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# $K$-CONTINUITY OF $K$-SUBQUADRATIC SET-VALUED FUNCTIONS

*KATARZYNA TROCZKA-PAWELEC, IWONA TYRALA*

Abstract

Let $X = (X, +)$ be an arbitrary topological group. A set-valued function $F\colon X \to n(Y)$ is called $K$-subquadratic if

$$2F(s) + 2F(t) \subset F(s + t) + F(s - t) + K,$$

for all $s$, $t \in X$, where $Y$ denotes a topological vector space and where $K$ is a cone in this space.

In this paper the $K$-continuity problem of multifunctions of this kind will be considered with respect to weakly $K$-boundedness. The case where $Y = \mathbb{R}^N$ will be considered separately.

## 1. Introduction

Let $X = (X, +)$ be an arbitrary topological group. A real-valued function $F$, is called subquadratic, if it fulfils inequality

$$F(x + y) + F(x - y) \leq 2F(x) + 2F(y), \qquad x, y \in X. \tag{1}$$

If the sign "$\leq$" in (1) is replaced by "$\geq$" then $F$ is called superquadratic. The continuity problem of functions of this kind was considered in [1]. This problem can be also considered in the class of set-valued functions. Then we have two inclusions

$$F(x + y) + F(x - y) \subset 2F(x) + 2F(y), \qquad x,\ y \in X \tag{2}$$

and

$$2F(x) + 2F(y) \subset F(x + y) + F(x - y), \qquad x,\ y \in X. \tag{3}$$

• *Katarzyna Troczka-Pawelec* — e-mail: k.troczka@ajd.czest.pl
Jan Długosz University in Częstochowa.
• *Iwona Tyrala* — e-mail: i.tyrala@ajd.czest.pl
Jan Długosz University in Częstochowa.

where $F\colon X \to n(Y)$ and where $Y$ denotes a topological vector space. The continuity problem of set-valued functions defined by inclusions (2) and (3) was considered in [4] and [5].

Adding a cone $K$ in the space of values let us consider a $K$-subquadratic set-valued function $F$, that is solution of the inclusion

$$2F(x) + 2F(y) \subset F(x+y) + F(x-y) + K, \qquad x, y \in X \tag{4}$$

which is defined on 2-divisible topological group $X$ with non-empty, compact and convex values in a locally convex topological vector space $Y$. The $K$-continuity problem of multifunctions of this kind was considered in [6]. Here the $K$-continuity problem of $K$-subquadratic set-valued functions will be considered with respect to the weakly $K$-boundedness. In the last part of this paper we will present some conditions which imply $K$-continuity of $K$-subquadratic multifunctions which values are in $n(\mathbb{R}^N)$.

The concept of $K$-subquadraticity is related to real-valued subquadratic functions. In case when $F$ is a real single-valued function and $K = [0, \infty)$, we obtain the standard definition of subquadratic functionals (1). Assuming $K = \{0\}$ in (4) we obtain the inclusion (3).

Let us start with the notations used in this paper. Let $Y$ be a topological vector space. Let $n(Y)$ denotes the family of all non-empty subsets of $Y$, $cc(Y)$ – the family of all compact and convex members of $n(Y)$, $B(Y)$ – the family of all bounded members of $n(Y)$ and $Bcc(Y)$ – the family of all bounded, compact and convex members of $n(Y)$. The term set-valued function will be abbreviated to the form s.v.f.

First of all we shall present some definitions for the sake of completeness. Recall that a set $K \subset Y$ is called a cone if $K + K \subset K$ and $sK \subset K$ for all $s \in (0, \infty)$.

