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Dedicated to Professor Jan Stochel
on the occasion of his 70th birthday

REDUCTION OF POSITIVE SELF-ADJOINT EXTENSIONS

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Abstract. We revise Krein's extension theory of semi-bounded Hermitian operators by reducing the problem to finding all positive and contractive extensions of the "resolvent operator" $(I + T)^{-1}$ of T. Our treatment is somewhat simpler and more natural than Krein's original method which was based on the Krein transform $(I-T)(I+T)^{-1}$. Apart from being positive and symmetric, we do not impose any further constraints on the operator T: neither its closedness nor the density of its domain is assumed. Moreover, our arguments remain valid in both real or complex Hilbert spaces.

Keywords: positive selfadjoint contractive extension, nonnegative selfadjoint extension, Friedrichs and Krein–von Neumann extension.

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1. INTRODUCTION

The theory of self-adjoint extensions of semi-bounded operators dates back at least to the seminal work of J. von Neumann [16] in 1929. He stated the conjecture that every semi-bounded symmetric operator admits a self-adjoint extension whose bound is the same as that of the original operator. Neumann himself gave a so called ε-solution to the problem, that is, he showed the existence of an extension whose bound only differs by an (arbitrarily given) $\varepsilon > 0$ from the given one. Today, the extension originating from Friedrichs [10] is mainly known, although Stone [29] and Freudenthal [9] also provided a solution to the raised problem. The proof of the latter can even be found in any textbooks and monographs (see e.g. [19–21]).

M.G. Krein in [15] developed a whole theory in which he showed that the bound-preserving extension is not unique in general. In fact, there are two "extremes" among those extensions which have several remarkable extremal properties. It is also important to mention here the paper [3] of Ando and Nishio who where the first to discuss the self-adjoint positive extendibility of *non-densely defined closed* operators.

The most general case of non-densely defined operators appeared in the paper [24] of J. Stochel and the first named author (see also [17, 18, 25]).

The starting point of Krein's treatment is the observation that every positive symmetric operator T satisfies the inequality

$$\|(I-T)f\|^2 \leq \|(I+T)f\|^2, \quad f \in \operatorname{dom} T.$$

This in turn means that

$$A := (I-T)(I+T)^{-1}$$

is a well defined symmetric contraction, defined on its natural domain $\operatorname{dom} A = \operatorname{ran}(I+T)$. (In the literature, A is called the Krein transform of T). Then the formula

$$\widetilde{T} := (I-\widetilde{A})(I+\widetilde{A})^{-1}$$

gives a positive self-adjoint extension of T, where $\widetilde{A}$ is any contractive self-adjoint extension of A such -1 does not belong to the point spectrum of $\widetilde{A}$. (If T is densely defined, then $\widetilde{A}$ automatically fulfills the latter requirement.) Thus Krein reduced the original question to the less involved problem of finding all contractive extensions of a bounded symmetric operator (this procedure and also an explicit parametrization of positive self-adjoint extensions of a given positive operator is found in [6], and also in the monograph [11]).

The main goal of our present article is to propose another approach to Neumann's extension problem. Namely, we reduce the question of positive self-adjoint extendibility to the problem of finding a contractive positive extension B of a given positive and contractive operator A. In fact, the inequality

$$\|h\|^2 \leq ((I+T)h, h), \quad h \in \operatorname{dom} T,$$

gives rise to define the positive and contractive operator

$$A := (I+T)^{-1},$$

defined on the (not necessarily closed or dense) subspace $\mathcal{D} := \operatorname{ran}(I+T)$. If $B \in \mathcal{B}(\mathcal{H})$, $\|B\| \leq 1$, is any positive extension of A such that $\ker B = \{0\}$, then

$$S = B^{-1} - I$$

is apparently a positive and self-adjoint extension of T. Accordingly, the positive extendibility of T depends on whether A has any *injective* contractive positive extension.

The question of extendibility of a bounded positive operator appears in classical papers, for example, of Y. Shmulian [28], and P. R. Halmos [12]. The approach of the present article follows the factorization procedure of the first author [23] and serves as the cornerstone of our reasoning. This result enables us to give a simple description of the set of all positive self-adjoint extensions of a densely defined positive operator T. On the other hand, that construction makes it also possible to give a fairly short proof of Krein's theorem about contractive self-adjoint extensions of a contractive symmetric transformation.

