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THERMODYNAMICS, MECHANICAL, AERONAUTICAL ENGINEERING AND ROBOTICS

Elastic buckling of a generalized cylindrical sandwich panel under axial compression

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Abstract. The paper is devoted to the buckling problem of an axially compressed generalized cylindrical sandwich panel and rectangular sandwich plate. The continuous variation of mechanical properties in the thickness direction of the structures is assumed. The generalized theory of deformation of the straight line normal to the neutral surface is applied. The analytical model of this sandwich panel is elaborated. Three differential equations of equilibrium of this panel based on the principle of stationary potential energy are obtained. This system of equations is analytically solved and the critical load is derived. Moreover, the limit transformation of the sandwich panel to a sandwich rectangular plate is presented. The critical loads of the example cylindrical panels and rectangular plates are derived.

Key words: cylindrical sandwich panel; rectangular sandwich plate; elastic buckling.

1. INTRODUCTION

The sandwich structures are intensively perfected in the 21st century. The basis of this activity is primarily analytical and numerical modelling of these structures. Zenkour [1] used the generalized shear deformation theory to study the static response of a simply supported functionally graded rectangular plate subjected to a transverse uniform load. Szyc *et al.* [2] considered the problem of elastic buckling of an open sandwich cylindrical thin-walled panel with three edges simply supported and one edge free under axial compression. Pandit *et al.* [3] proposed the improved higher-order zigzag plate theory for the accurate prediction of the buckling load of a softcore sandwich plate. Carrera and Brischetto [4] described a large variety of plate theories and assessed them to evaluate the bending and vibration of sandwich plates. Shen [5] described the geometrically nonlinear response of inhomogeneous isotropic and functionally graded plates and shells to nonlinear bending, post-buckling and nonlinear vibration. Reddy [6] reformulated the classical and shear deformation beam and plate theories using the nonlocal differential constitutive relations of Eringen and the von Kármán nonlinear strains. Carrera *et al.* [7] studied the transverse normal strain effects in classical and higher-order two-dimensional theories for one-layered and multi-layered plates and shells embedding functionally graded material (FGM) layers. The proposed higher-order theories have been implemented by referring to Carrera Unified Formulation (CUF). Belica *et al.* [8] and Belica and Magnucki [9] investigated the stability of an isotropic metal foam circular cylindrical shell subjected to combined loads. Malinowski *et al.* [10] discussed the problems of elastic stability and post-buckling behaviour of a seven-layered cylindrical shell subjected to uniformly distributed external pressure and simply supported at circular edges. Magnucka-Blandzi and Magnucki [11] studied analytically critical stresses and equilibrium paths of axially compressed shallow cylindrical panels. Fazzolari [12] conducted a stability analysis of functionally graded isotropic and sandwich plates by using the advanced hierarchical trigonometric Ritz formulation.

Rezaei and Saidi [13] employed Carrera Unified Formulation (CUF) for the free vibration analysis of Levy-type thick rectangular porous–cellular plates. The variation of porosity through the thickness caused mechanical properties to change along the thickness. The effects of the coefficient of plate porosity and the thickness–length ratio as well as the aspect ratio, on the frequencies were investigated for Levy-type boundary conditions. Abrate and Di Sciuva [14] presented an overview of the field of equivalent single-layer theories for beams, plates and shells. Birman and Kardomateas [15] outlined the major trends in the research of sandwich structures concentrating on theoretical developments, novel designs and modern applications.

Magnucki [16] analyzed analytically the elastic buckling of the cylindrical panel with symmetrically varying mechanical properties in the thickness direction. Eslami [17] covered the stability of beams, plates, and shells made of homogeneous/isotropic material or functionally graded materials under both mechanical and thermal loads. Magnucki *et al.* [18] studied critical loads and fundamental natural frequencies of a simply supported rectangular plate with symmetrically varying mechanical properties in the thickness direction. The plate was subjected to compression in two directions with a uniformly distributed load. Magnucki *et al.* [19] examined a circular plate with symmetrically thickness-wise varying mechanical properties. The plate was simply supported and carried a concentrated force located in its centre. The axisymmetric bending problem of the plate with consideration of the shear effect was analytically and numerically studied. Magnucki *et al.* [20] analyzed

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simply supported beams with bi-symmetrical cross-sections under a generalized load. The novel shear deformation theory of a planar beam cross-section was formulated based on the Zhuravsky shear stress formula.

Magnucki and Magnucka-Blandzi [21] presented a generalization of the analytical models of the sandwich structures. This generalization concerns the continuity of symmetrical variation of mechanical properties in the thickness direction of the structures and the individual nonlinear function of deformation of the straight normal line to the neutral surface. Carrera *et al.* [22] investigated the nonlinear displacement and stress distributions in the cross-section effect of the isotropic beam, composite beam placed at different angles, and FG beams by CUF theory. Wu *et al.* [23] developed a unified formulation of full geometrically nonlinear refined shell theory based on the CUF and the total Lagrangian approach to predict the post-buckling, large-deflection, snap-through and snap-back nonlinear responses with high accuracy.

Twinkle and Pitchaimani [24] investigated the buckling and free vibration response of graphene nanoplatelets (GPL) reinforced porous sandwich cylindrical panels subjected to different types of in-plane compressive loads. Different combinations of porosity and GPL distributions for the sandwich core were considered. Magnucki *et al.* [25] studied analytically and numerically the problem of bending a circular sandwich plate under a concentrated force.

The subject of the paper is an axially compressed cylindrical sandwich panel of radius R, angle β, length L and total thickness h with symmetrically continuous varying mechanical properties in the thickness direction (Fig. 1). This panel is compressed along generating line in the middle surface with a uniformly distributed load of intensity and simply supported along four edges. The main goal of the research is to develop an analytical model of the cylindrical sandwich panel and to determine the critical load for this panel and for a rectangular sandwich plate as the limit transformation of this panel.

