THE STURM-LIOUVILLE EIGENVALUE PROBLEM - A NUMERICAL SOLUTION USING THE CONTROL VOLUME METHOD

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Abstract. The solution of the 1D Sturm-Liouville problem using the Control Volume Method is discussed. The second order linear differential equation with homogeneous boundary conditions is discretized and converted to the system of linear algebraic equations. The matrix associated with this system is tridiagonal and eigenvalues of this system are an approximation of the real eigenvalues of the boundary value problem. The numerical results of the eigenvalues for various cases and the experimental rate of convergence are presented.

Keywords: *Sturm-Liouville problem, eigenvalues, numerical methods, Control Volume Method*

1. Introduction

This paper is concerned with the computation of eigenvalues of regular eigenvalue problems occurring in ordinary differential equations. The Sturm-Liouville problem arises within in many areas of science, engineering and applied mathematics. It has been studied for more than two decades. Many physical, biological and chemical processes are described using models based on the Sturm-Liouville equations.

The Sturm-Liouville problem appears directly as the eigenvalue problem in a one-dimensional space. It also arises when linear partial differential equations are separable in a certain coordinate system. For a more detailed study of the integer order Sturm-Liouville theory, we refer the reader to [1-5].

The Sturm-Liouville problem can be solved by using either analytical or numerical methods. One of the most common approaches to a numerical solution of the considered problem is the finite difference method [3, 6-8] where each derivative is discretized at each grid point with an adequate difference scheme. Apart from the

finite difference method, several analytical ones, such as the variational or decomposition methods, are proposed to find an approximate solution.

2. Statement of the problem

We consider the following problem defined on the bounded interval $x \in [a, b]$

$$-\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = \lambda w(x)y(x) \tag{1}$$

$$\begin{aligned}
\alpha_1 y(a) + \alpha_2 y'(a) &= 0 \\
\beta_1 y(b) + \beta_2 y'(b) &= 0
\end{aligned} \tag{2}$$

where $p(x) > 0$, dp/dx and $q(x)$ are continuous, $w(x) > 0$ on $[a, b]$, $|\alpha_1| + |\alpha_2| \neq 0$ and $|\beta_1| + |\beta_2| \neq 0$. The above problem is called the regular Sturm-Liouville Problem (SLP). The solution to SLP consists of a pair λ and y, where λ is a constant - called an eigenvalue, while y is a nontrivial (nonzero) function - called an eigenfunction, and together they satisfy the given SLP. For each SLP, the eigenvalues form an infinite increasing sequence: $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ and $\lim_{k\to\infty}\lambda_k = \infty$.

For arbitrary choices of the functions $p(x)$, $q(x)$ and $w(x)$ in Eq. (1), the computation of the exact values of the eigenvalues for which SLP (1)-(2) has a nontrivial eigensolution $y(x)$ which is very complicated or it is practically impossible to determine. Numerical methods should, therefore, be used for computing the approximate values of λ. In many cases, the importance of a numerical approximation to the SLP described by a differential eigenvalue problem is to reduce the problem to that of solving the eigenvalue problem of a matrix equation (an algebraic problem).

In this paper, we apply the Control Volume Method (also known as the Finite Volume Method) to compute the eigenvalues of the Sturm-Liouville problem numerically.

3. Control Volume Method

In this numerical method [9] the considered domain of SLP: $x \in [a, b]$ is divided into N control volumes Ω_i for $i = 1,\ldots,N$ with the central nodes ξ_i. The mesh is presented in Figure 1. The auxiliary nodes $x_i = a + i\,\Delta x$, for $i = 0,\ldots,N$ and $\Delta x = (b - a) / N$ have also been introduced. Then, the positions of central nodes are the following: $\xi_i = a + (i - 0.5)\Delta x$, $i = 1,\ldots,N$.
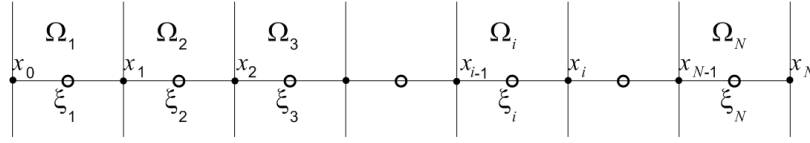
Fig. 1. Mesh of control volumes