2. POSITIVE EXTENSIONS OF CONTRACTIONS

We begin this section with a result on contractive positive extensions of bounded positive operators with a non-dense domain. The next lemma and its consequence Theorem 2.2 will serve as the basis of our further study of positive self-adjoint extensions of symmetric operators.

Lemma 2.1. *Let A be a positive symmetric contraction in the real or complex Hilbert space $\mathcal{H}$, defined on the (closed) linear subspace $\mathcal{D}$. Then the following statements are equivalent:*

(i) *there exists a positive symmetric contraction B that extends A,*
(ii) *A fulfills the inequality*

$$\|Ah\|^2 \leq (Ah, h), \quad h \in \mathcal{D}.$$

Proof. Implication (i)⇒(ii) is straightforward from inequality

$$\|Bh\|^2 \leq \|B^{1/2}\|^2 \|B^{1/2}h\|^2 = \|B\|(Bh, h), \quad h \in \mathcal{H}.$$

The proof of the converse implication can be found e.g. in [23]. Nevertheless, for the sake of completeness we include a short proof here.

Consider the range space $\operatorname{ran} A$ of A as a pre-Hilbert space endowed with the inner product

$$\langle Ah, Ak \rangle_A := (Ah, k), \quad h, k \in \mathcal{D}.$$

Indeed, inequality (ii) guarantees that $\langle \cdot, \cdot \rangle_A$ is well-defined inner product. Let H_A denote the completion of that pre-Hilbert space. Note that $\operatorname{ran} A$ is then a dense linear subspace of H_A. The inequality in (ii) enables us then to introduce the following linear contraction J_A from $\operatorname{ran} A \subseteq H_A$ to $\mathcal{H}$ by setting

$$J_A(Ah) := Ah, \quad h \in \mathcal{D}.$$

Indeed, we have

$$\|J_A(Ah)\|^2 = \|Ah\|^2 \leq (Ah, h) \leq \langle Ah, Ah \rangle_A,$$

thanks to (ii). J_A extends now to contraction acting between $\mathcal{H}_A$ and $\mathcal{H}$. For simplicity, we continue to denote this operator by J_A. We claim that the adjoint J_A^* of J_A has the following characteristic property:

$$J_A^* h = Ah, \quad h \in \mathcal{D}. \tag{2.1}$$

This follows from

$$(J_A(Ak), h) = (Ak, h) = \langle Ak, Ah \rangle_A, \quad h, k \in \mathcal{D}.$$

As a consequence, we conclude that the positive contraction $J_A J_A^* \in \mathcal{B}(\mathcal{H})$ extends A:

$$J_A J_A^* h = J_A(Ah) = Ah, \quad h \in D.$$

This proves implication (ii)⇒(i). □

If the positive symmetric operator $A:\mathcal{H}\supseteq\mathcal{D}\to\mathcal{H}$ fulfills any of the equivalent statements of Lemma 2.1 then we shall use the notation

$$A_m \coloneqq J_A J_A^*. \tag{2.2}$$

In the following, we are going to prove some extremal properties of the operator A_m. More precisely, we will prove that the positive contractions extending A form an operator interval whose smallest element is A_m, while the largest one is $I-(I-A)_m$.

Theorem 2.2. *Let $A:\mathcal{H}\supseteq\mathcal{D}\to\mathcal{H}$ be a positive and symmetric operator in the real or complex Hilbert space $\mathcal{H}$ such that*

$$\|Ah\|^2 \leq (Ah,h), \quad h\in\mathcal{D}. \tag{2.3}$$

The set of positive symmetric contractions extending A form an operator interval $[A_m, A_M]$, where $A_M = I-(I-A)_m$.