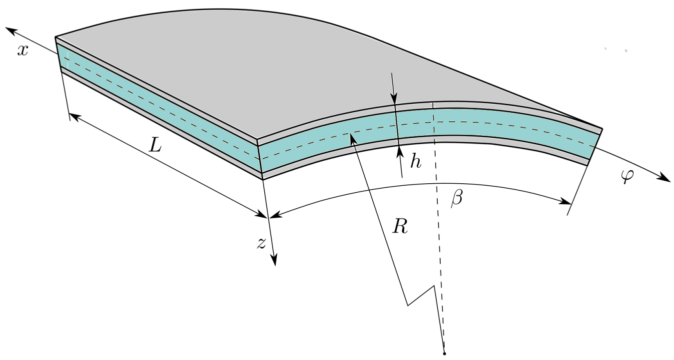

Fig. 1. Scheme of the sandwich cylindrical panel

2. ANALYTICAL MODEL OF THE CYLINDRICAL SANDWICH PANEL

The structures of the cylindrical sandwich panel are a composite of two faces of thicknesses h_f and one core of thickness h_c and so the total thickness $h = 2h_f + h_c$ (Fig. 2).

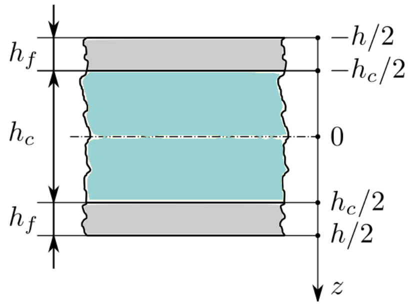

Fig. 2. Scheme of the structures of the sandwich cylindrical panel

The variation of mechanical properties in the thickness direction of the cylindrical sandwich panel, with consideration of the paper [21], is continuous. Young's modulus E_f is the constant of two faces; however, this modulus of the core varies

$$E_c(\zeta) = E_f f_e(\zeta), \tag{1}$$

where dimensionless function

$$f_e(\zeta) = e_0 + (1-e_0)\left(\frac{2}{\chi_c}\zeta\right)^{k_e}, \tag{2}$$

and $e_0 = E_0/E_f$ – dimensionless coefficient, $\chi_c = h_c/h$ – relative thickness of the core, $\zeta = z/h$ – dimensionless coordinate, k_e – even exponent.

The example graph of the dimensionless function (2) is shown in Fig. 3.

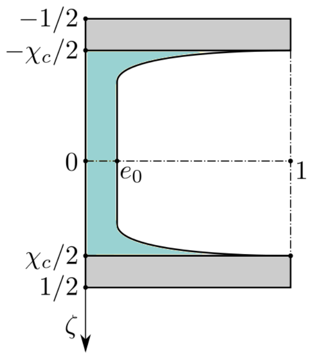

Fig. 3. The example graph of the dimensionless function (2)

The analytical model of the cylindrical sandwich panel is formulated with consideration of the papers [16] and [21]. A straight line normal to the neutral surface of the sandwich structure before bending is a curve after bending (Fig. 4).

Taking into account the above scheme, the displacements in the x and φ directions in the particular layers are as follows:

- the upper face $(-1/2 \le \zeta \le -\chi_c/2)$

$$u(x,\varphi,\zeta) = u_0(x,\varphi) - h\left[\zeta\frac{\partial w}{\partial x} + \psi(x,\varphi)\right], \tag{3}$$

$$v(x,\varphi,\zeta) = v_0(x,\varphi) - h\left[\zeta\frac{\partial w}{R\partial\varphi} + \phi(x,\varphi)\right]; \tag{4}$$

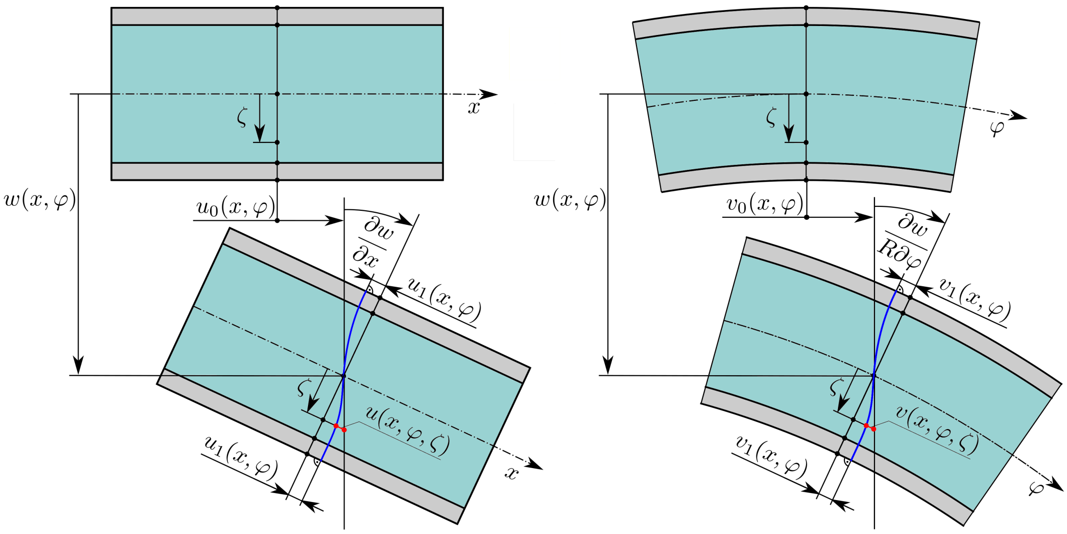

Fig. 4. Scheme of the deformation of the straightnormal line – the nonlinear hypothesis

