \sin f_2) + q_2^2 \sec^4 \frac{f_2}{2} + 2\gamma q_1 q_2 \sec^2 \frac{f_1}{2} \tan \frac{f_1}{2} + 4 \left(q_1^2 \tan^2 \frac{f_1}{2} + \zeta q_1 q_2 \tan \frac{f_1}{2} \tan \frac{f_2}{2} + q_2^2 \tan^2 \frac{f_2}{2} \right) \quad (60)$$

$$S(f_1, f_2) = \sqrt{\left(2\frac{q_1^3}{\mu}\right)} \left(\tan\frac{f_1}{2} + \frac{1}{3}\tan^3\frac{f_1}{2}\right) - \sqrt{\left(2\frac{q_2^3}{\mu}\right)} \left(\tan\frac{f_2}{2} + \frac{1}{3}\tan^3\frac{f_2}{2}\right) + \tau_1 - \tau_2 = 0 \quad (61)$$

Eqs. (60) and (61) represent the objective function (S) and constraint (G), respectively, of two parabolic orbits. The basic equations of our special parabolic orbit systems are as follows:

$$\frac{\partial S}{\partial f_1} = q_1 \sec^2\left(\frac{f_1}{2}\right) \left\{2q_1 \tan^3\left(\frac{f_1}{2}\right) + \gamma\xi_2 + \zeta\eta_2 - (\alpha\xi_2 + \beta\eta_2) \tan\left(\frac{f_1}{2}\right)\right\} \quad (62)$$

$$\frac{\partial S}{\partial f_2} = q_2 \sec^2\left(\frac{f_2}{2}\right) \left\{2q_2 \tan^3\left(\frac{f_2}{2}\right) + \beta\xi_1 + \zeta\eta_1 - (\alpha\xi_1 + \gamma\eta_1) \tan\left(\frac{f_2}{2}\right)\right\} \quad (63)$$

$$\frac{\partial G}{\partial f_1} = \sqrt{\frac{q_1^3}{2\mu}} \sec^4\left(\frac{f_1}{2}\right) \quad (64)$$

$$\frac{\partial G}{\partial f_2} = \sqrt{\frac{q_2^3}{2\mu}} \sec^4\left(\frac{f_2}{2}\right) \quad (65)$$

The formula in Eq. (61) is solved using Eqs. (21) and (22); thus, we obtain the following:

$$L(f_1, f_2) = \frac{\partial S}{\partial f_1} \frac{\partial G}{\partial f_2} - \frac{\partial S}{\partial f_2} \frac{\partial G}{\partial f_1} = 0 \quad (66)$$

$$G(f_1, f_2) = 0 \quad (67)$$

3.1.6. *The point of closest approach between parabolic and hyperbolic conic sections ($k_1 = 2$, $k_2 = 3$)*

$$S(f_1, H_2) = \frac{2a_2 q_1}{1 + \cos f_1} \left\{-e_2(\alpha + \gamma \sin f_1) - \sinh H_2 \sqrt{e_2^2 - 1}(\beta + \zeta \sin f_1) + \cosh H_2(\alpha + \gamma \sin f_1)\right\} + a_2^2 (e_2 \cosh H_2 - 1)^2 - \frac{1}{2} q_1^2 (\cos 2f_1 - 3) \sec^2 \frac{f_1}{2} \quad (68)$$

$$G(f_1, H_2) = \sqrt{\left(2\frac{q_1^3}{\mu}\right)} \left(\tan\frac{f_1}{2} + \frac{1}{3}\tan^3\frac{f_1}{2}\right) - \sqrt{\frac{a_2^2}{\mu}} (e_2 \sinh H_2 - H_2) + \tau_1 - \tau_2 = 0 \quad (69)$$

Here, Eqs. (68) and (69) represent the objective function (S) and constraint (G), respectively, between parabolic and hyperbolic orbits. The basic equations of our special hyperbolic and parabolic orbit systems are as follows:

$$\frac{\partial S}{\partial f_1} = q_1 \sec^2\left(\frac{f_1}{2}\right) \left\{2q_1 \tan^3\left(\frac{f_1}{2}\right) + \gamma\xi_2 + \zeta\eta_2 - (\alpha\xi_2 + \beta\eta_2) \tan\left(\frac{f_1}{2}\right)\right\} \quad (70)$$

$$\frac{\partial S}{\partial H_2} = 2a_2^2 e_2^2 \sinh H_2 \cosh H_2 + a_2(-2a_2 e_2 + \alpha\xi_1 + \gamma\eta_1) \sinh H_2 - a_2 \sqrt{e_2^2 - 1}(\beta\xi_1 + \zeta\eta_1) \cosh H_2 \quad (71)$$

$$\frac{\partial G}{\partial H_2} = -\sqrt{-\frac{a_2^3}{\mu}} (1 - e_2 \cosh(H_2)) \quad (72)$$

$$\frac{\partial G}{\partial f_1} = \sqrt{\frac{q_1^3}{2\mu}} \sec^4\left(\frac{f_1}{2}\right) \quad (73)$$

