Opuscula Math. 42, no. 1 (2022), 31–54
https://doi.org/10.7494/OpMath.2022.42.1.31 Opuscula Mathematica

γ-PAIRED DOMINATING GRAPHS OF CYCLES

Pannawat Eakawinrujee and Nantapath Trakultraipruk

Communicated by Dalibor Fronček

Abstract. A paired dominating set of a graph G is a dominating set whose induced subgraph contains a perfect matching. The paired domination number, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a paired dominating set of G. A $\gamma_{pr}(G)$-set is a paired dominating set of cardinality $\gamma_{pr}(G)$. The γ-paired dominating graph of G, denoted by $PD_\gamma(G)$, as the graph whose vertices are $\gamma_{pr}(G)$-sets. Two $\gamma_{pr}(G)$-sets D_1 and D_2 are adjacent in $PD_\gamma(G)$ if there exists a vertex $u \in D_1$ and a vertex $v \notin D_1$ such that $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$. In this paper, we present the γ-paired dominating graphs of cycles.

Keywords: paired dominating graph, paired dominating set, paired domination number.

Mathematics Subject Classification: 05C69, 05C38.

1. INTRODUCTION

For notation and terminology, we refer the reader to [9]. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $v \in V(G)$, the *open neighborhood* and *closed neighborhood* of v are denoted by $N(v)$ and $N[v]$, respectively. For a set $D \subseteq V(G)$, the *open neighborhood* of D is $N(D) = \bigcup_{v \in D} N(v)$, and the *closed neighborhood* of D is $N[D] = N(D) \cup D$. The subgraph of G induced by D is denoted by $G[D]$. The vertices in D *dominate* the vertices in $S \subseteq V(G)$ if $S \subseteq N[D]$. We denote the graph obtained from G by deleting all vertices in D and all edges incident with them by $G - D$. A *path*, a *cycle*, and a *complete graph* with n vertices are denoted by P_n, C_n, and K_n, respectively.

A set $D \subseteq V(G)$ is a *dominating set* of G if $N[D] = V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$*-set*. For a detailed literature on domination, see [5, 6].

The *gamma graph* $\gamma.G$ of a graph G, defined by Lakshmanan and Vijayakumar [7], as the graph whose vertices are $\gamma(G)$-sets, and $\gamma(G)$-sets D_1 and D_2 are adjacent in $\gamma.G$ if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1$ and $v \notin D_1$. In 2011, Fricke *et al.* [2]

also defined the *gamma graph* $G(\gamma)$ with different meaning. The only difference is that two $\gamma(G)$-sets D_1 and D_2 are adjacent in $G(\gamma)$ if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1, v \notin D_1$, and $uv \in E(G)$. Notice that $G(\gamma)$ is a subgraph of $\gamma.G$ with the same vertex set. In 2014, Haas and Seyffarth [3] introduced the *k-dominating* graph of G, denoted by $D_k(G)$, as the graph whose vertices are dominating sets of cardinality at most k. Two dominating sets D_1 and D_2 are adjacent in $D_k(G)$ if $D_2 = D_1 \cup \{v\}$ for some $v \notin D_1$. They gave conditions that ensure $D_k(G)$ is connected.

In 2017, Wongsriya and Trakultraipruk [10] defined the *γ-total dominating graph* of G, denoted by $TD_\gamma(G)$, as the graph whose vertices are $\gamma_t(G)$-sets, which are total dominating sets of minimum cardinality. Two $\gamma_t(G)$-sets D_1 and D_2 are adjacent in $TD_\gamma(G)$ if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1$ and $v \notin D_1$. They determined the γ-total dominating graphs of paths and cycles. In 2019, Samanmoo *et al.* [8] introduced the *γ-independent dominating graph* of G, denoted by $ID_\gamma(G)$, as the graph whose vertices are $\gamma_i(G)$-sets, which are independent dominating sets of minimum cardinality. Two $\gamma_i(G)$-sets D_1 and D_2 are adjacent in $ID_\gamma(G)$ if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1, v \notin D_1$. They provided the γ-independent dominating graphs of paths and cycles.