The integration of Eq. (1) with respect to volume Ω_i leads to

$$-\int_{\Omega_i} \frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)dV + \int_{\Omega_i} q(x)y(x)dV = \lambda \int_{\Omega_i} w(x)y(x)dV \qquad (3)$$

or written in the form (assuming that Ω_i: [x_{i-1}, x_i])

$$-\int_{x_{i-1}}^{x_i} \frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)dx + \int_{x_{i-1}}^{x_i} q(x)y(x)dx = \lambda \int_{x_{i-1}}^{x_i} w(x)y(x)dx \qquad (4)$$

All the components in Eq. (4) can be approximated as follows:

$$\int_{x_{i-1}}^{x_i} q(x)y(x)dx \cong q(\xi_i)y(\xi_i)\Delta x \qquad (5)$$

$$\int_{x_{i-1}}^{x_i} w(x)y(x)dx \cong w(\xi_i)y(\xi_i)\Delta x \qquad (6)$$

$$\int_{x_{i-1}}^{x_i} \frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)dx = -p(x)\frac{dy(x)}{dx}\Big|_{x=x_{i-1}} + p(x)\frac{dy(x)}{dx}\Big|_{x=x_i}$$

$$\cong -p(x_{i-1})\begin{cases} \dfrac{y(\xi_i)-y(\xi_{i-1})}{\Delta x} & \text{for } i>1 \\[2mm] \dfrac{y(\xi_1)-y_a}{0.5\Delta x} & \text{for } i=1 \end{cases} + p(x_i)\begin{cases} \dfrac{y(\xi_{i+1})-y(\xi_i)}{\Delta x} & \text{for } i<N \quad (7) \\[2mm] \dfrac{y_b-y(\xi_N)}{0.5\Delta x} & \text{for } i=N \end{cases}$$

The values of $y_a = y(a)$ and $y_b = y(b)$ are determined on the basis of approximations of the boundary conditions (2)

$$\alpha_1 y_a + \alpha_2 \frac{y(\xi_1)-y_a}{0.5\Delta x} = 0, \quad \beta_1 y_b + \beta_2 \frac{y_b-y(\xi_N)}{0.5\Delta x} = 0 \qquad (8)$$

and hence, these values are equal to

$$y_a = \frac{2\alpha_2}{2\alpha_2 - \alpha_1\Delta x}y(\xi_1), \quad y_b = \frac{2\beta_2}{2\beta_2 + \beta_1\Delta x}y(\xi_N) \qquad (9)$$

Substituting (9) into (8), the following approximation of (7) has been obtained

$$\int_{x_{i-1}}^{x_i} \frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)dx \cong -\frac{p(x_{i-1})}{\Delta x}\begin{cases} y(\xi_i)-y(\xi_{i-1}), & \text{for } i>1 \\ \gamma_a y(\xi_1), & \text{for } i=1 \end{cases}$$

$$+\frac{p(x_i)}{\Delta x}\begin{cases} y(\xi_{i+1})-y(\xi_i), & \text{for } i<N \\ -\gamma_b y(\xi_N), & \text{for } i=N \end{cases} \tag{10}$$

where

$$\gamma_a = 2 - \frac{4\alpha_2}{2\alpha_2 - \alpha_1 \Delta x}, \quad \gamma_b = 2 - \frac{4\beta_2}{2\beta_2 + \beta_1 \Delta x} \tag{11}$$

For example, in the case of the Dirichlet boundary conditions at boundaries $x = a$ (i.e. $\alpha_1 = 1$, $\alpha_2 = 0$) and/or $x = b$ (i.e. $\beta_1 = 1$, $\beta_2 = 0$), the coefficients γ_a and γ_b are equal to 2, and in the case of the Neumann boundary conditions at $x = a$ ($\alpha_1 = 0$, $\alpha_2 = 1$) and/or $x = b$ ($\beta_1 = 0$, $\beta_2 = 1$), these coefficients take the values of 0, respectively.