We solve the system of Eqs. (70), (71), (72), and (73) using Eqs. (21), (22), (23), and (24); thus, we obtain the following:

$$L(f_1, H_2) = \frac{\partial S}{\partial f_1} \frac{\partial G}{\partial H_2} - \frac{\partial S}{\partial H_2} \frac{\partial G}{\partial f_1} = 0 \quad (74)$$

$$G(f_1, H_2) = 0 \quad (75)$$

3.1.7. *The point of closest approach between hyperbolic and elliptic conic sections ($k_1 = 3$, $k_2 = 1$)*

$$\begin{aligned} S(H_1, E_2) = a_1 a_2 \{ & -e_1(\alpha \cos E_2 - \alpha e_2 + \beta \sqrt{1 - e_2^2} \sin E_2) + e_2(\gamma \sinh H_1 \sqrt{e_1^2 - 1} \\ & - \alpha \cosh H_1) + \cos E_2(\alpha \cosh H_1 - \gamma \sinh H_1 \sqrt{e_1^2 - 1}) \\ & + \beta \sqrt{1 - e_2^2} \sin E_2 \cosh H_1 - \zeta \sqrt{e_1^2 - 1} \sinh H_1 \} \\ & + a_1^2 (e_1 \cosh H_1 - 1)^2 + a_2^2 (e_2 \cos E_2 - 1)^2 \end{aligned} \quad (76)$$

$$G(H_1, E_2) = \sqrt{\frac{-a_1^3}{\mu}} (e_1 \sinh(H_1) - H_1) - \sqrt{\frac{a_2^3}{\mu}} (E_2 - e_2 \sin E_2) + \tau_1 - \tau_2 = 0 \quad (77)$$

Here, Eqs. (76) and (77) represent the objective function (S) and constraint (G), respectively, between hyperbolic and elliptic orbits. The basic equations of our special hyperbolic and elliptic orbit systems are as follows:

$$\begin{aligned} \frac{\partial S}{\partial H_1} = 2a_1^2 e_1^2 \operatorname{sh} H_1 \operatorname{ch} H_1 + a_1 (-2a_1 e_1 + \alpha \xi_2 + \beta \eta_2) \operatorname{sh} H_1 \\ + a_1 \sqrt{e_1^2 - 1} (\gamma \xi_2 + \zeta \eta_2) \operatorname{ch} H_1 \end{aligned} \quad (78)$$

$$\begin{aligned} \frac{\partial S}{\partial E_2} = -2a_2^2 e_2^2 \sin E_2 \cos E_2 + a_2 (2a_2 e_2 - \alpha \xi_1 - \gamma \eta_1) \sin E_2 \\ + a_2 \sqrt{(1 - e_2^2)} (\beta \xi_1 + \zeta \eta_1) \cos E_2 \end{aligned} \quad (79)$$

$$\frac{\partial G}{\partial H_1} = -\sqrt{-\frac{a_1^3}{\mu}} (1 - e_1 \cosh H_1) \quad (80)$$

$$\frac{\partial G}{\partial E_2} = -\sqrt{\left(\frac{a_2^3}{\mu}\right)} (1 - e_2 \cos E_2) \quad (81)$$

The system of Eqs. (78), (79), (80), and (81) are solved using Eqs. (21), (22), (23), and (24); we write as follows:

$$L(H_1, E_2) = \frac{\partial S}{\partial H_1} \frac{\partial G}{\partial E_2} - \frac{\partial S}{\partial E_2} \frac{\partial G}{\partial H_1} = 0 \quad (82)$$

$$G(H_1, E_2) = 0 \quad (83)$$

3.1.8. The point of closest approach between hyperbolic and parabolic conic sections ($k_1 = 3$, $k_2 = 2$)

$$\begin{aligned} S(H_1, f_2) = & a_1^2 (\cosh(H_1) - e_1)^2 + \alpha a_1 (\cosh(H_1) - e_1) q_2 \sec\left(\frac{f_2}{2}\right)^2 \\ & + q_2^2 \sec\left(\frac{f_2}{2}\right)^4 - \frac{2\gamma a_1 \sqrt{-1 + e_1^2} q_2 \sinh(H_1)}{1 + \cos(f_2)} + a_1^2 (-1 \\ & + e_1^2) \sinh(H_1)^2 + 2\beta a_1 (\cosh(H_1) - e_1) q_2 \tan\left(\frac{f_2}{2}\right) \\ & - 2\zeta a_1 \sqrt{-1 + e_1^2} q_2 \sinh(H_1) \tan\left(\frac{f_2}{2}\right) + 4q_2^2 \tan\left(\frac{f_2}{2}\right)^2 \end{aligned} \quad (84)$$

$$\begin{aligned} G(H_1, f_2) = & \sqrt{\frac{-a_1^3}{\mu}} (e_1 \sinh(H_1) - H_1) - \sqrt{\frac{2q_2^3}{\mu}} \left(\tan\left(\frac{f_2}{2}\right) + \frac{1}{3} \tan^3\left(\frac{f_2}{2}\right) \right) + \tau_1 - \tau_2 \\ = & 0 \end{aligned} \quad (85)$$