A *matching* in G is a set of independent edges in G. A *perfect matching* M in G is a matching such that every vertex of G is incident to an edge of M. A set $D \subseteq V(G)$ is a *paired dominating set* of G if it is a dominating set and the induced subgraph $G[D]$ has a perfect matching. The set $\{u, v\} \subseteq D$ is called *paired* if uv is an edge in a perfect matching of $G[D]$. The *paired domination number* $\gamma_{pr}(G)$ is the minimum cardinality of a paired dominating set of G. A *$\gamma_{pr}(G)$-set* is a paired dominating set of cardinality $\gamma_{pr}(G)$. Paired domination was introduced by Haynes and Slater [4] as a model for assigning backups to guards for security purposes.

In [1], we introduced the *γ-paired dominating graph* of G, denoted by $PD_\gamma(G)$, as the graph whose vertices are $\gamma_{pr}(G)$-sets, and two $\gamma_{pr}(G)$-sets D_1 and D_2 are adjacent in $PD_\gamma(G)$ if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1$ and $v \notin D_1$. We determined the γ-paired dominating graphs of paths. In this paper, we present the γ-paired dominating graphs of cycles. For example, the γ-paired dominating graphs of cycles $C_4 : v_0v_1v_2v_3v_0$ and $C_5 : v_0v_1v_2v_3v_4v_0$ are shown in Figure 1. We see that $PD_\gamma(C_4) \cong C_4$ and $PD_\gamma(C_5) \cong K_5$.

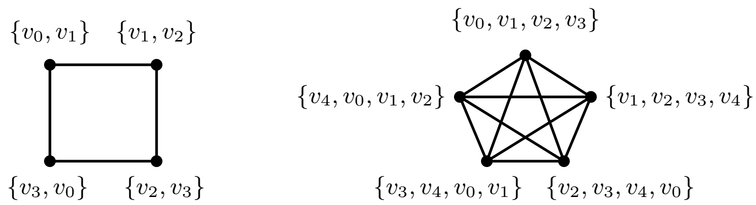

Fig. 1. The γ-paired dominating graphs of C_4 and C_5, respectively

2. PRELIMINARY RESULTS

In this section, we recall some definitions, notations, and results used in the main results.

Haynes and Slater [4] established the following useful lemma.

Lemma 2.1. *For any integer* $n \geq 3$, $\gamma_{pr}(P_n) = \gamma_{pr}(C_n) = 2\lceil \frac{n}{4} \rceil$.

The *Cartesian product* of graphs G and H, denoted by $G\Box H$, is the graph with vertex set $V(G) \times V(H)$ whose vertices (u, v) and (u', v') are adjacent if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.

For any positive integers p and q, let $P_p : u_1u_2u_3 \cdots u_p$ and $P_q : v_1v_2v_3 \cdots v_q$ be two paths with p and q vertices, respectively. Fricke *et al.* [2] defined a *stepgrid* $SG_{p,q}$ to be the subgraph of $P_p\Box P_q$ induced by

$$\{(u_x, v_y) \in V(P_p\Box P_q) : 1 \leq x \leq p, 1 \leq y \leq q, x - y \leq 1\}.$$

We call the vertex (u_x, v_y) in the stepgrid as the *vertex at the position* (x, y). For example, the stepgrids $SG_{1,1}, SG_{2,2}$, and $SG_{4,3}$ are shown in Figure 2.

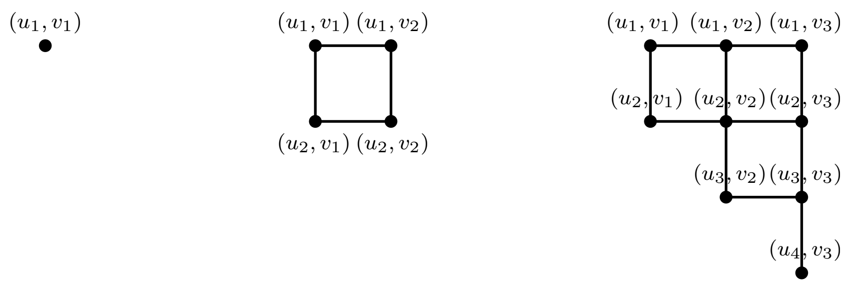

Fig. 2. The stepgrids $SG_{1,1}, SG_{2,2}$, and $SG_{4,3}$, respectively

For any positive integers p, q, and r, let $P_p : u_1u_2u_3 \cdots u_p$, $P_q : v_1v_2v_3 \cdots v_q$, and $P_r : w_1w_2w_3 \cdots w_r$ be three paths with p, q, and r vertices, respectively. We define a *stepgrid* $SG_{p,q,r}$ as the graph with the vertex set