Proof. First we are going to check that $A_m = J_A J_A^*$ is the smallest positive symmetric extension of A. To do so, by using the denseness of $\operatorname{ran} A$ in the energy space H_A we conclude that, for every $g\in\mathcal{H}$,

$$\begin{aligned}0 &= \inf\{\langle J_A^*g + Ah, J_A^*g+Ah\rangle_{A} \,:\, h\in\mathcal{D}\}\\ &= (J_AJ_A^*g,g) + \inf\{(g,Ah)+(Ah,g)+(Ah,h) \,:\, h\in\mathcal{D}\}\end{aligned}$$

Now if B is any positive contractive symmetric extension of A then

$$\begin{aligned}0 &= (J_AJ_A^*g,g) + \inf\{(g,Bh)+(Bh,g)+(Bh,h) \,:\, h\in\mathcal{D}\}\\ &\geq (J_AJ_A^*g,g) + \inf\{(g,Bh)+(Bh,g)+(Bh,h) \,:\, h\in\mathcal{H}\}\\ &= (J_AJ_A^*g,g) + \inf\{\langle J_B^*g,h\rangle_{B} + \langle h, J_B^*g\rangle_{B} + \langle Bh,Bh\rangle_{B} \,:\, h\in\mathcal{H}\}\\ &= (J_AJ_A^*g,g) - (J_BJ_B^*g,g) + \inf\{\langle J_B^*g+Bh, J_B^*g+Bh\rangle_{B} \,:\, h\in\mathcal{H}\}\\ &= (J_AJ_A^*g,g) - (Bg,g),\end{aligned}$$

thanks to the argument of the proof of Lemma 2.1, by applying it to B instead of A. As a consequence, $J_AJ_A^*\leq B$.

To construct the maximal contractive extension we start by observing that

$$\|(I-A)h\|^2 \leq ((I-A)h,h), \quad h\in\mathcal{D}.$$

Thus the positive symmetric operator $I-A$ fulfills the conditions of Lemma 2.1. Let now B be any positive contractive extension of A, then our reasoning above yields

$$(I-A)_m \leq I-B.$$

Hence $B = I-(I-B)\leq I-(I-A)_m$ which means just that $A_M \coloneqq I-(I-A)_m$ is the maximum of all contractive positive extensions of A.

Let now B be any operator such that $A_m \leq B \leq A_M$. Applying the Cauchy–Schwarz inequality to the positive operator $A_M - B$ and to the vectors $h \in \mathcal{D}$ and $g \in \mathcal{H}$ we get just

$$\begin{aligned}|((A_M - B)h, g)|^2 &\leq ((A_M - B)h, h)((A_M - B)g, g) \\ &\leq ((A_M - A_m)h, h)((A_M - B)g, g) = 0,\end{aligned}$$

whence $Bh = A_M h = Ah$, indeed. □

3. REDUCTION OF THE KREIN-VON NEUMANN EXTENSION

In this section we deal with positive self-adjoint extendibility of possibly unbounded operators. The essence of the argument we are going to use here is that the positive self-adjoint extendibility of T will be reduced to the analysis of the positive symmetric contraction $A := (I + T)^{-1}$.

The main idea of our procedure is based on the observation that if S is an arbitrary positive self-adjoint extension of T, then $B := (I+S)^{-1} \in \mathcal{B}(\mathcal{H})$ is a positive contractive extension of $A = (I + T)^{-1}$, such that

$$S = B^{-1} - I.$$

Conversely, if B is any contractive positive extension of A then $S = B^{-1} - I$ is a positive self-adjoint extension of T, provided it exists, i.e., $\ker B = \{0\}$. Taking this into account, the positive extendibility of T depends on whether A has any *injective* contractive positive extension.

Lemma 3.1. *Let $A : \mathcal{D} \to \mathcal{H}$ be a positive symmetric operator satisfying* (2.3) *and denote by A_M the maximal contractive positive extension of A. For a vector $g \in \mathcal{H}$ the following statements are equivalent:*

(i) $g \in \ker A_M$,
(ii) *there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ such that*

$$(I - A)g_n \to g \quad \textit{and} \quad ((I - A)g_n, Ag_n) \to 0.$$

Proof. First of all observe that $A_M g = 0$ if and only if $(A_M g, g) = 0$, that is,

$$\|g\|^2 = ((I - A)_m g, g).$$

Here the right hand side can be calculated as follows:

$$\begin{aligned}((I - A)_m g, g) &= -\inf_{h\in\mathcal{D}}\{(g, (I - A)h) + (g, (I - A)h) + ((I - A)h, h)\} \\ &= -\inf_{h\in\mathcal{D}}\{\|(I - A)h + g\|^2 - \|g\|^2 - \|(I - A)h\|^2 + ((I - A)h, h)\} \\ &= \|g\|^2 - \inf_{h\in\mathcal{D}}\{\|(I - A)h + g\|^2 + ((I - A)h, Ah)\}.\end{aligned}$$