Here, Eqs. (84) and (85) represent the objective function (S) and constraint (G), respectively, between hyperbolic and parabolic orbits. The basic equations of our special hyperbolic and parabolic orbits systems are as follows:

$$\begin{aligned} \frac{\partial S}{\partial H_1} = & 2a_1^2 e_1^2 \operatorname{shin}H_1 \operatorname{cosh}H_1 + a_1 (-2a_1 e_1 + \alpha \xi_2 + \beta \eta_2) \operatorname{shin}H_1 \\ & + a_1 \sqrt{e_1^2 - 1} (\gamma \xi_2 + \zeta \eta_2) \operatorname{cosh}H_1 \end{aligned} \quad (86)$$

$$\frac{\partial S}{\partial f_2} = q_2 \sec^2\left(\frac{f_1}{2}\right) \left\{ 2q_2 \tan^3\left(\frac{f_2}{2}\right) + \beta \xi_1 + \zeta \eta_1 - (\alpha \xi_1 + \gamma \eta_1) \tan\left(\frac{f_2}{2}\right) \right\} \quad (87)$$

$$\frac{\partial G}{\partial H_1} = \sqrt{-\frac{a_2^3}{\mu}} (1 - e_1 \cosh(H_1)) \quad (88)$$

$$\frac{\partial G}{\partial f_2} = \sqrt{\frac{q_2^3}{2\mu}} \sec^4\left(\frac{f_2}{2}\right) \quad (89)$$

The formula in Eq. (86) is solved using Eqs. (21) and (22); thus, we write as follows:

$$L(H_1, f_2) = \frac{\partial S}{\partial H_1} \frac{\partial G}{\partial f_2} - \frac{\partial S}{\partial f_2} \frac{\partial G}{\partial H_1} = 0 \quad (90)$$

$$G(H_1, f_2) = 0 \quad (91)$$

3.1.9. The point of closest approach between two hyperbolic conic sections ($k_1 = 3, k_2 = 3$)

$$\begin{aligned}
S(H_1, H_2) = & a_1^2(e_1 \cosh H_1 - 1)^2 + a_2^2(e_2 \cosh H_2 - 1)^2 \\
& + a_1 a_2 \left(\cosh H_2 (\alpha \cosh H_1 - \gamma \sinh H_1 \sqrt{e_1^2 - 1}) + (-\alpha \cosh H_1 \right. \\
& + \gamma \sinh H_1 \sqrt{e_1^2 - 1}) e_2 + \sinh H_2 (-\beta \cosh H_1 \\
& + \zeta \sinh H_1 \sqrt{e_1^2 - 1}) \sqrt{e_2^2 - 1} + e_1 (-\alpha \cosh H_2 + \alpha e_2 \\
& \left. + \beta \sinh H_2 \sqrt{e_2^2 - 1}) \right)
\end{aligned} \tag{92}$$

$$\begin{aligned}
G(H_1, H_2) = & \sqrt{\left(-\frac{a_1^3}{\mu}\right)} (e_1 \sinh H_1 - H_1) - \sqrt{\left(-\frac{a_2^3}{\mu}\right)} (e_2 \sinh H_2 - H_2) + \tau_1 - \tau_1 \\
= & 0
\end{aligned} \tag{93}$$

Here, Eqs. (92) and (93) represent the objective function (S) and constraint (G), respectively, of two hyperbolic orbits. The basic equations of our special hyperbolic orbit systems are as follows:

$$\begin{aligned}
\frac{\partial S}{\partial H_1} = & 2a_1^2 e_1^2 \sinh H_1 \cosh H_1 + a_1 (-2a_1 e_1 + \alpha \xi_2 + \beta \eta_2) \sinh H_1 \\
& + a_1 \sqrt{e_1^2 - 1} (\gamma \xi_2 + \zeta \eta_2) \cosh H_1
\end{aligned} \tag{94}$$

$$\begin{aligned}
\frac{\partial S}{\partial H_2} = & 2a_2^2 e_2^2 \sinh H_2 \cosh H_2 + a_1 (-2a_2 e_2 + \alpha \xi_1 + \gamma \eta_1) \sinh H_2 \\
& - a_2 \sqrt{e_2^2 - 1} (\beta \xi_1 + \zeta \eta_1) \cosh H_2
\end{aligned} \tag{95}$$

$$\frac{\partial G}{\partial H_1} = \sqrt{-\frac{a_1^3}{\mu}} (1 - e_1 \cosh(H_1)) \tag{96}$$

$$\frac{\partial G}{\partial H_2} = -\sqrt{-\frac{a_2^3}{\mu}} (1 - e_2 \cosh(H_2)) \tag{97}$$

The formula in Eq. (94) is solved using Eqs. (21) and (22); accordingly, we write as follows:

$$L(H_1, H_2) = \frac{\partial S}{\partial H_1} \frac{\partial G}{\partial H_2} - \frac{\partial S}{\partial H_2} \frac{\partial G}{\partial H_1} = 0 \tag{98}$$

$$G(H_1, H_2) = 0 \tag{99}$$

3.2. Hessian matrix derivatives

Now, we find the Hessian matrix for elliptic, parabolic, and hyperbolic conics.