After substitution of (5), (6) and (10) into (4), the following system of the discrete equations for every control volume:

- for Ω_1:

$$\left(\frac{\gamma_a p(x_0) + p(x_1)}{(\Delta x)^2} + q(\xi_1)\right)y(\xi_1) - \frac{p(x_1)}{(\Delta x)^2}y(\xi_2) = \lambda w(\xi_1)y(\xi_1) \tag{12}$$

- for Ω_i, for $i = 2,\ldots,N-1$:

$$-\frac{p(x_{i-1})}{(\Delta x)^2}y(\xi_{i-1}) + \left(\frac{p(x_{i-1}) + p(x_i)}{(\Delta x)^2} + q(\xi_i)\right)y(\xi_i) - \frac{p(x_i)}{(\Delta x)^2}y(\xi_{i+1})$$
$$= \lambda w(\xi_i)y(\xi_i) \tag{13}$$

- for Ω_N:

$$-\frac{p(x_{N-1})}{(\Delta x)^2}y(\xi_{N-1}) + \left(\frac{\gamma_b p(x_N) + p(x_{N-1})}{(\Delta x)^2} + q(\xi_N)\right)y(\xi_N) = \lambda w(\xi_N)y(\xi_N) \tag{14}$$

is obtained.

This system can be also written in the matrix form as

$$(\mathbf{P} + \mathbf{Q})\mathbf{y} = \lambda \mathbf{W}\mathbf{y} \tag{15}$$

where

$$
\mathbf{P} = \frac{1}{(\Delta x)^2}
\begin{bmatrix}
\gamma_a p(x_0) \\ + p(x_1) & -p(x_1) & 0 & 0 & \cdots & 0 \\
-p(x_1) & \begin{matrix} p(x_1) \\ + p(x_2) \end{matrix} & -p(x_2) & 0 & & 0 \\
0 & -p(x_{i-1}) & \begin{matrix} p(x_{i-1}) \\ + p(x_i) \end{matrix} & -p(x_i) & & 0 \\
\vdots & & & \ddots & & \\
0 & \cdots & 0 & -p(x_{N-2}) & \begin{matrix} p(x_{N-2}) \\ + p(x_{N-1}) \end{matrix} & -p(x_{N-1}) \\
0 & \cdots & 0 & 0 & -p(x_{N-1}) & \begin{matrix} p(x_{N-1}) \\ + \gamma_b p(x_N) \end{matrix}
\end{bmatrix}
\tag{16}
$$

$$
\mathbf{Q} = \text{diag}\big(q(\xi_1), q(\xi_2), ..., q(\xi_N)\big) \tag{17}
$$

$$
\mathbf{W} = \text{diag}\big(w(\xi_1), w(\xi_2), ..., w(\xi_N)\big) \tag{18}
$$

and $\mathbf{y} = \big[y(\xi_1), y(\xi_2), ..., y(\xi_N) \big]^T$. Then, SLP (1)-(2) is equivalent to the matrix eigenvalue problem (15). The ordered eigenvalues of (15) are denoted by $\lambda_i^{(N)}$, $i = 1, ..., N$.

In order to evaluate eigenvalues of large matrix eigenvalue problem (15), the numerical methods implemented in mathematical software can be used.