From the above calculations, the equivalence between (i) and (ii) can be easily deduced. □

Let now T be a (not necessarily bounded) positive symmetric operator acting in the Hilbert space $\mathcal{H}$ with domain $\operatorname{dom} T$. It is straightforward that $I+T$ is injective, and that its inverse

$$A := (I+T)^{-1} \tag{3.1}$$

acts as a contractive positive symmetric operator on $\mathcal{D} := \operatorname{ran}(I+T)$. Furthermore, an immediate calculation shows that

$$\|Ah\|^2 \le (Ah, h), \quad h \in \mathcal{D}.$$

This in turn means that A satisfies condition (ii) of Theorem 2.2, hence both the minimal and maximal positive contractive extensions A_m and A_M of A do exist. In what follows, we are going to analyze the relation between the positive self-adjoint extensions of T and the operators A_m and A_M.

Proposition 3.2. *Let T be a positive operator in the real or complex Hilbert space $\mathcal{H}$ and let $A := (I+T)^{-1}$. Then*

$$\ker A_M = \{g \in \mathcal{H} : \exists (g_n)_{n\in\mathbb{N}} \subset \operatorname{dom} T, Tg_n \to g, (Tg_n, g_n) \to 0\}$$

Proof. Let $g \in \ker A_M$ be any vector. According to Lemma 3.1, there exists $(h_n)_{n\in\mathbb{N}}$ in $\mathcal{D} := \operatorname{ran}(I+T)^{-1}$ such that

$$(I-A)h_n \to g \quad \text{and} \quad ((I-A)h_n), Ah_n) \to 0.$$

Since we have $I - A = T(I+T)^{-1}$ it follows that $g_n = (I+T)^{-1}h_n \in \operatorname{dom} T$, $Tg_n \to g$ and $(Tg_n, g_n) \to 0$, which completes the proof. □

The next theorem gives a characterization of positive self-adjoint extendibility of (not necessarily densely defined) positive operators (cf. also [3, 24, 25]).

Theorem 3.3. *Let T be a positive symmetric operator in the Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(i) *there is a positive self-adjoint extension S of T,*
(ii) $\ker A_M = \{0\}$,
(iii) *for every sequence $(g_n)_{n\in\mathbb{N}}$ of $\operatorname{dom} T$ such that $Tg_n \to g$ and $(Tg_n, g_n) \to 0$ it follows that $g = 0$.*

In any case,

$$T_m := A_M^{-1} - I \tag{3.2}$$

is the smallest positive self-adjoint extension of T (i.e., $(I+S)^{-1} \le (I+T_m)^{-1}$ holds for every positive self-adjoint extension S of T).

Proof. Let S be a positive self-adjoint extension of T and consider a sequence $(g_n)_{n\in\mathbb{N}}$ of $\operatorname{dom} T$ such that $Tg_n \to g$ and $(Tg_n, g_n) \to 0$. Letting $f_n := S^{1/2}g_n$ we gain a sequence $(f_n)_{n\in\mathbb{N}}$ of $\operatorname{dom} S^{1/2}$ such that $f_n \to 0$ and $S^{1/2}f_n \to g$, hence $g = 0$ due to closability of $S^{1/2}$. This proves implication (i)⇒(iii).

The equivalence between (ii) and (iii) is established in Proposition 3.2.

Assume now that $\ker A_M = \{0\}$ so that A_M^{-1} is a (densely defined) positive and self-adjoint operator. In fact, $A_M \leq I$ implies $(A_M)^{-1} \geq I$, hence $T_m := (A_M)^{-1} - I$ is a positive and self-adjoint operator in $\mathcal{H}$. A straightforward calculation shows that T_m extends T. Hence (iii) implies (i).