$$\begin{aligned}
\frac{\partial^2 S}{\partial E_1^2} = & -a_1 \cos E_1 (\beta \eta_2 - 2a_1 e_1 + \alpha \xi_2) - 2a_1^2 e_1^2 \cos 2E_1 \\
& - a_1 \sin E_1 \sqrt{(1 - e_1^2)} (\gamma \xi_2 + \zeta \eta_2)
\end{aligned} \tag{100}$$

$$\begin{aligned} \frac{\partial^2 S}{\partial E_2^2} = & -a_2 \cos E_2 (\alpha \xi_1 - 2a_2 e_2 + \gamma \eta_1) - 2a_2^2 e_2^2 \cos 2E_2 \\ & - a_2 \sin E_2 \sqrt{(1 - e_2^2)} (\beta \xi_1 + \zeta \eta_2) \end{aligned} \quad (101)$$

$$\begin{aligned} \frac{\partial^2 S}{\partial^2 H_1} = & a_1 \cosh(H_1) (\beta \eta_2 - 2a_1 e_1 + \alpha \xi_2) + 2a_1^2 e_1^2 \cosh(2H_1) \\ & + a_1 \sinh(H_1) \sqrt{(e_1^2 - 1)} (\xi_2 \gamma + \zeta \eta_2) \end{aligned} \quad (102)$$

$$\begin{aligned} \frac{\partial^2 S}{\partial^2 H_2} = & 2a_2^2 e_2^2 \cosh(2H_2) + a_2 \cosh(H_2) (\gamma \eta_1 - 2a_2 e_2 + \alpha \xi_1) \\ & + a_2 \sinh(H_2) \sqrt{(e_2^2 - 1)} (\beta \xi_1 + \zeta \eta_1) \end{aligned} \quad (103)$$

$$\begin{aligned} \frac{\partial^2 S}{\partial f_1^2} &= \frac{q_1 \sin\left(\frac{f_1}{2}\right) \left(2q_1 \tan^3\left(\frac{f_1}{2}\right) - \tan\left(\frac{f_1}{2}\right) (\beta \eta_2 + \alpha \xi_2) + \gamma \xi_2 + \zeta \eta_2\right)}{\cos^3\left(\frac{f_1}{2}\right)} \\ &= \frac{q_1 \left((\beta \eta_2 + \alpha \xi_2) \left(\frac{\tan\left(\frac{f_1}{2}\right)^2}{2} + \frac{1}{2}\right) - 6q_1 \tan^2\left(\frac{f_1}{2}\right) \left(\frac{\tan^2\left(\frac{f_1}{2}\right)}{2} + \frac{1}{2}\right) \right)}{\cos^2\left(\frac{f_1}{2}\right)} \end{aligned} \quad (104)$$

$$\begin{aligned} \frac{\partial^2 S}{\partial f_2^2} &= \frac{q_2 \sin\left(\frac{f_2}{2}\right) \left(2q_2 \tan^3\left(\frac{f_2}{2}\right) + \beta \xi_1 - \tan\left(\frac{f_2}{2}\right) (\eta_1 \gamma + \alpha \xi_1) + \zeta \eta_1\right)}{\cos^3\left(\frac{f_2}{2}\right)} \\ &= \frac{q_2 \left(\left(\frac{\tan^2\left(\frac{f_2}{2}\right)}{2} + \frac{1}{2}\right) (\eta_1 \gamma + \alpha \xi_1) - 6q_2 \tan^2\left(\frac{f_2}{2}\right) \left(\frac{\tan^2\left(\frac{f_2}{2}\right)}{2} + \frac{1}{2}\right) \right)}{\cos^2\left(\frac{f_2}{2}\right)} \end{aligned} \quad (105)$$

$$\frac{\partial^2 G}{\partial E_1^2} = e_1 \sqrt{\left(\frac{a_1^3}{\mu}\right)} \sin(E_1) \quad (106)$$

$$\frac{\partial^2 G}{\partial E_2^2} = -e_2 \sqrt{\left(\frac{a_2^3}{\mu}\right)} \sin(E_2) \quad (107)$$

$$\frac{\partial^2 G}{\partial f_1^2} = \frac{2 \sin\left(\frac{f_1}{2}\right) \sqrt{\left(\frac{\mu q_1^3}{2}\right)}}{\cos^5\left(\frac{f_1}{2}\right)} \quad (108)$$

$$\frac{\partial^2 G}{\partial f_2^2} = -\frac{2 \sin\left(\frac{f_2}{2}\right) \sqrt{\frac{\mu q_2^3}{2}}}{\cos^5\left(\frac{f_2}{2}\right)} \quad (109)$$

$$\frac{\partial^2 G}{\partial^2 H_1} = \sqrt{-\frac{a_1^3}{\mu}} e_1 \sinh(H_1) \quad (110)$$