$$V(SG_{p,q,r}) = \{(u_x, v_y, w_z) \in V(P_p\Box P_q\Box P_r) : 1 \leq x \leq p, 1 \leq y \leq q, \\ 1 \leq z \leq r, x - y \leq 0, x - z \leq 1, y - z \geq 0\},$$

and the edge set

$$E(SG_{p,q,r}) = E(P_p\Box P_q\Box P_r) \cup \{(u_x, v_x, w_x)(u_{x+1}, v_{x+1}, w_x) : 1 \leq x \leq p - 1\}.$$

The vertex (u_x, v_y, w_z) is called the *vertex at the position* (x, y, z) in $SG(p, q, r)$. For example, the stepgrids $SG_{2,2,1}$ and $SG_{3,3,2}$ are shown in Figure 3, and the stepgrid $SG_{4,4,3}$ is shown in Figure 4, where we write (x, y, z) as (u_x, v_y, w_z).

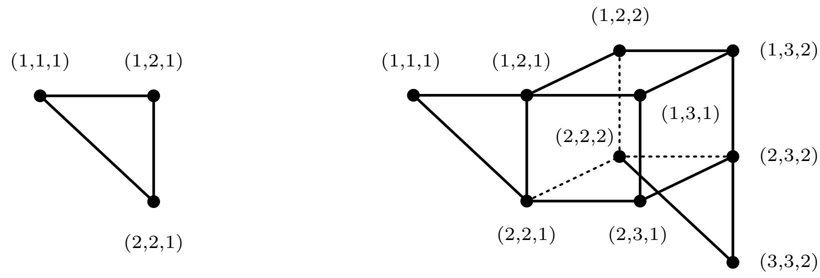

Fig. 3. The stepgrids $SG_{2,2,1}$ and $SG_{3,3,2}$, respectively

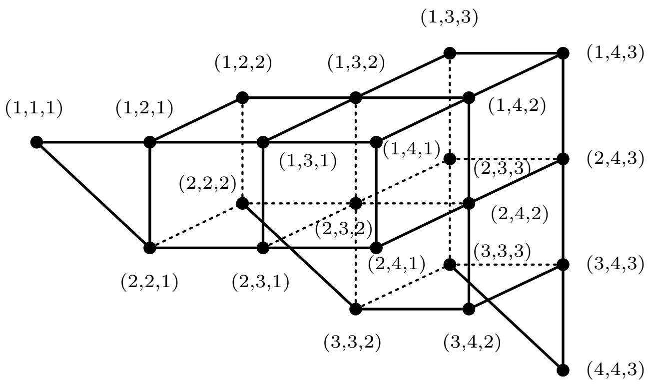

Fig. 4. The stepgrid $SG_{4,4,3}$

Let $P_n : v_1v_2v_3 \cdots v_n$ be a path with n vertices. In [1], we determined the γ-paired dominating graphs of paths, and gave the following results.

Lemma 2.2. *Let $k \geq 1$ be an integer. Then there is only one $\gamma_{pr}(P_{4k-1})$-set containing the pair $\{v_{4k-2}, v_{4k-1}\}$, and there is only one $\gamma_{pr}(P_{4k-1})$-set containing the pair $\{v_1, v_2\}$.*

Lemma 2.3. *Let $k \geq 2$ be an integer. Then all $\gamma_{pr}(P_{4k-2})$-sets containing the pair $\{v_{4k-3}, v_{4k-2}\}$ form a path $A_1A_2 \cdots A_k$ with k vertices, where A_1 and A_k are of degree two, the others are of degree three, and A_k has a neighbor of degree two in $PD_\gamma(P_{4k-2})$. Moreover, A_k contains the pair $\{v_{4k-6}, v_{4k-5}\}$, and the others contain the pair $\{v_{4k-7}, v_{4k-6}\}$. The similar results also hold for the $\gamma_{pr}(P_{4k-2})$-sets containing the pair $\{v_1, v_2\}$.*