• the core ($-\chi_c/2 \le \zeta \le \chi_c/2$)

$$u(x,\varphi,\zeta) = u_0(x,\varphi) - h\left[\zeta\frac{\partial w}{\partial x} - f_d(\zeta)\psi(x,\varphi)\right], \qquad (5)$$

$$v(x,\varphi,\zeta) = v_0(x,\varphi) - h\left[\zeta\frac{\partial w}{R\partial\varphi} - f_d(\zeta)\phi(x,\varphi)\right]; \qquad (6)$$

• the lower face ($\chi_c/2 \le \zeta \le 1/2$)

$$u(x,\varphi,\zeta) = u_0(x,\varphi) - h\left[\zeta\frac{\partial w}{\partial x} - \psi(x,\varphi)\right], \qquad (7)$$

$$v(x,\varphi,\zeta) = v_0(x,\varphi) - h\left[\zeta\frac{\partial w}{R\partial\varphi} - \phi(x,\varphi)\right], \qquad (8)$$

where $w(x,\varphi)$ – deflection, $\psi(x,\varphi) = u_1(x,\varphi)/h$, $\phi(x,\varphi) = v_1(x,\varphi)/h$ – dimensionless functions.

The function of deformation of the straight normal line for the core is elaborated with consideration of the [25] in the following form:

$$f_d(\zeta) = \frac{1}{C_0}\int\frac{1-\chi_c^2+\left(\chi_c^2-4\zeta^2\right)e_0+8\left(1-e_0\right)J_c(\zeta)}{f_e(\zeta)}\,\mathrm{d}\zeta, \qquad (9)$$

where

$$J_c(\zeta) = \frac{\chi_c^2}{4\left(k_2+2\right)}\left[1-\left(\frac{2}{\chi_c}\zeta\right)^{k_e+2}\right],$$

$$C_0 = \int_0^{\chi_2/2}\frac{1-\chi_c^2+\left(\chi_c^2-4\zeta^2\right)e_0+8\left(1-e_0\right)J_c(\zeta)}{f_e(\zeta)}\,\mathrm{d}\zeta.$$

The values of the deformation function (9) for the upper and lower surfaces of the core are as follows: $f_d(-\chi_c/2) = -1$ and $f_d(\chi_c/2) = 1$.

Thus, the strains in the particular layers are as follows:

• the upper face ($-1/2 \le \zeta \le -\chi_c/2$)

$$\varepsilon_x^{(uf)}(x,\varphi,\zeta) = \frac{\partial u_0}{\partial x} - h\left[\zeta\frac{\partial^2 w}{\partial x^2} + \frac{\partial\psi}{\partial x}\right], \qquad (10)$$

$$\varepsilon_\varphi^{(uf)}(x,\varphi,\zeta) = \frac{\partial v_0}{R\partial\varphi} - h\left[\zeta\frac{\partial^2 w}{R^2\partial^2\varphi} + \frac{\partial\phi}{R\partial\varphi}\right] - \frac{w(x,\varphi)}{R}, \qquad (11)$$

$$\gamma_{x\varphi}^{(uf)}(x,\varphi,\zeta) = \frac{\partial u_0}{R\partial\varphi} + \frac{\partial v_0}{\partial x} - h\left[2\zeta\frac{\partial^2 w}{\partial xR\partial\varphi} + \frac{\partial\psi}{R\partial\varphi} + \frac{\partial\phi}{\partial x}\right], \qquad (12)$$

$$\gamma_{xz}^{(uf)}(x,\varphi,\zeta) = 0, \quad \gamma_{\varphi z}^{(uf)}(x,\varphi,\zeta) = 0; \qquad (13)$$

• the core ($-\chi_c/2 \le \zeta \le \chi_c/2$)

$$\varepsilon_x^{(c)}(x,\varphi,\zeta) = \frac{\partial u_0}{\partial x} - h\left[\zeta\frac{\partial^2 w}{\partial x^2} - f_d(\zeta)\frac{\partial\psi}{\partial x}\right], \qquad (14)$$

$$\varepsilon_\varphi^{(c)}(x,\varphi,\zeta) = \frac{\partial v_0}{R\partial\varphi} - h\left[\zeta\frac{\partial^2 w}{R^2\partial^2\varphi} - f_d(\zeta)\frac{\partial\phi}{R\partial\varphi}\right] - \frac{w(x,\varphi)}{R}, \qquad (15)$$

$$\gamma_{x\varphi}^{(c)}(x,\varphi,\zeta) = \frac{\partial u_0}{R\partial\varphi} + \frac{\partial v_0}{\partial x} - h\left[2\zeta\frac{\partial^2 w}{\partial xR\partial\varphi} - f_d(\zeta)\left(\frac{\partial\psi}{R\partial\varphi} + \frac{\partial\phi}{\partial x}\right)\right], \qquad (16)$$

$$\gamma_{xz}^{(c)}(x,\varphi,\zeta) = \frac{df_d}{d\zeta}\psi(x,\varphi),$$
$$\gamma_{\varphi z}^{(c)}(x,\varphi,\zeta) = \frac{df_d}{d\zeta}\phi(x,\varphi); \qquad (17)$$

• the lower face ($\chi_c/2 \le \zeta \le 1/2$)

$$\varepsilon_x^{(lf)}(x,\varphi,\zeta) = \frac{\partial u_0}{\partial x} - h\left[\zeta\frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x}\right], \tag{18}$$

$$\varepsilon_\varphi^{(lf)}(x,\varphi,\zeta) = \frac{\partial v_0}{R\partial\varphi} - h\left[\zeta\frac{\partial^2 w}{R^2\partial^2\varphi} - \frac{\partial \phi}{R\partial\varphi}\right] - \frac{w(x,\varphi)}{R}, \tag{19}$$

$$\gamma_{x\varphi}^{(lf)}(x,\varphi,\zeta) = \frac{\partial u_0}{R\partial\varphi} + \frac{\partial v_0}{\partial x} - h\left[2\zeta\frac{\partial^2 w}{\partial x R\partial\varphi} - \frac{\partial \psi}{R\partial\varphi} - \frac{\partial\phi}{\partial x}\right], \tag{20}$$

$$\gamma_{xz}^{(lf)}(x,\varphi,\zeta) = 0, \quad \gamma_{\varphi z}^{(lf)}(x,\varphi,\zeta) = 0. \tag{21}$$