3.1. Particular case

Assuming the functions $p(x) = 1$, $q(x) = 0$ and $w(x) = 1$ in the SLP problem, then matrices $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{W}$ are reduced to the following forms: $\mathbf{Q} = \mathbf{O}$ (Zero matrix), $\mathbf{W} = \mathbf{I}$ (Identity matrix) and

$$
\mathbf{P} = \frac{1}{(\Delta x)^2}
\begin{bmatrix}
\gamma_a + 1 & -1 & 0 & 0 & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 \\
0 & -1 & 2 & -1 & & 0 \\
\vdots & & & & \ddots & \vdots \\
0 & \cdots & 0 & -1 & 2 & -1 \\
0 & \cdots & 0 & 0 & -1 & \gamma_b + 1
\end{bmatrix}
\tag{19}
$$

In the case of the mixed Dirichlet and/or Neumann boundary conditions at both boundaries, one can find in literature (e.g. [4]) the explicit formulas for the *i*-th eigenvalue λ_i of the SLP (1)-(2) and they are presented in Table 1. For the discrete case, the eigenvalues of the matrix **P** can be determined in an analytical way. The eigenvalue problems of tridiagonal matrices (in a similar form as the matrix **P**) are considered in [10, 11]. On the basis of these results, the eigenvalues of the matrix **P**: $\lambda_i^{(N)}$, $i = 1,...,N$ are adopted to the analysed problem for four mixed boundary conditions and they are also given in Table 1.

Table 1

**Eigenvalues of Eq. (1) for functions $p(x) = 1$, $q(x) = 0$ and $w(x) = 1$
and four mixed cases of boundary conditions
(explicit formulas for λ_i, $i = 1,...,\infty$ and discrete one for $\lambda_i^{(N)}$, $i = 1,...,N$)**

	The Dirichlet B.C. at $x = b$ $\gamma_b = 2$ (for $\beta_1 = 1$, $\beta_2 = 0$)	The Neumann B.C. at $x = b$ $\gamma_b = 0$ (for $\beta_1 = 0$, $\beta_2 = 1$)
The Dirichlet B.C. at $x = a$ $\gamma_a = 2$ (for $\alpha_1 = 1$, $\alpha_2 = 0$)	$\lambda_i = \left(\dfrac{i\pi}{b-a} \right)^2$ $\lambda_i^{(N)} = \dfrac{4N^2}{(b-a)^2} \sin^2 \left(\dfrac{i\pi}{2N} \right)$	$\lambda_i = \left(\dfrac{(i-0.5)\pi}{b-a} \right)^2$ $\lambda_i^{(N)} = \dfrac{4N^2}{(b-a)^2} \sin^2 \left(\dfrac{(i-0.5)\pi}{2N} \right)$
The Neumann B.C. at $x = a$ $\gamma_a = 0$ (for $\alpha_1 = 0$, $\alpha_2 = 1$)	$\lambda_i = \left(\dfrac{(i-0.5)\pi}{b-a} \right)^2$ $\lambda_i^{(N)} = \dfrac{4N^2}{(b-a)^2} \sin^2 \left(\dfrac{(i-0.5)\pi}{2N} \right)$	$\lambda_i = \left(\dfrac{(i-1)\pi}{b-a} \right)^2$ $\lambda_i^{(N)} = \dfrac{4N^2}{(b-a)^2} \sin^2 \left(\dfrac{(i-1)\pi}{2N} \right)$

For more complicated cases such as non-constant functions $p(x)$, $q(x)$, $w(x)$ occurring in Eq. (1), the eigenvalues of system (15) should be determined in a numerical way (i.e. using mathematical software - here the Maple is used).