Finally we prove that T_m is the smallest among the set of all positive self-adjoint extensions of T. For let S be any positive self-adjoint extension of T. Then $(I+S)^{-1} \in \mathcal{B}(\mathcal{H})$ is a positive contraction which extends A, hence $(I+S)^{-1} \leq A_M$. Consequently, $I + S \geq (A_M)^{-1}$, thus $T_m \leq S$. □

In the last corollary of this section we present an application which essentially uses the fact that Theorem 3.3 is valid for non-densely defined operators too (see also [26, 27]):

Corollary 3.4. *Let L be a densely defined linear operator between the real or complex Hilbert space $\mathcal{H}$ and $\mathcal{K}$. Then L^*L has (at least one) positive and self-adjoint extension.*

Proof. We show that L^*L fulfills (iii) of Theorem 3.3. For take a sequence $(g_n)_{n\in\mathbb{N}}$ from $\operatorname{dom} L^*L$ such that $(L^*Lg_n, g_n) \to 0$ and $L^*Lg_n \to g$ for some vector g. The first condition entails $Lg_n \to 0$, hence $g = 0$ because of closedness of L^*. Theorem 3.3 completes then the proof. □

4. REDUCTION OF THE FRIEDRICHS EXTENSION

In the preceding section we characterized those positive symmetric operators which can be extended to a positive self-adjoint operator. In particular, we proved that if the set of the positive self-adjoint extensions has a smallest element (provided it is not empty). It is not hard to see that the largest positive self-adjoint extensions may exist only when the domain of T is a total subspace. According to Krein's famous result [15], in the densely defined case, in addition to the smallest positive self-adjoint extension there exists the largest one too. While the smallest extension of T was constructed from the largest contractive extension of the bounded positive operator $A = (I+T)^{-1}$, we will use the smallest extension A_m of A to gain the largest (Friedrichs) extension of T.

Lemma 4.1. *Let $A : \mathcal{H} \supseteq \mathcal{D} \to \mathcal{H}$ be a positive operator satisfying* (2.3). *Then*

$$\ker A_m = \operatorname{ran} A^{\perp}.$$

Proof. Based on the construction presented in the proof of Lemma 2.1 we have

$$\ker A_m^{\perp} = \overline{\operatorname{ran} A_m} = \overline{\operatorname{ran} J_A} = \overline{\operatorname{ran} A}$$

which entails the desired identity. □

Theorem 4.2. *Let T be a densely defined positive symmetric operator in the Hilbert space $\mathcal{H}$. Then* $\ker A_m = \{0\}$ *and*

$$T_M := (A_m)^{-1} - I$$

is a positive and self-adjoint extension of T. In fact, T_M is the largest among the set of all positive self-adjoint extensions of T.

Proof. Taking into account of identity

$$\operatorname{ran} A = \operatorname{dom} T,$$

the injectivity of A_m follows from the preceding Lemma. Since $A_m \leq I$, it follows that $T_M = (A_m)^{-1} - I$ is a positive and self-adjoint operator which apparently extends T. Finally, if S is any positive self-adjoint extension of T then $(I+S)^{-1}$ is a contractive positive extension of A. Hence $A_m \leq (I+S)^{-1}$ from which we infer that $(A_m)^{-1} - I \geq S$. This proves the theorem. □

Corollary 4.3. *Let T be a densely defined positive symmetric operator in the Hilbert space $\mathcal{H}$. A positive self-adjoint operator S is an extension of T if and only if $T_m \leq S \leq T_M$.*

Proof. If S is a positive self-adjoint extension of T, then $T_m \leq S \leq T_M$, according to Theorems 3.3 and 4.2. On the contrary, if $T_m \leq S \leq T_M$ then $(T_M+I)^{-1} \leq (S+I)^{-1} \leq (T_m+I)^{-1}$ from which we infer that the positive contraction $B := (S+I)^{-1} - I$ extends A due to Theorem 2.2. This in turn means that $T \subset S$, as claimed. □

Remark 4.4. From the proof of Corollary 4.3 it turns out that the map

$$\mathcal{S} \ni S \mapsto (S+I)^{-1} \in \mathcal{A} \tag{4.1}$$

is a bijective correspondence between the set $\mathcal{S}$ of all positive self-adjoint extensions of T and the set $\mathcal{A}$ of all contractive positive extensions of $A = (I+T)^{-1}$. If we consider the Krein transform $K(T) := (I-T)(I+T)^{-1}$ instead, then a similar bijective correspondence can be given between the set of contractive symmetric extensions of $K(T)$ and the set $\mathcal{S}$, see e.g. [15]. With the help of an accurate description of all contractive extensions of a given sub-operator [6, Theorem 1.3], T. Constantinescu and A. Gheondea provided an explicit and versatile parametrization of the set $\mathcal{S}$, see [7, Theorem 3.3] and also [11, Theorem 2.3.10].