Consequently, the stresses – Hooke's law:
• the upper face ($-1/2 \le \zeta \le -\chi_c/2$)

$$\sigma_x^{(uf)}(x,\varphi,\zeta) = \frac{E_f}{1-\nu^2}\left[\varepsilon_x^{(uf)}(x,\varphi,\zeta) + \nu\varepsilon_\varphi^{(uf)}(x,\varphi,\zeta)\right], \tag{22}$$

$$\sigma_\varphi^{(uf)}(x,\varphi,\zeta) = \frac{E_f}{1-\nu^2}\left[\varepsilon_\varphi^{(uf)}(x,\varphi,\zeta) + \nu\varepsilon_x^{(uf)}(x,\varphi,\zeta)\right], \tag{23}$$

$$\tau_{x\varphi}^{(uf)}(x,\varphi,\zeta) = \frac{E_f}{2(1+\nu)}\gamma_{x\varphi}^{(uf)}(x,\varphi,\zeta), \tag{24}$$

$$\tau_{xz}^{(uf)}(x,\varphi,\zeta) = 0, \quad \tau_{\varphi z}^{(uf)}(x,\varphi,\zeta) = 0; \tag{25}$$

• the core ($-\chi_c/2 \le \zeta \le \chi_c/2$)

$$\sigma_x^{(c)}(x,\varphi,\zeta) = \frac{E_f}{1-\nu^2}\left[\varepsilon_x^{(c)}(x,\varphi,\zeta) + \nu\varepsilon_\varphi^{(c)}(x,\varphi,\zeta)\right] f_e(\zeta), \tag{26}$$

$$\sigma_\varphi^{(c)}(x,\varphi,\zeta) = \frac{E_f}{1-\nu^2}\left[\varepsilon_\varphi^{(c)}(x,\varphi,\zeta) + \nu\varepsilon_x^{(c)}(x,\varphi,\zeta)\right] f_e(\zeta), \tag{27}$$

$$\tau_{x\varphi}^{(c)}(x,\varphi,\zeta) = \frac{E_f}{2(1+\nu)}\gamma_{x\varphi}^{(c)}(x,\varphi,\zeta) f_e(\zeta), \tag{28}$$

$$\tau_{xz}^{(c)}(x,\varphi,\zeta) = \frac{E_f}{2(1+\nu)}\gamma_{xz}^{(c)}(x,\varphi,\zeta) f_e(\zeta), \tag{29}$$

$$\tau_{\varphi z}^{(c)}(x,\varphi,\zeta) = \frac{E_f}{2(1+\nu)}\gamma_{\varphi z}^{(c)}(x,\varphi,\zeta) f_e(\zeta); \tag{30}$$

• the lower face ($\chi_c/2 \le \zeta \le 1/2$)

$$\sigma_x^{(lf)}(x,\varphi,\zeta) = \frac{E_f}{1-\nu^2}\left[\varepsilon_x^{(lf)}(x,\varphi,\zeta) + \nu\varepsilon_\varphi^{(lf)}(x,\varphi,\zeta)\right], \tag{31}$$

$$\sigma_\varphi^{(lf)}(x,\varphi,\zeta) = \frac{E_f}{1-\nu^2}\left[\varepsilon_\varphi^{(lf)}(x,\varphi,\zeta) + \nu\varepsilon_x^{(lf)}(x,\varphi,\zeta)\right], \tag{32}$$

$$\tau_{x\varphi}^{(lf)}(x,\varphi,\zeta) = \frac{E_f}{2(1+\nu)}\gamma_{x\varphi}^{(lf)}(x,\varphi,\zeta), \tag{33}$$

$$\tau_{xz}^{(lf)}(x,\varphi,\zeta) = 0, \quad \tau_{\varphi z}^{(lf)}(x,\varphi,\zeta) = 0, \tag{34}$$

where $\nu_f = \nu_c = \nu$ is Poisson's ratio of the faces and the core is the same.

The normal and shear forces are as follows:

$$N_x(x,\varphi) = h\left\{\int_{-1/2}^{-\chi_2/2}\sigma_x^{(uf)}(x,\varphi,\zeta)\,d\zeta + \int_{-\chi_c/2}^{\chi_c/2}\sigma_x^{(c)}(x,\varphi,\zeta)\,d\zeta + \int_{\chi_2/2}^{1/2}\sigma_x^{(lf)}(x,\varphi,\zeta)\,d\zeta\right\}, \tag{35}$$

$$N_\varphi(x,\varphi) = h\left\{\int_{-1/2}^{-\chi_2/2}\sigma_\varphi^{(uf)}(x,\varphi,\zeta)\,d\zeta + \int_{-\chi_c/2}^{\chi_c/2}\sigma_\varphi^{(c)}(x,\varphi,\zeta)\,d\zeta + \int_{\chi_2/2}^{1/2}\sigma_\varphi^{(lf)}(x,\varphi,\zeta)d\zeta\right\}, \tag{36}$$

$$S_{x\varphi}(x,\varphi) = h\left\{\int_{-1/2}^{-\chi_2/2}\tau_{x\varphi}^{(uf)}(x,\varphi,\zeta)\,d\zeta + \int_{-\chi_c/2}^{\chi_c/2}\tau_{x\varphi}^{(c)}(x,\varphi,\zeta)\,d\zeta + \int_{\chi_2/2}^{1/2}\tau_{x\varphi}^{(lf)}(x,\varphi,\zeta)\,d\zeta\right\}. \tag{37}$$