Lemma 2.4. *Let $k \geq 3$ be an integer. Then all $\gamma_{pr}(P_{4k-3})$-sets containing the pair $\{v_{4k-4}, v_{4k-3}\}$ form a stepgrid $SG_{k,k-1}$, where $B_{1,1}, B_{2,1}, B_{1,k-1}$ are of degree three, $B_{2,k-1}, B_{3,k-1}, \ldots, B_{k-1,k-1}$ are of degree four, and $B_{k,k-1}$ is of degree two in $PD_\gamma(P_{4k-3})$. Moreover, $B_{1,k-1}, B_{2,k-1}, \ldots, B_{k-1,k-1}$, contain the pair $\{v_{4k-7}, v_{4k-6}\}$, and $B_{k,k-1}$ contains the pair $\{v_{4k-6}, v_{4k-5}\}$ (see Figure 5). The similar results also hold for the $\gamma_{pr}(P_{4k-3})$-sets containing the pair $\{v_1, v_2\}$.*

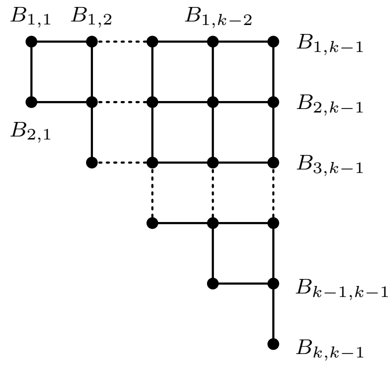

Fig. 5. The stepgrid $SG_{k,k-1}$

Theorem 2.5. *Let $k \geq 1$ be an integer. Then $PD_\gamma(P_{4k}) \cong P_1$.*

Theorem 2.6. *Let $k \geq 1$ be an integer. Then $PD_\gamma(P_{4k-1}) \cong P_{k+1}$.*

Theorem 2.7. *Let $k \geq 1$ be an integer. Then $PD_\gamma(P_{4k-2}) \cong SG_{k,k}$.*

Corollary 2.8. *Let $k \geq 2$ be an integer, and $A_{x,y}$ the $\gamma_{pr}(P_{4k-2})$-set at the position (x, y) in $PD_\gamma(P_{4k-2}) \cong SG_{k,k}$ (see Figure 6) for all $x, y \in \{1, 2, \ldots, k\}$ with $x-y \leq 1$. If $A_{x,k}$ contains the pair $\{v_{4k-3}, v_{4k-2}\}$, then we get the following properties.*

- (A1) *If $y = k$, then $A_{x,y}$ contains the pair $\{v_{4k-3}, v_{4k-2}\}$; otherwise, it contains the pair $\{v_{4k-4}, v_{4k-3}\}$.*
  - (A1.1) *$A_{x,k}$ contains the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-3}, v_{4k-2}\}$ for all $x \in \{1, 2, \ldots, k-1\}$, and $A_{k,k}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-3}, v_{4k-2}\}$.*
- (A2) *If $x = 1$, then $A_{x,y}$ contains the pair $\{v_1, v_2\}$; otherwise, it contains the pair $\{v_2, v_3\}$.*
  - (A2.1) *$A_{1,1}$ contains the pairs $\{v_1, v_2\}$, $\{v_4, v_5\}$, and $A_{1,y}$ contains the pairs $\{v_1, v_2\}$, $\{v_5, v_6\}$ for all $y \in \{2, 3, \ldots, k\}$.*

Theorem 2.9. *Let $k \geq 2$ be an integer. Then $PD_\gamma(P_{4k-3}) \cong SG_{k,k,k-1}$.*

Corollary 2.10. *Let $k \geq 3$ be an integer and $B_{x,y,z}$ the $\gamma_{pr}(P_{4k-3})$-set at the position (x, y, z) in $PD_\gamma(P_{4k-3}) \cong SG_{k,k,k-1}$ (see Figure 7) for all $x, y \in \{1, 2, \ldots, k\}$, $z \in \{1, 2, \ldots, k-1\}$ with $x - y \leq 0$, $x - z \leq 1$, $y - z \geq 0$. If $B_{x,k,z}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$, then we get the following properties.*

(B1) *If* $y = k$, *then* $B_{x,y,z}$ *contains the pair* $\{v_{4k-4}, v_{4k-3}\}$*; otherwise, it contains the pair* $\{v_{4k-5}, v_{4k-4}\}$.