$$\frac{\partial^2 G}{\partial^2 H_2} = -\sqrt{\frac{-a_2^3}{\mu}} e_2 \sinh(H_2) \quad (111)$$

$$\begin{aligned} \frac{\partial^2 S}{\partial E_1 \partial E_2} &= \frac{\partial^2 S}{\partial E_2 \partial E_1} = a_1 a_2 \left(\alpha \sin E_1 + \gamma \cos E_1 \sqrt{1 - e^2_1} \right) \sin E_2 \\ &\quad + a_1 a_2 \left(\sqrt{1 - e^2_2} \right) \left\{ -\beta \sin E_1 + \zeta \sqrt{(1 - e^2_1)} \cos E_1 \right\} \cos E_2 \end{aligned} \quad (112)$$

$$\begin{aligned} \frac{\partial^2 S}{\partial f_2 \partial f_1} &= \frac{\partial^2 S}{\partial f_1 \partial f_2} = q_1 q_2 \sec\left(\frac{f_1}{2}\right)^2 \sec\left(\frac{f_2}{2}\right)^2 \left(\zeta + \gamma \tan\left(\frac{f_2}{2}\right) \right. \\ &\quad \left. - \left\{ \beta + \alpha \tan\left(\frac{f_2}{2}\right) \right\} \tan\left(\frac{f_1}{2}\right) \right) \end{aligned} \quad (113)$$

$$\begin{aligned} \frac{\partial^2 S}{\partial H_1 \partial H_2} &= \frac{\partial^2 S}{\partial H_2 \partial H_1} \\ &= a_1 a_2 \left\{ \alpha \sinh H_2 - \beta \cosh H_2 \sqrt{(e^2_2 - 1)} \right\} \sinh H_1 \\ &\quad - a_1 a_2 \left(\cosh H_1 \sinh H_2 \sqrt{(e^2_1 - 1)} \left\{ \gamma + \zeta \sqrt{(e^2_2 - 1)} \right\} \right) \end{aligned} \quad (114)$$

$$\frac{\partial^2 G}{\partial E_1 \partial E_2} = \frac{\partial^2 G}{\partial E_2 \partial E_1} = 0 \quad (115)$$

$$\frac{\partial^2 S}{\partial f_2 \partial f_1} = \frac{\partial^2 S}{\partial f_1 \partial f_2} = 0 \quad (116)$$

$$\frac{\partial^2 S}{\partial H_2 \partial H_1} = \frac{\partial^2 S}{\partial H_1 \partial H_2} = 0 \quad (117)$$

The equation systems from Eq. (100) to Eq. (117) represent the coefficients of a Hessian matrix. We write one conic equation as an example so that the other formulas can be easily derived; all cases are the same. Selecting the condition $k_1 = 1$ and $k_2 = 1$, use Eqs. (26), (100), and (101). The equations for an elliptic orbit are Eqs. (30), (32), (33), and (34); we calculate λ (known as the Lagrange multiplier) from Eq. (24), as follows:

$$H_{(E_1, E_2)} = \begin{bmatrix} 0 & \frac{\partial G}{\partial E_1} & \frac{\partial G}{\partial E_2} \\ \frac{\partial G}{\partial E_1} & \frac{\partial^2 S}{\partial E_1^2} + \lambda \frac{\partial^2 G}{\partial E_1^2} & \frac{\partial^2 S}{\partial E_1 \partial E_2} + \lambda \frac{\partial^2 G}{\partial E_1 \partial E_2} \\ \frac{\partial G}{\partial E_2} & \frac{\partial^2 S}{\partial E_1 \partial E_2} + \lambda \frac{\partial^2 G}{\partial E_1 \partial E_2} & \frac{\partial^2 S}{\partial E_2^2} + \lambda \frac{\partial^2 G}{\partial E_2^2} \end{bmatrix} \quad (118)$$

where $H_{(E_1, E_2)} < 0$, and it is a local minimum.

4. RESULTS AND DISCUSSION

In this work, we take only different cases of elliptical orbits. We compare the elliptic case with the parabolic and hyperbolic cases, as shown in Table 1; MATLAB software is used for comparison. These data can be obtained from https://ssd.jpl.nasa.gov/?sb_elem#legend.

First, we study the closest approach between elliptic and elliptic orbits ($K_1 = 1, K_2 = 1$). Table 2 demonstrates the analysis of data between two elliptical orbits, which achieve minimum distance function, as shown in Figure 3. The validation of the local minimum (S) is applied using the determinant of the Hessian matrix. The results are shown in Table 2 (E_1 and E_2). The (G) constraint was achieved, as was the objective function (S) (minimal distance between two orbits). Moreover, in Figure 3, the black line is the constraint between two elliptical orbits. Furthermore, (S) is defined as a global minimum.

Table 3 shows the data between elliptic and parabolic cases, which achieve minimal distance function, as shown in Figure 4. The validation of the local minimum (S) is applied by determination of the Hessian matrix. The results are shown in Table 3 (E_1 and f_1). The (G) constraint required is achieved, and the objective function (S) is determined (minimal distance between two orbits). Besides, in Figure 4, the black line is the constraint between the two elliptical and parabolic orbits. Furthermore, the function (S) has been defined as a global minimum between two upper local minimums and a lower local minimum.