Substituting the expressions (22)–(24), (26)–(28) and (31)–(33) into above (35)–(37), after integration and simply transformations, one obtains:

$$N_x(x,\varphi) = \frac{E_f h}{1-\nu^2}C_N\left[\frac{\partial u_0}{\partial x} + \nu\left(\frac{\partial v_0}{R\partial\varphi} - \frac{w(x,\varphi)}{R}\right)\right], \tag{38}$$

$$N_\varphi(x,\varphi) = \frac{E_f h}{1-\nu^2}C_N\left[\nu\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{R\partial\varphi} - \frac{w(x,\varphi)}{R}\right], \tag{39}$$

$$S_{x\varphi}(x,\varphi) = \frac{E_f h}{2(1+\nu)}C_N\left(\frac{\partial u_0}{R\partial\varphi} + \frac{\partial v_0}{\partial x}\right), \tag{40}$$

where $C_N = 1-(1-e_0)\dfrac{k_e}{k_e+1}\chi_c$ – dimensionless coefficient, from which the strains in the middle surface of this sandwich panel are as follows:

$$\varepsilon_x^o = \frac{\partial u_0}{\partial x} = \frac{1}{E_f h C_N}\left[N_x(x,\varphi) - \nu N_\varphi(x,\varphi)\right], \tag{41}$$

$$\varepsilon_\varphi^o = \frac{\partial v_0}{R\partial\varphi} - \frac{w(x,\varphi)}{R} = \frac{1}{E_f h C_N}\left[N_\varphi(x,\varphi) - \nu N_x(x,\varphi)\right], \tag{42}$$

$$\gamma_{x\varphi}^o = \frac{\partial u_0}{R\partial\varphi} + \frac{\partial v_0}{\partial x} = 2\frac{1+\nu}{E_f h C_N}S_{x\varphi}(x,\varphi). \tag{43}$$

Consequently, the equation of compatibility of deformation in the middle surface of this sandwich panel is in the form:

$$\nabla^4 F(x,\varphi) + \frac{E_f h}{R} C_N \frac{\partial^2 w}{\partial x^2} = 0, \tag{44}$$

where: $F(x,\varphi)$ – force function, $\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 R^2 \partial \varphi^2} + \frac{\partial^4}{R^4 \partial \varphi^4}$ – differential operator, and the forces

$$N_x(x,\varphi) = \frac{\partial^2 F}{R^2 \partial \varphi^2}, \quad N_\varphi(x,\varphi) = \frac{\partial^2 F}{\partial x^2}, \quad S_{x\varphi}(x,\varphi) = -\frac{\partial^2 F}{\partial x R \partial \varphi}. \tag{45}$$

The elastic strain energy of the sandwich panel

$$U_\varepsilon = \frac{1}{2}\int_0^L \int_0^\beta \left\{ \Re^{(uf)}(x,\varphi) + \Re^{(c)}(x,\varphi) + \Re^{(lf)}(x,\varphi) \right\} R \,\mathrm{d}\varphi\, \mathrm{d}x, \tag{46}$$

where

$$\Re^{(uf)}(x,\varphi) = \int_{-1/2}^{-\chi_c/2} \left[\sigma_x^{(uf)} \varepsilon_x^{(uf)} + \sigma_\varphi^{(uf)} \varepsilon_\varphi^{(uf)} + \tau_{x\varphi}^{(uf)} \gamma_{x\varphi}^{(uf)} \right] h \,\mathrm{d}\zeta, \tag{47}$$

$$\Re^{(c)}(x,\varphi) = \int_{-\chi_c/2}^{\chi_c/2} \left[\sigma_x^{(c)} \varepsilon_x^{(c)} + \sigma_\varphi^{(c)} \varepsilon_\varphi^{(c)} + \tau_{x\varphi}^{(c)} \gamma_{x\varphi}^{(c)} + \tau_{xz}^{(c)} \gamma_{xz}^{(c)} + \tau_{\varphi z}^{(c)} \gamma_{\varphi z}^{(c)} \right] h \,\mathrm{d}\zeta, \tag{48}$$

$$\Re^{(lf)}(x,\varphi) = \int_{\chi_c/2}^{1/2} \left[\sigma_x^{(lf)} \varepsilon_x^{(lf)} + \sigma_\varphi^{(lf)} \varepsilon_\varphi^{(lf)} + \tau_{x\varphi}^{(lf)} \gamma_{x\varphi}^{(lf)} \right] h \,\mathrm{d}\zeta. \tag{49}$$

The work of the load

$$W = \frac{1}{2} N_x^o \int_0^L \int_0^\beta \left(\frac{\partial w}{\partial x} \right)^2 R \,\mathrm{d}\varphi\, \mathrm{d}x. \tag{50}$$

Therefore, based on the principle of stationary total potential energy $\delta(U\varepsilon - W) = 0$, the system of five partial differential equations of equilibrium of the cylindrical panel is obtained. Considering the expressions (41)–(43) and the force function (45), the system is reduced to three equations of equilibrium. Moreover, assuming the displacement function $\theta(x,\varphi)$ two unknown dimensionless functions $\psi(x,\varphi)$ and $\phi(x,\varphi)$ are formulated in the form:

$$\psi(x,\varphi) = \frac{\partial \theta}{\partial x}, \quad \phi(x,\varphi) = \frac{\partial \theta}{R \partial \varphi}. \tag{51}$$