- (B1.1) $B_{x,k,k-1}$ *contains the pairs* $\{v_{4k-7}, v_{4k-6}\}, \{v_{4k-4}, v_{4k-3}\}$ *for all* $x \in \{1, 2, \ldots, k-1\}$, *and* $B_{k,k,k-1}$ *contains the pairs* $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-4}, v_{4k-3}\}$.
- (B1.2) $B_{x,k,z}$ *contains the pairs* $\{v_{4k-8}, v_{4k-7}\}, \{v_{4k-4}, v_{4k-3}\}$ *for all* $z \neq k-1$.

(B2) *If* $x = 1$, *then* $B_{x,y,z}$ *contains the pair* $\{v_1, v_2\}$*; otherwise, it contains the pair* $\{v_2, v_3\}$.

- (B2.1) $B_{1,1,1}$ *contains the pairs* $\{v_1, v_2\}, \{v_3, v_4\}$, *and* $B_{1,y,1}$ *contains the pairs* $\{v_1, v_2\}, \{v_4, v_5\}$ *for all* $y \in \{2, 3, \ldots, k\}$.
- (B2.2) $B_{1,y,z}$ *contains the pairs* $\{v_1, v_2\}, \{v_5, v_6\}$ *for all* $z \neq 1$.

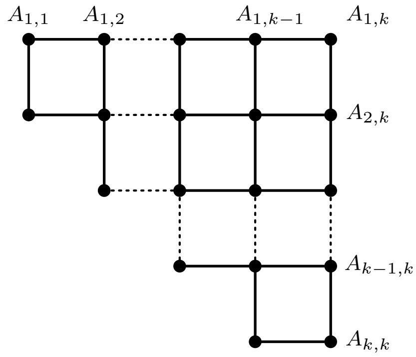

Fig. 6. The stepgrid $SG_{k,k}$

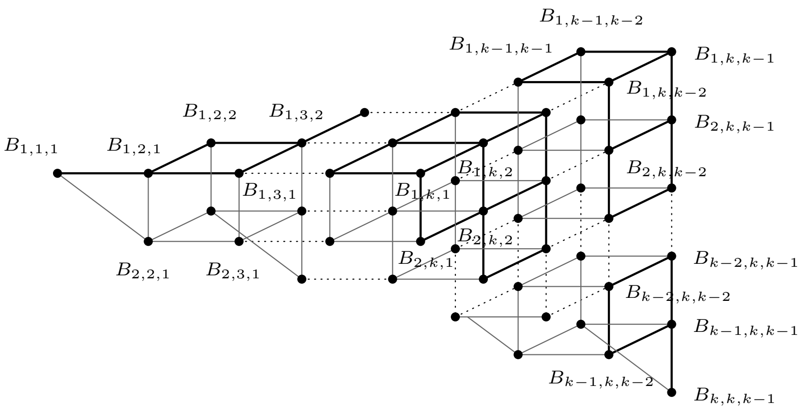

Fig. 7. The stepgrid $SG_{k,k,k-1}$

3. γ-PAIRED DOMINATING GRAPHS OF CYCLES

In this section, we present the γ-paired dominating graphs of cycles. We always let $C_n : v_0v_1v_2 \cdots v_{n-1}v_0$ be a cycle with n vertices. We first consider the γ-paired dominating graph of C_{4k}, as stated the following theorem.