Table 1. Classic orbital elements from different orbits

Full_name	e	a (AU)	q (AU)	I (deg)	Om (deg)	W (deg)	Epoch	tp
<i>Elliptic</i>								
158P/Kowal-LINEAR	0.0291	4.719	4.5817	7.9084	137.3	231.57	2,455,007.50	2,456,186.60
331P/Gibbs	0.0420	3.003	2.8773	9.7396	216.8	177.40	2,455,961.50	2,455,283.50
<i>Parabolic</i>								
C/1998 V2 (SOHO)	1		0.0053	142.73	358.3	78.92	2,451,125.10	2,451,122.42
C/1998 V3 (SOHO)	1		0.0055	147.39	351.1	76.95	2,451,125.20	2,451,125.20
<i>Hyperbolic</i>								
C/1999 H3 (LINEAR)	1.002	1181.3	3.50	115.83	332.7	101.91	2,451,581.50	2,451,408.70
C/1999 T3 (LINEAR)	1.003	1658.4	5.36	104.75	223.5	211.08	2,451,580.50	2,451,786.70

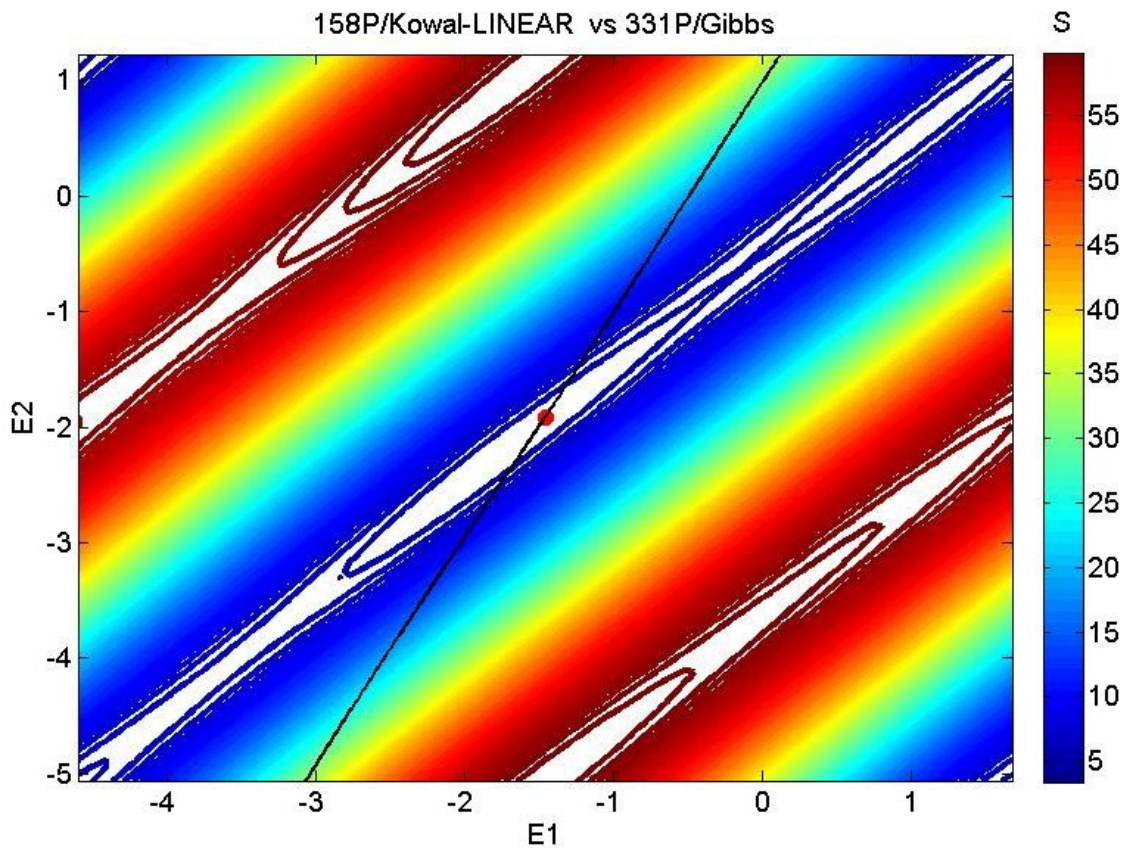


Figure 3. Closest approach between two Elliptic orbits (MOID)

Table 2. Orbit analysis of elliptic with elliptic orbits

λ	$-1.123966415967069 * 10^{-7}$
Hessian matrix	$10^6 * \begin{bmatrix} 0 & 2.720730319241977 & -1.407449249628583 \\ 2.720730319241977 & 0.000028815678919 & -0.000027821455392 \\ -1.407449249628583 & -0.000027821455392 & 0.000027997751636 \end{bmatrix}$
Eigenvalues of Hessian matrix	$10^6 * \begin{bmatrix} 3.063240793691692 \\ 0.000005462728107 \\ -3.063189442989243 \end{bmatrix}$
Determinant of Hessian matrix	$-5.125834486699852 * 10^{13}$
E_1	276.274609317199°
E_2	250.177611066412°
G (constraint)	0
\sqrt{S} (distance)	1.68821698778736

The second case is the comparison between elliptic and parabolic orbits ($k_1 = 1$ and $k_2 = 2$).