Thus, the system of three equations of equilibrium of the cylindrical panel is reduced to two equations of equilibrium – the governing differential equations of the elastic buckling problems of the cylindrical sandwich panel in the following form:

$$\frac{E_f h^3}{1-\nu^2}\left[C_{ww} \nabla^8 w(x,\varphi) - C_{w\psi\varphi} \nabla^4 \theta(x,\varphi) \right] + \frac{E_f h}{R^2} C_N \frac{\partial^4 w}{\partial x^4} + N_x^o \frac{\partial^2}{\partial x^2} \nabla^4 w(x,\varphi) = 0, \tag{52}$$

$$C_{w\psi\varphi} \nabla^4 w(x,\varphi) - C_{\psi\varphi} \nabla^4 \theta(x,\varphi) + \frac{1}{2}(1-\nu)\frac{J_3}{h^2} \nabla^2 \theta(x,\varphi) = 0, \tag{53}$$

where

$$C_{ww} = \frac{1}{12}\left[1 - (1-e_0)\frac{k_e}{k_e+3}\chi_c^3 \right],$$

$$C_{w\psi\varphi} = \frac{1}{4}\left(1-\chi_c^2\right) + J_1,$$

$$C_{\psi\varphi} = 1 - \chi_c + J_2,$$

$$J_1 = \int_{-\chi_c/2}^{\chi_c/2} f_d(\zeta) f_e(\zeta) \zeta \,\mathrm{d}\zeta,$$

$$J_2 = \int_{-\chi_c/2}^{\chi_c/2} f_d^2(\zeta) f_e(\zeta) \,\mathrm{d}\zeta,$$

$$J_3 = \frac{1}{C_0^2} \int_{-\chi_c/2}^{\chi_2/2} \frac{\left[1-\chi_c^2 + (\chi_c^2 - 4\zeta^2) e_0 + 8(1-e_0) J_c(\zeta) \right]^2}{f_e(\zeta)} \mathrm{d}\zeta$$

are dimensionless coefficients.

3. CRITICAL LOAD OF THE CYLINDRICAL SANDWICH PANEL

The system of two differential equations (52) and (53) is approximately solved. Two unknown functions of these equations are assumed in the following form:

$$w(x,\phi) = w_a \sin\left(m\pi \frac{x}{L} \right) \sin\left(n\pi \frac{\phi}{\beta} \right), \quad \theta(x,\phi) = \theta_a \sin\left(m\pi \frac{x}{L} \right) \sin\left(n\pi \frac{\phi}{\beta} \right), \tag{54}$$

where: w_a, θ_a – coefficients of these functions, m, n – natural numbers.

Substituting these functions (54) into equations (52) and (53), after simple transformation, one obtains

$$\theta_a = \frac{mnC_{mn}C_{w\psi\phi}}{mnC_{mn}C_{\psi\phi} + \frac{1-\nu}{2\pi^2}J_3\frac{LR\beta}{h^2}}w_a\,, \tag{55}$$

and the critical load

$$N^o_{x,CR} = \min_{m,n}\left[(C_{ww} - C_s)\left(\frac{n\pi}{\beta}\frac{h}{R}\right)^2 C^2_{mn} + (1-\nu^2)\left(\frac{\beta}{n\pi}\right)^2\frac{C_N}{C^2_{mn}}\right]\frac{E_f h}{1-\nu^2}\,, \tag{56}$$

where

$$C_s = \frac{mnC_{mn}C^2_{w\psi\phi}}{mnC_{mn}C_{\psi\phi} + \frac{1-\nu}{2\pi^2}J_3\frac{LR\beta}{h^2}},$$

$C_{mn} = \frac{m}{n}\frac{R\beta}{L} + \frac{n}{m}\frac{L}{R\beta}$ are dimensionless coefficients.

For the particular case of this cylindrical sandwich panel as the homogeneous structure ($k_e = 0$, $e_0 = 1$, $f_e(\zeta) = 1$, $\chi_c = 1$, $C_N = 1$, $C_{ww} = 1/12$, without shear effect $C_s = 0$) based on the expressions (56), one obtains the Lorenz-Timoshenko-Southwell critical stress

$$\sigma_{x,CR} = \frac{N^o_{x,CR}}{h} = \frac{E_f}{\sqrt{3(1-\nu^2)}}\frac{h}{R}\,. \tag{57}$$

The detailed calculations of the exemplary generalized cylindrical sandwich panels, with consideration of paper [25], where the material of the faces is aluminium and the material of the core is aluminium foam, are conducted for the following data:

Example 1. The total thickness $h = 20$ mm, radius $R = 4000$ mm, length $L = 4000$ mm, angle $\beta = \pi/3$, Young's modulus $E_f = 72000$ MPa, Poisson's ratio $\nu = 0.3$, the relative thickness of the core $\chi_c = 17/20$ and dimensionless coefficient $e_0 = 1/20$. The results of the calculations are specified in Table 1 and shown in Figs. 5 and 6.

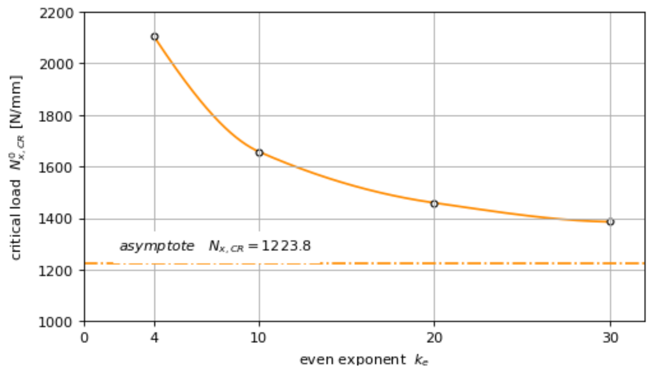

Fig. 5. The graph of the critical load of the exemplary cylindrical sandwich panel ($e_0 = 1/20$)

Table 1

The results of calculations of the exemplary cylindrical sandwich panel for $e_0 = 1/20$

k_e	4	10	20	30	∞
m	6	5	5	5	6
n	3	3	3	3	2
C_s	0.00118	0.00081	0.00071	0.00067	0.00066
$N^o_{x,CR}$ [N/mm]	2103.9	1656.8	1458.8	1385.4	1223.8