**Definition 1.** (cf. [2]) *A cone $K$ in a topological vector space $Y$ is said to be a normal cone if there exists a base $\mathfrak{W}$ of zero in $Y$ such that*

$$W = (W + K) \cap (W - K)$$

*for all $W \in \mathfrak{W}$.*

**Definition 2.** (cf. [2]) *An s.v.f. $F\colon X \to n(Y)$ is said to be $K$-upper semicontinuous (abbreviated $K$-u.s.c.) at $x_0 \in X$ if for every neighbourhood $V$ of zero in $Y$ there exists a neighbourhood $U$ of zero in $X$ such that*

$$F(x) \subset F(x_0) + V + K$$

*for every $x \in x_0 + U$.*

**Definition 3.** (cf. [2]) *An s.v.f. $F\colon X \to n(Y)$ is said to be $K$-lower semicontinuous (abbreviated $K$-l.s.c.) at $x_0 \in X$ if for every neighbourhood $V$*

*of zero in $Y$ there exists a neighbourhood $U$ of zero in $X$ such that*

$$F(x_0) \subset F(x) + V + K$$

*for every $x \in x_0 + U$.*

**Definition 4.** (cf. [2]) *An s.v.f. $F\colon X \to n(Y)$ is said to be $K$-continuous at $x_0 \in X$ if it is both $K$-u.s.c. and $K$-l.s.c. at $x_0$. It is said to be $K$-continuous if it is $K$-continuous at each point of $X$.*

Note that in the case where $K = \{0\}$ the $K$-continuity of $F$ means its continuity with respect to the Hausdorff topology on $n(Y)$.

In our proofs we use known following lemma.

**Lemma 1.** (cf. [6]) *Let $Y$ be a topological vector space and $K$ be a cone in $Y$. Let $A$, $B$, $C$ be non-empty subsets of $Y$ such that $A + C \subset B + C + K$. If $B$ is convex and $C$ is bounded then $A \subset \overline{B + K}$.*

In our proofs we will also use two known lemmas (see Lemma 1.6 and Lemma 1.5 in [2]). The first lemma says that if $A \subset Y$ is a closed set and $B \subset Y$ is a compact set, where $Y$ denotes a real topological vector space, then the set $A + B$ is closed. The second lemma says that for any bounded sets $A, B \subset Y$, where $Y$ denotes the same space as above, the set $A + B$ is bounded.

Let us adopt the following three definitions which are natural extension of the concept of the boundedness for real-valued functions.

**Definition 5.** *An s.v.f. $F\colon X \to n(Y)$ is said to be $K$-lower bounded on a set $A \subset X$ if there exists a bounded set $B \subset Y$ such that $F(x) \subset B + K$ for all $x \in A$.*

**Definition 6.** *An s.v.f. $F\colon X \to n(Y)$ is said to be $K$-upper bounded on a set $A \subset X$ if there exists a bounded set $B \subset Y$ such that $F(x) \subset B - K$ for all $x \in A$.*

**Definition 7.** *An s.v.f. $F\colon X \to n(Y)$ is said to be locally $K$-bounded in $X$ if for every $x \in X$ there exists a neighbourhood $U_x$ of zero in $X$ such that $F$ is $K$-lower and $K$-upper bounded on a set $x + U_x$.*

## 2. The main result connected with weakly $K$-boundedness

Let us introduce the following definitions:

**Definition 8.** *An s.v.f. $F\colon X \to n(Y)$ is said to be weakly $K$-lower bounded on a set $A \subset X$ if there exists a bounded set $B \subset Y$ such that*

$$F(x) \cap (B + K) \neq \emptyset$$

*for all $x \in A$.*

**Definition 9.** *An s.v.f. $F\colon X \to n(Y)$ is said to be weakly $K$-upper bounded on a set $A \subset X$ if there exists a bounded set $B \subset Y$ such that*

$$F(x) \cap (B - K) \neq \emptyset$$

*for all $x \in A$.*

**Definition 10.** *An s.v.f. $F\colon X \to n(Y)$ is said to be locally weakly $K$-bounded in $X$ if for every $x \in X$ there exists a neighbourhood $U_x$ of zero in $X$ such that $F$ is weakly $K$-lower and weakly $K$-upper bounded on a set $x + U_x$.*

Clearly, if $F$ is $K$-upper ($K$-lower) bounded on a set $A$, then it is weakly $K$-upper ($K$-lower) bounded on a set $A$. In the case of single-valued functions these definitions coincide.