Table 3. Orbit analysis of elliptic with parabolic ($k_1 = 1$ and $k_2 = 2$)

λ	$1.6010 * 10^{-7}$
Hessian matrix	$10^6 * \begin{bmatrix} 0 & 2.784247599338849 & -0.001199656749179 \\ 2.784247599338849 & -0.000001080182139 & 0.000000380247510 \\ -0.001199656749179 & 0.000000380247510 & 0.000000207030237 \end{bmatrix}$
Eigenvalues of Hessian matrix	$10^6 * \begin{bmatrix} -2.784248398043477 \\ 0.000000207357675 \\ 2.784247317533900 \end{bmatrix}$
Determinant of Hessian matrix	$-1.607444186041507 * 10^{12}$
E_1	207.789°
E_2	119.534°
G (constraint)	0
\sqrt{S} (distance)	4.830880

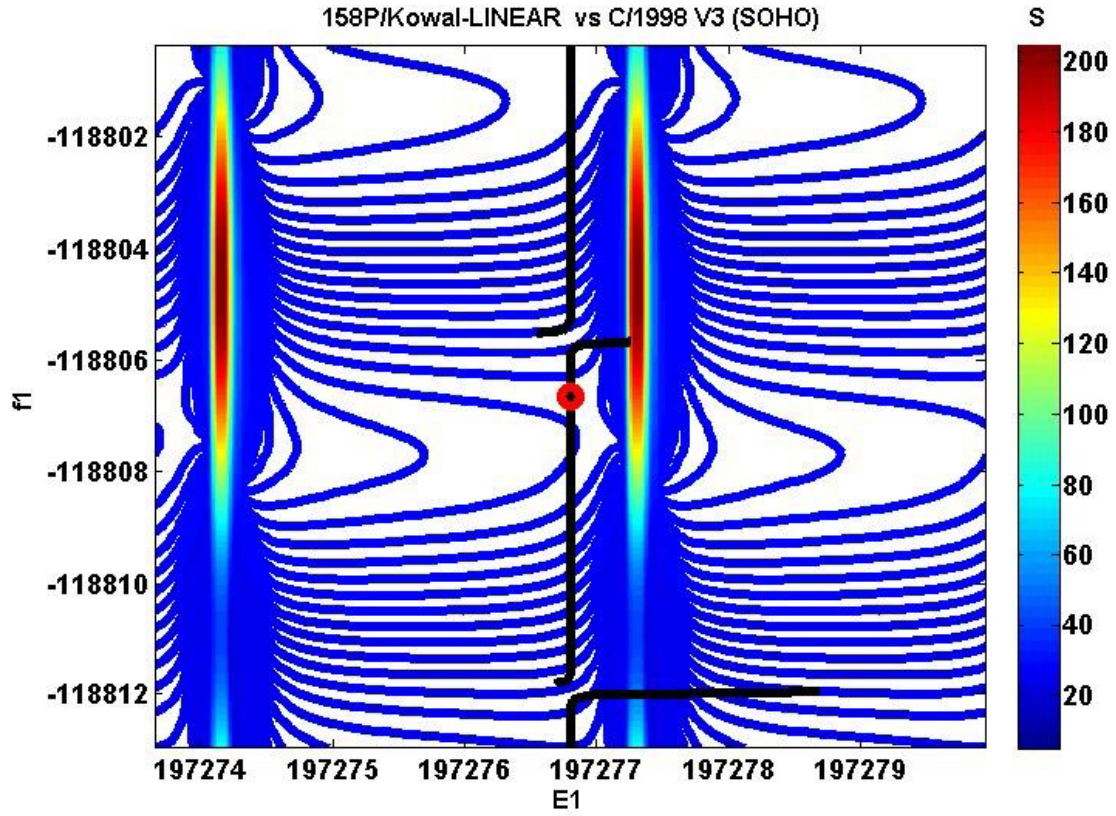


Figure 4. Closest approach between elliptic and parabolic orbits (MOID)

The third case is the comparison of elliptic and hyperbolic orbits ($k_1 = 1$ and $k_2 = 3$).