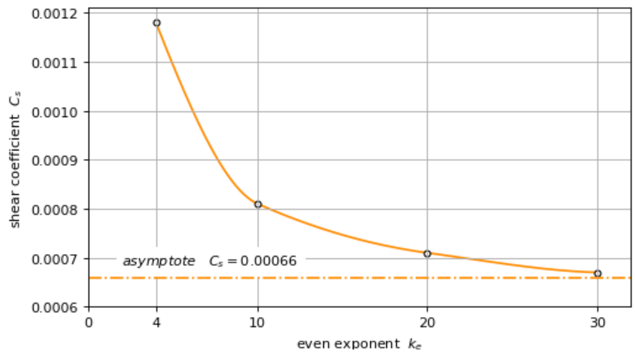

Fig. 6. The graph of the shear coefficient of the exemplary cylindrical sandwich panel ($e_0 = 1/20$)

Example 2. The total thickness $h = 20$ mm, radius $R = 4000$ mm, length $L = 4000$ mm, angle $\beta = \pi/3$, Young's modulus $E_f = 72000$ MPa, Poisson's ratio $\nu = 0.3$, the relative thickness of the core $\chi_c = 17/20$ and dimensionless coefficient $e_0 = 1/30$. The results of the calculations are specified in Table 2 and shown in Figs. 7 and 8.

Table 2

The results of calculations of the exemplary cylindrical sandwich panel for $e_0 = 1/20$

k_e	4	10	20	30	∞
m	6	6	6	6	6
n	3	2	2	2	2
C_s	0.00157	0.00133	0.00118	0.00111	0.00093
$N^o_{x,CR}$ [N/mm]	2054.9	1594.8	1393.8	1319.5	1157.3

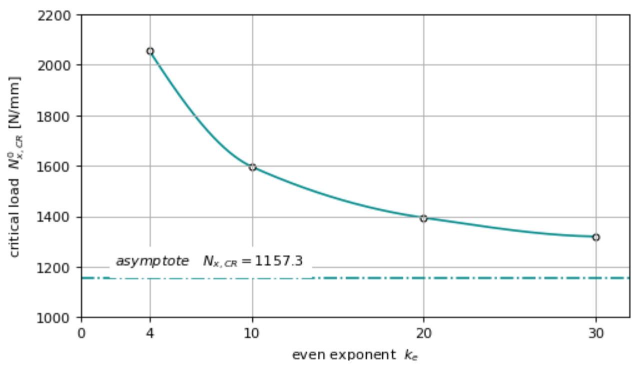

Fig. 7. The graph of the critical load of the exemplary cylindrical sandwich panel ($e_0 = 1/30$)

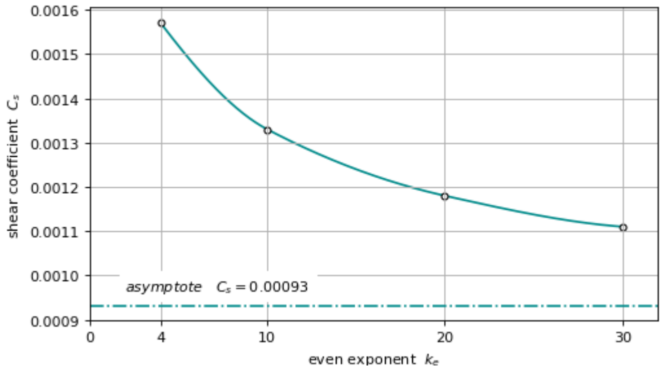

Fig. 8. The graph of the shear coefficient of the exemplary cylindrical sandwich panel ($e_0 = 1/30$)

4. CRITICAL LOAD OF THE RECTANGULAR SANDWICH PLATE

The parallel length of this sandwich panel is $b = R\beta$ (Fig. 1). The rectangular sandwich plate of width b, length L and total thickness h with symmetrically continuous varying mechanical properties is the limit transformation of this cylindrical sandwich panel, then $R \to \infty$ and $\beta \to 0$. Therefore, the critical load (56) for the rectangular sandwich plate is in the following form:

$$N^o_{x,CR} = \min_{m,n} \left[(C_{ww} - C_s) \left(n\pi \frac{h}{b} \right)^2 \left(\frac{m}{n}\frac{b}{L} + \frac{n}{m}\frac{L}{b} \right)^2 \right] \frac{E_f h}{1-\nu^2}, \tag{58}$$

where

$$C_s = \frac{mnC_{mn}C^2_{w\psi\varphi}}{mnC_{mn}C_{\psi\varphi} + \dfrac{1-\nu}{2\pi^2} J_3 \dfrac{Lb}{h^2}},$$

$$C_{mn} = \frac{m}{n}\frac{b}{L} + \frac{n}{m}\frac{L}{b}$$

are dimensionless coefficients.

For the particular case of this rectangular sandwich plate as the homogeneous structure ($k_e = 0$, $e_0 = 1$, $f_e(\zeta) = 1$, $\chi_c = 1$, $C_N = 1$, $C_{ww} = 1/12$, without shear effect $C_s = 0$) based on the expressions (58), one obtains well known in the literature the critical stress

$$\sigma_{x,CR} = \frac{N^o_{x,CR}}{h} = \frac{\pi^2}{3(1-\nu^2)} E_f \left(\frac{h}{b} \right)^2. \tag{59}$$

The detailed calculations of the exemplary generalized square sandwich plate are conducted for the following data.

Example 3. The total thickness $h = 20$ mm, width $b = 1\,000$ mm, length $L = 1\,000$ mm, $m = n = 1$, Young's modulus $E_f = 72\,000$ MPa, Poisson's ratio $\nu = 0.3$, the relative thickness of the core $\chi_c = 17/20$ and dimensionless coefficient $e_0 = 1/20$. The results of the calculations are specified in Table 3 and shown in Figs. 9 and 10.