**Definition 11.** *We say that 2-divisible topological group $X$ has the property $\left(\frac{1}{2}\right)$ if for every neighbourhood $V$ of zero there exists a neighbourhood $W$ of zero such that $\frac{1}{2}W \subset W \subset V$.*

For the $K$-subquadratic set-valued functions the following theorem holds.

**Theorem 1.** (cf. [6]) *Let $X$ be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, $Y$ – locally convex topological vector space and a subset $K$ of $Y$ – a closed normal cone. If a $K$-subquadratic s.v.f. $F\colon X \to cc(Y)$ is $K$-continuous at zero, locally $K$-bounded in $X$ and $F(0) = \{0\}$, then it is $K$-continuous in $X$.*

**Lemma 2.** *Let $X$ be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, $Y$ – topological vector space and $K \subset Y$ a cone. Let $F\colon X \to B(Y)$ be a $K$-subquadratic s.v.f. such that $F(0) = \{0\}$ and $G\colon X \to n(Y)$ be an s.v.f. with*

$$G(x) \subset F(x) + K \tag{5}$$

*for all $x \in X$.*

*If $F$ is $K$-lower bounded at zero and $G$ is locally weakly $K$-upper bounded in $X$, then $F$ is locally $K$-lower bounded in $X$.*

*Proof.* Let $x \in X$. There exist a bounded set $B_1 \subset Y$ and a symmetric neighbourhood $U_1$ of zero in $X$ such that

$$G(x - t) \cap (B_1 - K) \neq \emptyset, \qquad t \in U_1,$$

which implies that

$$0 \in G(x - t) - B_1 + K \tag{6}$$

for all $t \in U_1$. Since $F$ is $K$-lower bounded at zero, there exist a symmetric neighbourhood $U_2$ of zero in $X$ and a bounded set $B_2 \subset Y$ such that

$$F(t) \subset B_2 + K, \qquad t \in U_2. \tag{7}$$

Let $\widetilde{U}$ be a symmetric neighbourhood of zero in $X$ with $\frac{1}{2}\widetilde{U} \subset \widetilde{U} \subset U_1 \cap U_2$. Let $t \in \frac{1}{2}\widetilde{U}$. Using (5), (6) i (7), we obtain

$$F(x+t)+0 \subset F(x+t)+G(x-t)-B_1+K \subset F(x+t)+F(x-t)-B_1+K \subset$$
$$\subset \frac{1}{2}F(2x) + \frac{1}{2}F(2t) - B_1 + K \subset \frac{1}{2}F(2x) + \frac{1}{2}B_2 - B_1 + K.$$

Define $\widetilde{B} := \frac{1}{2}F(2x) + \frac{1}{2}B_2 - B_1$. Since $F(2x)$ is a bounded set, then the set $\widetilde{B}$ is also bounded as the sum of bounded sets. Therefore

$$F(x+t) \subset \widetilde{B} + K, \quad t \in \widetilde{U},$$

which means that $F$ is locally $K$-lower bounded in $X$. □

**Lemma 3.** *Let $X$ be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, $Y$ topological vector space and $K \subset Y$ a cone. Let $F\colon X \to B(Y)$ be a $K$-subquadratic s.v.f. such that $F(0) = \{0\}$ and $G\colon X \to n(Y)$ be an s.v.f. with*

$$G(x) \subset F(x) - K \tag{8}$$

*for all $x \in X$.*

*If $F$ is $K$-upper bounded at zero and $G$ is locally weakly $K$-lower bounded in $X$, then $F$ is locally $K$-upper bounded in $X$.*