Theorem 3.1. *Let $k \geq 1$ be an integer. Then*

$$PD_\gamma(C_{4k}) \cong \begin{cases} C_4 & \text{if } k = 1; \\ 4P_1 & \text{if } k \geq 2. \end{cases}$$

Proof. From Figure 1, we get that $PD_\gamma(C_4) \cong C_4$. Let $k \geq 2$. By Theorem 2.1, we have $\gamma_{pr}(C_{4k}) = 2k$. It is easy to check that

$$\{v_0, v_1, v_4, v_5, \ldots, v_{4k-4}, v_{4k-3}\}, \quad \{v_1, v_2, v_5, v_6, \ldots, v_{4k-3}, v_{4k-2}\},$$
$$\{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\}, \quad \{v_3, v_4, v_7, v_8, \ldots, v_{4k-1}, v_{4k}\}$$

are the only $\gamma_{pr}(C_{4k})$-sets. Thus, $PD_\gamma(C_{4k}) \cong 4P_1$. □

Before we prove the result on the γ-paired dominating graph of a cycle with $4k+3$ vertices, we need the following lemma.

Lemma 3.2. *Let $k \geq 0$ be an integer and D a $\gamma_{pr}(C_{4k+3})$-set. Then there is exactly one vertex not in D dominated by two vertices of D.*

Proof. We can easily get that the lemma holds for $k = 0$. Let $k \geq 1$. Note that $|D| = 2k + 2$, so we can write $D = \bigcup_{x=1}^{k+1} D_x$, where D_x's are pairwise disjoint sets of paired vertices. Clearly, $|N[D_x]| = 4$ for all $x \in \{1, 2, 3, \ldots, k+1\}$, and $V(C_{4k+3}) = \bigcup_{x=1}^{k+1} N[D_x]$. If $N[D_x]$'s are pairwise disjoint sets, then

$$4k + 3 = |V(C_{4k+3})| = \sum_{x=1}^{k+1} |N[D_x]| = 4k + 4,$$

a contradiction. Therefore, without loss of generality, there are exactly two disjoint sets D_1 and D_2 such that $|N[D_1] \cap N[D_2]| = 1$. Thus, this common vertex is the only vertex not in D dominated by two vertices of D. □

Theorem 3.3. *Let $k \geq 0$ be an integer. Then $PD_\gamma(C_{4k+3}) \cong C_{4k+3}$.*

Proof. For convenience, we omit the modulo $4k+3$ in the subscript of each vertex. For example, we write v_{x+1} instead of $v_{(x+1) \pmod{4k+3}}$. For each $x \in \{0, 1, 2, \ldots, 4k+2\}$, let

$$D_x = \{v_{x+4i+1}, v_{x+4i+2} : 0 \leq i \leq k\}$$

as shown in Figure 8, where D_x contains the black vertices. It is easy to check that D_x is a $\gamma_{pr}(C_{4k+3})$-set such that $v_x \notin D_x$ is the only vertex dominated by two vertices of D_x. Hence, $D_0, D_1, D_2, \ldots, D_{4k+2}$ are all distinct. Similarly, we omit the modulo $4k+3$ in the subscript of each $\gamma_{pr}(C_{4k+3})$-set.

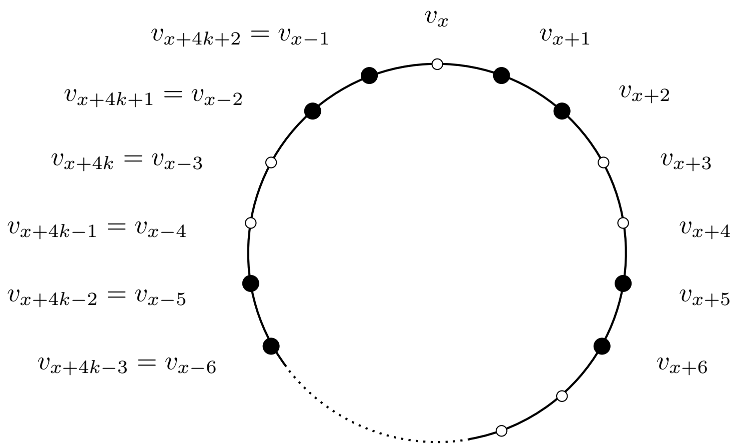

Fig. 8. The $\gamma_{pr}(C_{4k+3})$-set D_x

Claim that $D_0, D_1, D_2, \ldots, D_{4k+2}$ are the only $\gamma_{pr}(C_{4k+3})$-sets. Let D be any $\gamma_{pr}(C_{4k+3})$-set. By Lemma 3.2, there is a unique vertex $v_x \notin D$ for some $x \in \{0, 1, 2, \ldots, 4k+2\}$, dominated by two vertices of D, so $D = D_x$.