3.2. Error of numerical approximation of the eigenvalues

Theorem 1. For the Sturm-Liouville problem (1)-(2) there exists a constant C such that the *i*-th exact eigenvalue λ_i and the approximate eigenvalue $\lambda_i^{(N)}$ satisfy the following relation

$$\left| \lambda_i - \lambda_i^{(N)} \right| \le C (\Delta x)^2 i^4 = O\left((\Delta x)^2 i^4 \right), \quad \Delta x = (b-a)/N, \quad i = 1,...,N \quad (20)$$

Proof: For the presented formulas in Table 1 (in these cases, the exact and discrete eigenvalues are known), one can estimate the error of approximation of the discrete eigenvalues. Let us start from the Taylor series expansion for the function $\sin^2(x)$ at about point $x = 0$

$$\sin^2(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!} \qquad (21)$$

Next, if we take into account the first two terms in the Taylor series (21), then we estimate the following expression as

$$\left| x^2 - \sin^2(x) \right| = \left| x^2 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!} \right| < \left| x^2 - \left(x^2 - \frac{1}{3} x^4 \right) \right| = \frac{1}{3} x^4 \qquad (22)$$

For the case of the Dirichlet-Dirichlet boundary conditions, the error is evaluated by using the estimation (22) in the following way:

$$\left| \lambda_i - \lambda_i^{(N)} \right| = \left| \left(\frac{i\pi}{b-a} \right)^2 - \frac{4N^2}{(b-a)^2} \sin^2 \left(\frac{i\pi}{2N} \right) \right| = \frac{(2N)^2}{(b-a)^2} \left| \left(\frac{i\pi}{2N} \right)^2 - \sin^2 \left(\frac{i\pi}{2N} \right) \right|$$

$$< \frac{(2N)^2}{3(b-a)^2} \left(\frac{i\pi}{2N} \right)^4 = \frac{\pi^4}{12(b-a)^2} \frac{i^4}{N^2} = \frac{\pi^4}{12(b-a)^4} (\Delta x)^2 i^4 = C(\Delta x)^2 i^4 \qquad (23)$$

In the other cases (from Table 1), the results are similar.

Another method of error estimation is based on the investigation of the Experimental Rate of Convergence (ERC). The total error in the estimate of the eigenvalues is composed of both the error resulting from the discretization of the equation and the error of the numerical algorithm for finding eigenvalues of system (15). Here, we assume that the error is

$$\left| \lambda_i - \lambda_i^{(N)} \right| = O\left((\Delta x)^r i^s \right), \quad \Delta x = (b-a)/N, \quad i = 1,...,N \qquad (24)$$

where the parameters r and s are to be determined experimentally.

If the i-th exact eigenvalue λ to SLP is known, then the parameters r and s can be determined using the following formulas for the ERC for variable values of N:

$$r = ERC_r(N,i) = \log_2 \frac{\lambda_i - \lambda_i^{(N/2)}}{\lambda_i - \lambda_i^{(N)}} \qquad (25)$$

$$s = ERC_s(N,i) = \log_{\frac{i}{i+1}} \frac{\lambda_i - \lambda_i^{(N)}}{\lambda_{i+1} - \lambda_{i+1}^{(N)}} \qquad (26)$$

Whereas, if the exact eigenvalue is unknown then, we determine the parameter r from the following formula

$$r = \underline{ERC}_r(N,i) = \log_2 \frac{\lambda_i^{(N)} - \lambda_i^{(N/2)}}{\lambda_i^{(2N)} - \lambda_i^{(N)}} \qquad (27)$$

The parameter s can be estimated using (26) and assuming that λ_i is numerically determined for sufficiently high value of N.

4. Example of numerical simulations

In tests of verification of the numerical solutions, three cases are taken into account:
- **Example 1**: $p(x) = 1$, $q(x) = 0$, $w(x) = (1 + x)^{-2}$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 1$, $\beta_2 = 0$.
- **Example 2**: $p(x) = (x + 1)^2$, $q(x) = x^2 - 2$, $w(x) = \exp(x)$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 0$, $\beta_2 = 1$.
- **Example 3**: $p(x) = 2 + \sin(2\pi x)$, $q(x) = -10$, $w(x) = 1 + \mathrm{sqrt}(x)$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 5$, $\beta_2 = 1$.