Table 4. Orbit analysis of elliptic and hyperbolic orbits ($k_1 = 1$ and $k_2 = 3$)

λ	$1.080854776120083 \cdot 10^{-4}$
Hessian matrix	$10^9 \cdot \begin{bmatrix} 0 & 0.002665024194745 & -1.05342954347067663 \\ 0.002665024194745 & 0.000000866698732 & 0.000001966300143 \\ -1.053429543470676 & -0.000001966300143 & -0.000335155241692 \end{bmatrix}$
Eigenvalues of Hessian matrix	$10^6 \cdot \begin{bmatrix} -2.784248398043477 \\ 0.000000207357675 \\ 2.784247317533900 \end{bmatrix}$
Determinant of Hessian matrix	$-9.594071548339930 \cdot 10^{20}$
E_1	192.4844831°
E_2	0.9630342904889
G (constraint)	0
\sqrt{S} (distance)	833.04326

Table 4 demonstrates the data analyzed between elliptic and hyperbolic orbits, which achieve minimum distance function, which is shown in **Figure 5**. The validation of the local minimum (S) is applied using the determinant of the Hessian matrix. The results are shown in Table 4 (E_1 and H_1). The G constraint required is achieved, and the objective function (S) is determined (minimal distance between two orbits). In addition, this figure shows that the black line is the constraint between the two elliptic and hyperbolic orbits. Furthermore, the function (S) is defined as a global minimum. To sum up, the figures show the minimum distance (MOID) between an elliptic, a parabolic, and a hyperbolic orbits, with analysis of the equations. Furthermore, both the objective function (S) and the constraint (G) are minimized (MOID), and the constraint of the problem is achieved. Finally, the constraint (G) is passing through the minimum distance (MOID).

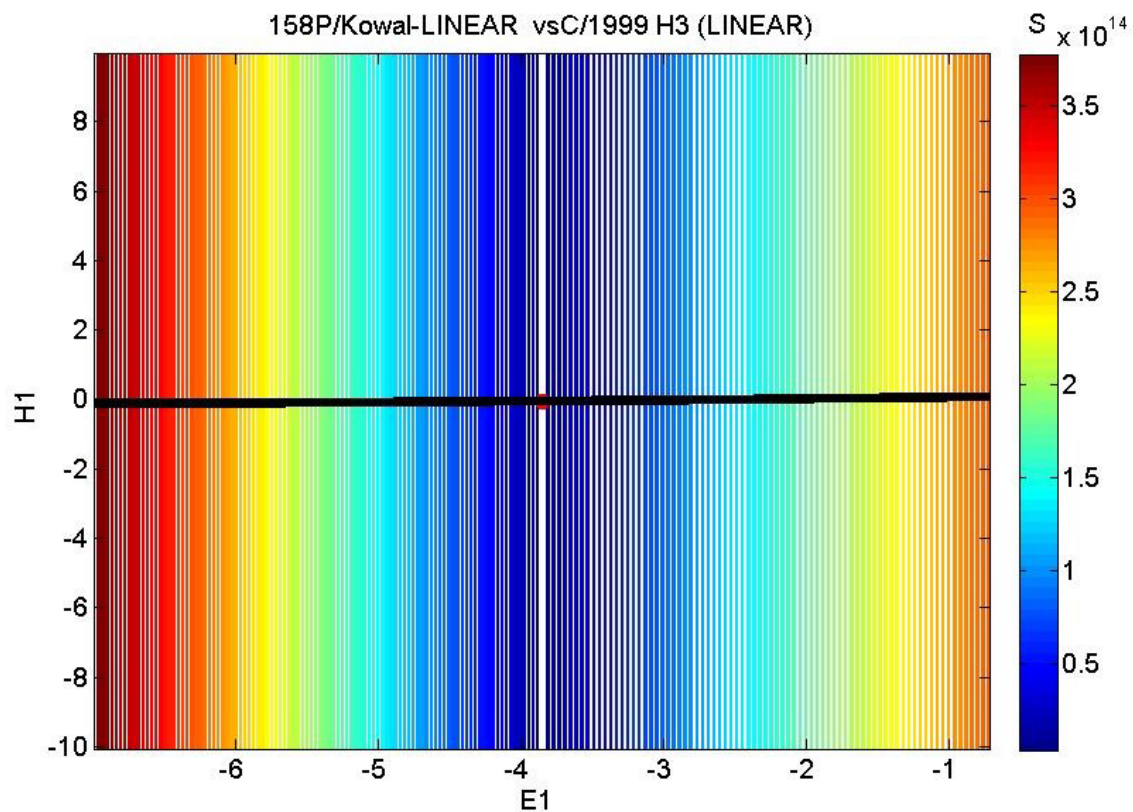


Figure 5. Closest approach between elliptic with hyperbolic orbits (MOID)

5. CONCLUSIONS

In the present paper, we have developed a constrained minimization technique to determine the point of optimal approach between two orbital conic sections, which are (elliptic–elliptic), (elliptic–parabolic), (elliptic–hyperbolic), (parabolic–elliptic), (parabolic–parabolic), (parabolic–hyperbolic), (hyperbolic–elliptic), (hyperbolic–parabolic), and (hyperbolic–hyperbolic). The developments were considered from two points of view, namely, analytical and computational. On the one hand, in the analytical developments, two literal expressions for the functions (S) and (G) were developed: (i) the minimum distance equation (S); and (ii) the constraint equation (G) for each case. Moreover, their first and second derivatives were deduced to calculate the Hessian matrix for each case. The point of closest approach between the conic sections of each case was obtained by solving the typical nonlinear system of equations. We applied this algorithm to a sample of different types of comets.

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