Let $x \in \{0, 1, 2, \ldots, 4k+2\}$. To find all neighbors of D_x in $PD_\gamma(C_{4k+3})$, we can only replace v_{x+1} by v_{x+3}, or v_{x-1} by v_{x-3} since v_x is the only vertex dominated by v_{x+1} and v_{x-1} of D_x. Thus, $(D_x \setminus \{v_{x+1}\}) \cup \{v_{x+3}\}$ and $(D_x \setminus \{v_{x-1}\}) \cup \{v_{x-3}\}$ are the only two neighbors of D_x in $PD_\gamma(C_{4k+3})$. Note that

$$(D_x \setminus \{v_{x+1}\}) \cup \{v_{x+3}\} = D_{x+4}$$

since v_{x+4} is the only vertex dominated by two dominating vertices. Similarly,

$$(D_x \setminus \{v_{x-1}\}) \cup \{v_{x-3}\} = D_{x-4}.$$

Therefore, $D_0,\ D_4, \ldots, D_{4k-4},\ D_{4k},\ D_1,\ D_5, \ldots, D_{4k-3}, D_{4k+1}, D_2, D_6, \ldots,\ D_{4k-2}$, $D_{4k+2},\ D_3,\ D_7, \ldots, D_{4k-1}, D_0$ form a cycle with $4k+3$ vertices. This completes the proof. $\square$

Before we determine the γ-paired dominating graph of a cycle with $4k+2$ vertices, we define some notations and a new graph called a loopgrid.

For a nonnegative integer i, let $P_n(v_i : v_{i+n-1}) : v_i v_{i+1} v_{i+2} \cdots v_{i+n-1}$ be a path with n vertices.

For any positive integer k, let $G_1 : u_1 u_2 u_3 \cdots u_{2k-1}$ and $G_2 : v_1 v_2 v_3 \cdots v_{3k-1}$ be two paths with $2k-1$ and $3k-1$ vertices, respectively. We define a *loopgrid* of size k, denoted by LG_k, as the graph with the vertex set

$$V(LG_k) = \{(u_x, v_y) \in V(G_1 \Box G_2) : 1 \le x \le 2k-1, 1 \le y \le 3k-1, 0 \le y - x \le k\},$$

and the edge set

$$E(LG_k) = E(G_1 \square G_2) \cup \{(u_1, v_y)(u_{2k-1}, v_{y+2k-1}) : 1 \leq y \leq k\}.$$

For example, Figure 9 illustrates the loopgrids LG_1 and LG_2, where we use (x, y) as (u_x, v_y).

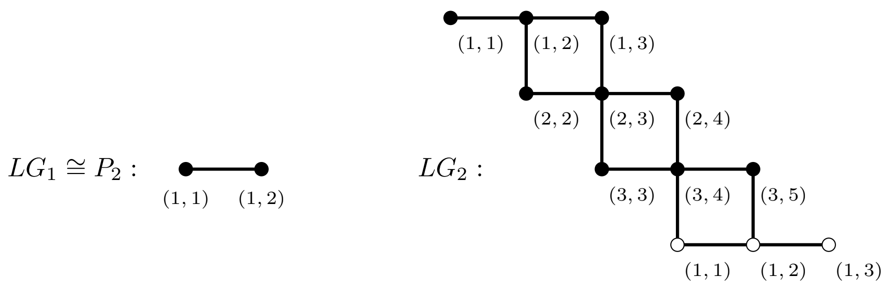

Fig. 9. The loopgrids LG_1 and LG_2, respectively

Lemma 3.4. *Let $k \geq 2$ be an integer.*

(1) *Each $\gamma_{pr}(C_{4k+2})$-set cannot contain any six or more consecutive vertices.*
(2) *For any fixed four consecutive vertices in C_{4k+2}, there is exactly one $\gamma_{pr}(C_{4k+2})$-set containing them.*