In all examples, the values of $a = 0$ and $b = 1$ have been assumed.

In the case of Example 1, the exact eigenvalues are given by $\lambda_i = 1/4 + (i\,\pi\,/\ln 2)^2$ [12], while for the remaining cases the exact eigenvalues are unknown.

In Tables 2, 4 and 5, the numerical values of the first 8 eigenvalues for different values of N and the calculated ERC_r ((25) or (27)) for all examples are presented, respectively. In addition, in Table 3, the calculated values of ERC_s (26) for Example 1 are shown.

Table 2

Eigenvalues and ERC_r for Example 1

N	$\lambda_1^{(N)}$	ERC_r	$\lambda_2^{(N)}$	ERC_r	$\lambda_3^{(N)}$	ERC_r	$\lambda_4^{(N)}$	ERC_r
100	20.7904297		82.3888961		184.976919		328.440408	
200	20.7918238	2.0001	82.4115895	2.0000	185.092175	1.9999	328.805046	1.9998
400	20.7921723	2.0000	82.4172627	2.0000	185.120991	2.0000	328.896222	1.9999
800	20.7922594	2.0000	82.4186811	2.0000	185.128195	2.0000	328.919017	2.0000
1600	20.7922812	2.0067	82.4190356	1.9995	185.129996	2.0001	328.924716	2.0000
3200	20.7922866	1.9994	82.4191243	2.0007	185.130446	2.0000	328.926140	2.0000
anal.	20.7922885		82.4191538		185.130596		328.926615	
N	$\lambda_5^{(N)}$	ERC_r	$\lambda_6^{(N)}$	ERC_r	$\lambda_7^{(N)}$	ERC_r	$\lambda_8^{(N)}$	ERC_r
100	512.619722		737.309749		1002.26001		1307.17479	
200	513.510262	1.9996	739.156478	1.9994	1005.68095	1.9992	1313.00956	1.9989
400	513.732969	1.9999	739.618392	1.9999	1006.53680	1.9998	1314.46964	1.9997
800	513.788651	2.0000	739.733885	2.0000	1006.75080	1.9999	1314.83475	1.9999
1600	513.802571	2.0000	739.762760	2.0000	1006.80430	2.0000	1314.92603	2.0000
3200	513.806051	2.0000	739.769978	2.0000	1006.81768	2.0000	1314.94885	2.0000
anal.	513.807211		739.772384		1006.82213		1314.95646	

Table 3

ERC_s for Example 1

N	$ERC_s(N,1)$	$ERC_s(N,2)$	$ERC_s(N,3)$	$ERC_s(N,4)$	$ERC_s(N,5)$	$ERC_s(N,6)$	$ERC_s(N,7)$	$ERC_s(N,8)$
100	4.0249	4.008	4.0037	4.0017	4.0006	3.9997	3.9989	3.9982
200	4.0250	4.0082	4.0040	4.0023	4.0014	4.0008	4.0004	4.0000
400	4.0250	4.0082	4.0041	4.0024	4.0016	4.0011	4.0007	4.0005
800	4.0250	4.0082	4.0041	4.0024	4.0016	4.0011	4.0008	4.0006
1600	4.0322	4.0073	4.0043	4.0024	4.0016	4.0012	4.0009	4.0007
3200	4.0309	4.0086	4.0042	4.0024	4.0017	4.0011	4.0008	4.0007

Table 4

Eigenvalues and $\underline{ERC_r}$ for Example 2

N	$\lambda_1^{(N)}$	ERC_r	$\lambda_2^{(N)}$	ERC_r	$\lambda_3^{(N)}$	ERC_r	$\lambda_4^{(N)}$	ERC_r
100	1.17035201	-	26.8573037	-	78.5053965	-	155.915777	-
200	1.17045757	2.0000	26.8618516	1.9999	78.5381317	1.9998	156.039051	1.9995
400	1.17048396	2.0000	26.8629886	2.0000	78.5463168	1.9999	156.069879	1.9999
800	1.17049056	2.0006	26.8632729	2.0001	78.5483631	2.0000	156.077587	2.0000
1600	1.17049221	1.9924	26.8633440	2.0000	78.5488747	1.9999	156.079514	2.0000
3200	1.17049262	-	26.8633617	-	78.5490026	-	156.079996	-