Table 3
The results of calculations of the exemplary square sandwich plate for $e_0 = 1/20$

k_e	4	10	20	30	∞
C_s	0.000859	0.000779	0.000684	0.000641	0.000537
$N^o_{x,CR}$ [N/mm]	1366.7	1128.4	1008.9	961.9	854.1

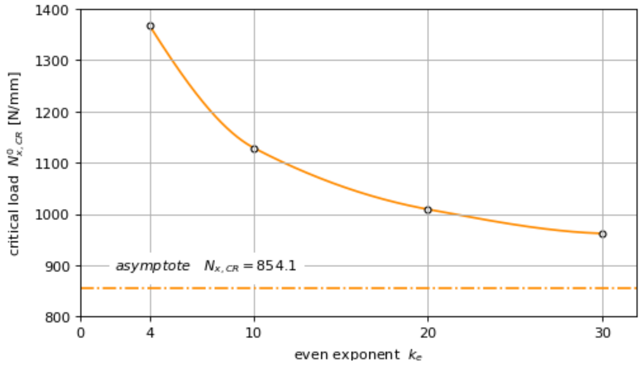

Fig. 9. The graph of the critical load of the exemplary square sandwich plate ($e_0 = 1/20$)

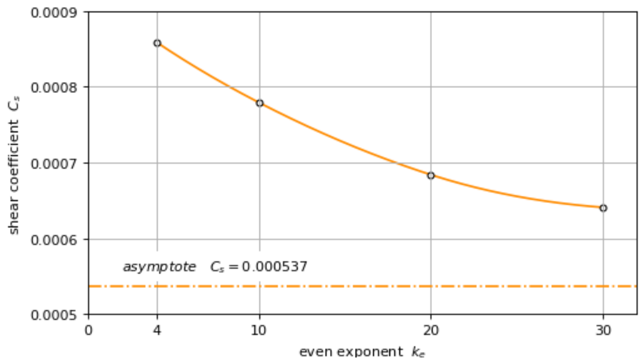

Fig. 10. The graph of the shear coefficient of the exemplary square sandwich plate ($e_0 = 1/20$)

Example 4. The total thickness $h = 20$ mm, width $b = 1\,000$ mm, length $L = 1\,000$ mm, $m = n = 1$, Young's modulus $E_f = 72\,000$ MPa, Poisson's ratio $\nu = 0.3$, the relative thickness of the core $\chi_c = 17/20$ and dimensionless coefficient $e_0 = 1/30$. The results of the calculations are specified in Table 4 and shown in Figs. 11 and 12.

Table 4
The results of calculations of the exemplary square sandwich plate for $e_0 = 1/30$

k_e	4	10	20	30	∞
C_s	0.001143	0.001078	0.000957	0.000900	0.000757
$N^o_{x,CR}$ [N/mm]	1347.4	1104.5	983.5	936.1	827.2

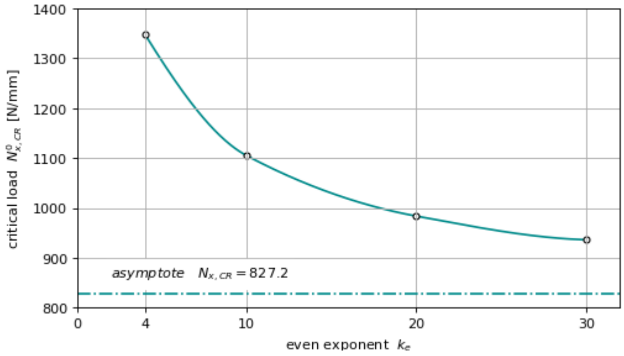

Fig. 11. The graph of the critical load the exemplary square sandwich plate ($e_0 = 1/30$)

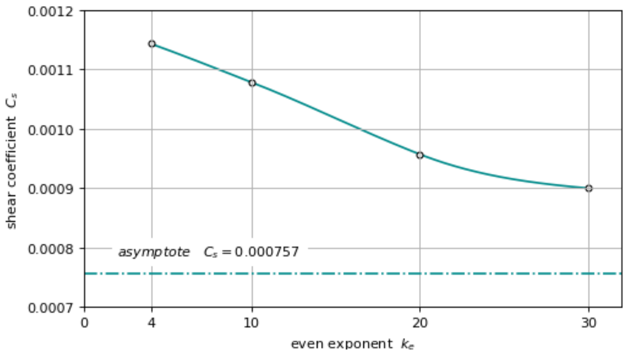

Fig. 12. The graph of the shear coefficient of the exemplary square sandwich plate ($e_0 = 1/30$)

5. CONCLUSIONS

The presented studies of the cylindrical sandwich panels and rectangular plates allow us to formulate the following conclusions:

1. The analytical expressions for a cylindrical sandwich panel formulated in this paper can be easily converted to expressions for a rectangular sandwich plate by calculating the limit with $R \to +\infty$.
2. Values of the critical load $N^0_{x,CR}$ [N/mm] of the cylindrical sandwich panel (56), (Table 1, Fig. 5 and Table 2, Fig. 7) and also of the square sandwich plate (58), (Table 3, Fig. 9 and Table 4, Fig. 11) are decreasing with increasing the exponent value k_s of the dimensionless function (2), i.e. the sandwich structure tends to be the classical one with a constant value of the core elasticity modulus.
3. Values of the shear coefficient C_s, similar to the critical load of the cylindrical sandwich panel or the square sandwich plate, are decreasing with increasing the exponent value k_s (Table 1, Fig. 6 and Table 2, Fig. 8) and (Table 3, Fig. 10 and Table 4, Fig. 12).
4. The presented analytical model of the cylindrical sandwich panel is an improvement of the classical analytical model of this structure.
5. Proposal of the topic of further research:
   a) analytical and numerical FEM studies of the dynamic stability of this cylindrical sandwich panel – linear and nonlinear problems,
   b) comparative analysis of the presented research results with the results of calculations determined with consideration of the other theories.

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