5. COMPLETION OF INCOMPLETE MATRICES

Let A^0 be an incomplete block operator of the form

$$A^0 := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & * \end{bmatrix} \tag{5.1}$$

on the orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of the Hilbert space $\mathcal{H}$ where $A_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$, $i,j = 1,2$, except $i = j = 2$. Our completion problem is to find

$A_{22} \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_2)$ (necessarily positive) operator such that the completed block operator is a positive contraction. of course we may assume that A_{11} is a positive contraction and $A_{12} = A_{21}^*$ because of the symmetry of A.

Taking

$$A^1 \coloneqq \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$$

we reduce the issue to our starting problem if A^1 admits a positive contractive extension. Moreover, the smallest extension $(A^1)_m$ will be identified as the smallest positive contractive completion of A^0. Accordingly, $A_{22} \coloneqq P_2(A^1)_m P_2$ is the smallest possible completion of (5.1). (Here P_2 denotes the orthogonal projection onto $\mathcal{H}_2$.)

Theorem 5.1. *The block operator* (5.1) *has a positive contractive completion if and only if*

$$((A_{11}^2 + A_{21}^* A_{21})h, h) \leq (A_{11}h, h) \quad (h \in \mathcal{H}_1). \tag{5.2}$$

Proof. Equality (5.2) simply expresses that the positive symmetric operator A^1 satisfies inequality (ii) of Lemma 2.1 and accordingly, A^1 extends to a positive contraction $B \in \mathcal{B}(\mathcal{H})$. It is clear that the block matrix representation of B along the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is of the form

$$B = \begin{bmatrix} A_{11} & A_{21}^* \\ A_{21} & A_{22} \end{bmatrix}$$

with some positive symmetric $A_{22} \in \mathcal{B}(\mathcal{H}_2)$. □

Under the assumptions of Theorem 4.2, the smallest extension $(A^1)_m$ of A^1 can be identified as the smallest positive contractive completion of A^0. Accordingly, $A_{22} \coloneqq P_2(A^1)_m P_2$ is the smallest possible completion of (5.1). (Here, P_2 denotes the orthogonal projection onto $\mathcal{H}_2$.) On the other hand, $(A^1)_M$ is obviously the largest contractive positive extension of A^1 and thus $P_2(A^1)_M P_2$ is identical with the largest block completing A^0 to a positive contraction. Reordering of inequality (5.2) expresses that

$$A_{21}^* A_{21} \leq (I - A_{11})A_{11}.$$

Using Douglas' factorization theorem [8], this holds if and only if there exists a contraction $D : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$A_{21} = D(I_{\mathcal{H}_1} - A_{11})^{1/2} A_{11}^{1/2}.$$

Taking into account the above considerations, as well as Theorem 2.2, we obtain the following result:

Corollary 5.2. *Let $A^0 : \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ be an incomplete block operator matrix of the form* (5.1) *and introduce the notation $H \coloneqq (I_{\mathcal{H}_1} - A_{11})^{1/2} A_{11}^{1/2}$. Then the following assertions are equivalent:*

(i) *there exists a* (*necesserily positive*) *$A_{22} : \mathcal{H}_2 \to \mathcal{H}_2$ such that the completed block operator is a positive contraction,*
(ii) *$A_{11} \geq 0$ and there exists a contraction $D : \mathcal{H}_1 \to \mathcal{H}_2$ such that $A_{21} = DH$.*

In any case, the set of positive contractive completions of A^0 is identical with

$$\left\{\begin{bmatrix} A_{11} & HD^* \\ DH & A_{22} \end{bmatrix} : P_2(A^1)_m|_{\mathcal{H}_2} \le A_{22} \le P_2(A^1)_M|_{\mathcal{H}_2}\right\}.$$

For contractive complementation of (symmetric) contractions, we refer the reader to [6, 7, 11].

Remark 5.3. Closing the section we note that the usual completion problem asks for a bounded (but not necessarily contractive) positive symmetric completion Of course, this means that A^1 must satisfy the following identity:

$$\|A^1h\|^2 \le \alpha(A^1h, h), \quad h \in \mathcal{H}_1.$$

Assuming that α is the smallest possible constant, we have that $(A^1)_M$ is equal to the smallest positive symmetric extension of A^1 with norm α. Accordingly,

$$(A^1)_M := \alpha - (\alpha - A^1)_m$$

is the largest such operator having norm α.