N	$\lambda_5^{(N)}$	ERC_r	$\lambda_6^{(N)}$	ERC_r	$\lambda_7^{(N)}$	ERC_r	$\lambda_8^{(N)}$	ERC_r
100	259.010064	-	387.685285	-	541.813108	-	721.239858	-
200	259.344073	1.9992	388.427244	1.9988	543.256336	1.9982	723.792747	1.9976
400	259.427622	1.9998	388.612893	1.9997	543.617583	1.9996	724.432011	1.9994
800	259.448513	1.9999	388.659316	1.9999	543.707922	1.9999	724.591893	1.9999
1600	259.453735	2.0000	388.670922	2.0000	543.730508	2.0000	724.631867	2.0000
3200	259.455041	-	388.673823	-	543.736155	-	724.641861	-

Table 5

Eigenvalues and $\underline{ERC_r}$ for Example 3

N	$\lambda_1^{(N)}$	ERC_r	$\lambda_2^{(N)}$	ERC_r	$\lambda_3^{(N)}$	ERC_r	$\lambda_4^{(N)}$	ERC_r
100	2.89692971	-	24.0701022	-	66.8696550	-	130.023860	-
200	2.89762086	1.9999	24.0778646	1.9998	66.9119064	1.9997	130.164588	1.9996
400	2.89779365	2.0000	24.0798055	1.9999	66.9224715	1.9999	130.199780	1.9999
800	2.89783685	1.9999	24.0802907	2.0000	66.9251129	2.0000	130.208579	2.0000
1600	2.89784765	1.9993	24.0804120	1.9999	66.9257733	2.0000	130.210778	2.0000
3200	2.89785036	-	24.0804424	-	66.9259384	-	130.211328	-

N	$\lambda_5^{(N)}$	ERC_r	$\lambda_6^{(N)}$	ERC_r	$\lambda_7^{(N)}$	ERC_r	$\lambda_8^{(N)}$	ERC_r
100	213.781678	-	318.196036	-	443.187196	-	588.625456	-
200	214.140243	1.9994	318.965739	1.9992	444.653772	1.9990	591.185500	1.9998
400	214.229921	1.9998	319.158267	1.9998	445.020663	1.9997	591.826047	1.9997
800	214.252343	2.0000	319.206405	1.9999	445.112402	1.9999	591.986219	1.9999
1600	214.257948	2.0000	319.218441	2.0000	445.135338	2.0000	592.026264	2.0000
3200	214.259350	-	319.221449	-	445.141072	-	592.036275	-

The analysis of the results presented in Tables 2-5 indicates that the rate r (ERC_r) is close to 2, while the rate s (ERC_s) is close to 4. Thus we can confirm that the relationship (20) is satisfied. The errors in the approximation of eigenvalues increase rapidly as the index of the eigenvalue grows.

5. Conclusions

In this paper, the new approach based on the control volume method for finding the eigenvalues of the Sturm-Liouville problem was discussed. The continuous problem described by the differential equation with the adequate boundary conditions was converted to the corresponding discrete one. The rate of convergence of the proposed numerical scheme is order 2. The presented results of the approximation of eigenvalues are in close agreement with the results obtained in an analytical way or in the mathematical software field. In the future, the presented approach can be extended to apply high order of accuracy difference schemes for approximating eigenvalues of the Sturm-Liouville problem and can be applied to the fractional Sturm-Liouville problem which is related to the corresponding fractional Euler--Lagrangian equation [13].

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