*Proof.* Let $x \in X$. Since $G$ is weakly $K$-lower bounded in $x$, then there exist a bounded set $B_1 \subset Y$ and a symmetric neighbourhood $U_1$ of zero in $X$ such that

$$G(x-t) \cap (B_1 + K) \neq \emptyset, \quad t \in U_1,$$

which implies that

$$0 \in G(x-t) - B_1 - K \tag{9}$$

for all $t \in U_1$. Since $F$ is $K$-upper bounded at zero, there exist a symmetric neighbourhood $U_2$ of zero in $X$ and a bounded set $B_2 \subset Y$ such that

$$F(t) \subset B_2 - K, \quad t \in U_2. \tag{10}$$

Let $\widetilde{U}$ be a symmetric neighbourhood of zero in $X$ with $\frac{1}{2}\widetilde{U} \subset \widetilde{U} \subset U_1 \cap U_2$. Let $t \in \frac{1}{2}\widetilde{U}$. Using (8), (9) i (10), we obtain

$$F(x+t)+0 \subset F(x+t)+G(x-t)-B_1-K \subset F(x+t)+F(x-t)-B_1-K \subset$$
$$\subset \frac{1}{2}F(2x) + \frac{1}{2}F(2t) - B_1 - K \subset \frac{1}{2}F(2x) + \frac{1}{2}B_2 - B_1 - K.$$

Define $\widetilde{B} := \frac{1}{2}F(2x) + \frac{1}{2}B_2 - B_1$. Since $F(2x)$ is a bounded set, then the set $\widetilde{B}$ is also bounded as the sum of bounded sets. Therefore

$$F(x+t) \subset \widetilde{B} - K, \quad t \in \widetilde{U},$$

which means that $F$ is locally $K$-upper bounded in $X$. □

As an immediate consequence of Lemma 2 and Lemma 3 we obtain the following lemma.

**Lemma 4.** *Let $X$ be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, $Y$ topological vector space and $K \subset Y$ a cone with zero. Let $F\colon X \to B(Y)$ be a $K$-subquadratic s.v.f. such that $F(0) = \{0\}$. If $F$ is $K$- bounded at zero and locally weakly $K$-bounded in $X$, then it is locally $K$-bounded in $X$.*

Let us note, that Theorem 1, Lemma 2 and Lemma 3 yield directly the following result.

**Theorem 2.** *Let $X$ be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, $Y$ locally convex topological vector space and $K \subset Y$ a closed normal cone. Let $F\colon X \to Bcc(Y)$ be a $K$-subquadratic s.v.f. with $F(0) = \{0\}$ and $G\colon X \to n(Y)$ be an s.v.f. with*

$$G(x) \subset (F(x) - K) \cap (F(x) + K)$$

*for all $x \in X$.*

*If $F$ is $K$-bounded at zero and $K$-continuous at zero, $G$ is locally weakly $K$-bounded in $X$, then $F$ is $K$-continuous everywhere in $X$.*

**Remark 1.** *Let $X$ be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, $Y$ locally convex topological vector space and $K \subset Y$ a closed normal cone. Let $F\colon X \to Bcc(Y)$ be a $K$-subquadratic s.v.f. with $F(0) = \{0\}$.*

*If $F$ is $K$-continuous at zero, $K$- bounded at zero and locally weakly $K$-bounded in $X$, then it is $K$-continuous in $X$.*

*Proof.* Note that a closed cone is a cone with zero. Then the following inclusion

$$F(x) \subset (F(x) - K) \cap (F(x) + K)$$

holds for all $x \in X$. Using Theorem 2 for $G = F$ we end the proof. □

## 3. The case $Y = \mathbb{R}^N$

Now we consider the case where the space of values is $n(\mathbb{R}^N)$. It is known that for $K$-subquadratic set-valued functions the following lemma holds.

**Lemma 5.** (cf. [6]) *Let $X$ be a 2-divisible topological group, $Y$ locally convex topological vector space and $K \subset Y$ a closed normal cone. If a $K$-subquadratic s.v.f. $F\colon X \to cc(Y)$ is $K$-continuous at zero, $F(0) = \{0\}$ and locally $K$-lower bounded in $X$, then it is $K$-u.s.c. in $X$.*

In this part of the paper $Y$ will be denote $\mathbb{R}^N$.