6. NORM PRESERVING SELF-ADJOINT EXTENSIONS

The key idea in Krein's method of finding all positive and self-adjoint extensions of a positive operator T was in reducing the original problem to construct all the extensions of the transformation

$$A := (I - T)(I + T)^{-1}$$

which are defined everywhere and which are self-adjoint and of norm at most 1. Although we choosed another way to describe all positive and self-adjoint extensions of T, Lemma 2.1 and Theorem 2.2 enable us to provide a quite short and simple proof of Krein's famous result [15, Theorem 2] on contractive self-adjoint extensions (cf. also [7, Theorem 3.1] and [11, Lemma 2.3.7]).

Theorem 6.1. *Let T be a norm one symmetric operator on a linear subspace $\mathcal{D}$ of a real or complex Hilbert soace $\mathcal{H}$. Then T always has a self-adjoint norm preserving contractive extension to the whole space $\mathcal{H}$. Moreover, the exist the smallest and the largest norm preserving self-adjoint extensions T_μ and T_M of T.*

Proof. Introduce the linear operators A^+, A^- by setting

$$A^+ := \frac{1}{2}(I + T), \quad A^- := \frac{1}{2}(I - T). \tag{6.1}$$

An immediate calculations shows that

$$\|A^{\pm}h\|^2 \le (A^{\pm}h, h), \quad h \in \mathcal{D}.$$

By Theorem 2.2, the corresponding smallest contractive (hence norm preserving) positive extensions $(A^{\pm})_m \in \mathcal{B}(\mathcal{H})$ exist and one easily verifiest that

$$T_\mu \coloneqq 2(A^+)_m - I \quad \text{and} \quad T_M \coloneqq I - 2(A^-)_m$$

are self-adjoint extensions of T having norm one. Furthermore, if $S \in \mathcal{B}(\mathcal{H})$ is any norm-one self-adjoint extension of T, then $\frac{1}{2}(I+S)$ and $\frac{1}{2}(I-S)$ are positive extensions of A^+ and A^-, respectively, whence

$$(A^+)_m \leq \frac{1}{2}(I+S) \quad \text{and} \quad (A^-)_m \leq \frac{1}{2}(I-S),$$

which apparently imply $T_\mu \leq S \leq T_M$. □

We conclude the paper with an interesting corollary of Theorem 6.1 on self-adjoint unitary extensions:

Corollary 6.2. *Assume that T is a symmetric isometry, then T_μ and T_M are self-adjoint unitary extensions of T.*

Proof. Being a symmetric isometry, T satisfies

$$\|(I \pm T)h\|^2 = 2((I \pm T)h, h), \quad h \in \mathcal{H}_1. \tag{6.2}$$

Let us apply now the proof of Lemma (2.1) to the positive operators $A^{\pm}$ in (6.1). Then (6.2) turns into

$$\|J_{\pm}(A^{\pm}h)\|^2 = (A^{\pm}h, h) = \langle A^{\pm}h, A^{\pm}h\rangle_{\pm}, \quad h \in \mathcal{D},$$

which means that $J_{\pm} : \mathcal{H}_{\pm} \to \mathcal{H}$ are isometries. According to Lemma 2.1 and Theorem 2.2 we have $J_{\pm}J_{\pm}^* = (A^{\pm})_m$, it follows that

$$\begin{aligned}
\|T_\mu g\|^2 &= \|2J_-J_-^*g - g\|^2 \\
&= 4\|J_-J_-^*g\|^2 - 4(J_-J_-^*g, g) + \|g\|^2 \\
&= 4(J_-(J_-^*J_-)J_-^*g, g) - 4(J_-J_-^*g, g) + \|g\|^2 = \|g\|^2,
\end{aligned}$$

and similarly,

$$\begin{aligned}
\|T_M g\|^2 &= \|g - 2J_+J_+^*g\|^2 \\
&= \|g\|^2 + 4\|J_+J_+^*g\|^2 - 4(J_+J_+^*g, g) \\
&= \|g\|^2 + 4(J_+(J_+^*J_+)J_+^*g, g) - 4(J_+J_+^*g, g) = \|g\|^2.
\end{aligned}$$

Hence the self-adjoint operators T_m, T_M are uniteries. □

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