**Theorem 3.** *Let $X$ be a 2-divisible topological group and $K$ be a closed normal cone in $Y$. Let $F\colon X \to cc(Y)$ be a $K$-subquadratic s.v.f. with $F(0) = \{0\}$. If $F$ is $K$-continuous at zero and locally $K$-lower bounded in $X$, then it is $K$-continuous in $X$.*

*Proof.* By Lemma 5 $F$ is $K$-u.s.c. in $X$. Now we will show that $F$ is $K$-l.s.c. in $X$. Let $x_0 \in X$ and let $V$ be a neighbourhood of zero in $Y$. There exists a convex neighbourhood $W$ of zero in $Y$ such that the set $\overline{W}$ is compact with $3\overline{W} \subset V$. Since $F$ is $K$-u.s.c. at $x_0$ then there exists a symmetric neighbourhood $U$ of zero in $X$ such that

$$F(x_0 + t) \subset F(x_0) + W + K, \tag{11}$$

$$F(x_0 - t) \subset F(x_0) + W + K, \tag{12}$$

for all $t \in U$.
Since $F$ is $K$-l.s.c. at zero and $F(0) = \{0\}$, there exists a neighbourhood $U_0$ of zero in $X$ such that

$$\{0\} \subset F(t) + W + K \qquad t \in U_0. \tag{13}$$

Consider a symmetric neighbourhood $\widetilde{U}$ of zero in $X$ with $\widetilde{U} \subset U \cap U_0$. Let $t \in \widetilde{U}$. Using (12) i (13), we obtain

$$F(x_0)+\{0\} \subset F(x_0)+F(t)+W+K \subset \frac{1}{2}F(x_0+t)+\frac{1}{2}F(x_0-t)+W+K \subset$$

$$\subset \frac{1}{2}F(x_0+t) + \frac{1}{2}F(x_0) + \frac{3}{2}\overline{W} + K$$

By convexity of the set $F(x_0)$, we have

$$\frac{1}{2}F(x_0) + \frac{1}{2}F(x_0) \subset \frac{1}{2}F(x_0) + \frac{1}{2}F(x_0+t) + \frac{3}{2}\overline{W} + K.$$

Note that the set $\frac{1}{2}F(x_0+t) + \frac{3}{2}\overline{W}$ is convex and compact. Therefore, the set $\frac{1}{2}F(x_0+t) + \frac{3}{2}\overline{W} + K$ is closed. Using Lemma 1

$$\frac{1}{2}F(x_0) \subset \overline{\frac{1}{2}F(x_0+t) + \frac{3}{2}\overline{W} + K} = \frac{1}{2}F(x_0+t) + \frac{3}{2}\overline{W} + K,$$

and consequently

$$F(x_0) \subset F(x_0+t) + 3\overline{W} + K \subset F(x_0+t) + V + K,$$

for all $t \in \widetilde{U}$. It means that $F$ is $K$-l.s.c. in $X$. □

**Theorem 4.** *Let $X$ be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$ and $K$ be a closed normal cone in $\mathbb{R}^N$. Let $F\colon X \to cc(Y)$ be a $K$-subquadratic s.v.f. with $F(0) = \{0\}$. If $F$ is $K$-continuous at zero and locally weakly $K$-upper bounded in $X$, then it is $K$-continuous in $X$.*

*Proof.* Let $V$ be a bounded neighbourhood of zero in $Y$. Since $F$ is $K$-u.s.c. at zero and $F(0) = \{0\}$ there exists a neighbourhood $U$ of zero in $X$ such that

$$F(t) \subset V + K, \qquad t \in U.$$

It means that $F$ is $K$-lower bounded at zero. By Lemma 2 ( with $G = F$), $F$ is locally $K$-lower bounded in $X$. Applying Theorem 3, $F$ is $K$-continuous in $X$. □

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*Katarzyna Troczka–Pawelec,*
Jan Długosz University in Częstochowa,
Institute of Mathematics and Computer Science,
al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
*E-mail address*: k.troczka@ajd.czest.pl

*Iwona Tyrala*
Jan Długosz University in Częstochowa,
Institute of Mathematics and Computer Science,
al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
*E-mail address*: i.tyrala@ajd.czest.pl