Proof. We prove the first claim by contradiction. Suppose that there is a $\gamma_{pr}(C_{4k+2})$-set D containing l consecutive vertices of C_{4k+2}, where $l \geq 6$ is an integer. Then these l vertices dominate $l + 2$ vertices in C_{4k+2}. Since $\gamma_{pr}(C_{4k+2}) = 2k + 2$, the other $2k + 2 - l$ vertices of D must dominate at least $4k + 2 - (l + 2) = 4k - l$ vertices in C_{4k+2}. We consider them as a path with $4k - l$ vertices. Note that the $2k + 2 - l$ remaining vertices of D can dominate at most $4k + 4 - 2l < 4k - l$ vertices in this path since $l \geq 6$. Thus, D cannot dominate all vertices in C_{4k+2}, a contradiction.

For the second claim, without loss of generality, we assume the four vertices are v_1, v_2, v_3, v_4. We find all $\gamma_{pr}(C_{4k+2})$-sets containing them. By the first claim, all such $\gamma_{pr}(C_{4k+2})$-sets cannot contain v_0 and v_5. The vertices v_1, v_2, v_3, v_4 dominate six vertices in C_{4k+2}. Note that $\gamma_{pr}(C_{4k+2}) = 2k + 2$, so the other $2k - 2$ vertices must dominate all vertices in $P_{4k-4}(v_6 : v_{4k+1})$. Since $\gamma_{pr}(P_{4k-4}(v_6 : v_{4k+1})) = 2k - 2$, these $2k - 2$ vertices form a $\gamma_{pr}(P_{4k-4}(v_6 : v_{4k+1}))$-set. Thus, each $\gamma_{pr}(C_{4k+2})$-set containing v_1, v_2, v_3, v_4 is a union of a $\gamma_{pr}(P_{4k-4}(v_6 : v_{4k+1}))$-set and $\{v_1, v_2, v_3, v_4\}$. By Theorem 2.5, there is a unique $\gamma_{pr}(P_{4k-4}(v_6 : v_{4k+1}))$-set. The claim follows. □

Theorem 3.5. *Let $k \geq 1$ be an integer. Then*

$$PD_\gamma(C_{4k+2}) \cong \begin{cases} C_3 \square C_3 & \text{if } k = 1, \\ LG_{k+1} & \text{if } k \geq 2. \end{cases}$$

Proof. Figure 10 shows that $PD_\gamma(C_6) \cong C_3 \square C_3$.

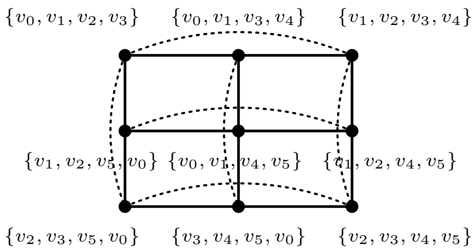

Fig. 10. The γ-paired dominating graph of C_6

Let $k \geq 2$. Since each $\gamma_{pr}(C_{4k+2})$-set must dominate the vertex v_0, we get it contains either the pair $\{v_{4k}, v_{4k+1}\}$, $\{v_{4k+1}, v_0\}$, $\{v_0, v_1\}$, or $\{v_1, v_2\}$. We first find all $\gamma_{pr}(C_{4k+2})$-sets containing the pair $\{v_{4k}, v_{4k+1}\}$. By Lemma 3.4(1), such a $\gamma_{pr}(C_{4k+2})$-set must satisfy one of the following:

(i) it contains the pair $\{v_{4k}, v_{4k+1}\}$ but not v_{4k-1}, v_0,
(ii) it contains the pairs $\{v_{4k-2}, v_{4k-1}\}$ and $\{v_{4k}, v_{4k+1}\}$,
(iii) it contains the pairs $\{v_{4k}, v_{4k+1}\}$ and $\{v_0, v_1\}$.

Note that each $\gamma_{pr}(C_{4k+2})$-set containing the pair $\{v_{4k}, v_{4k+1}\}$ but not v_{4k-1}, v_0 is a union of a $\gamma_{pr}(P_{4k-2}(v_1 : v_{4k-2}))$-set and $\{v_{4k}, v_{4k+1}\}$. By Theorem 2.7, we have

$$PD_\gamma(P_{4k-2}(v_1 : v_{4k-2})) \cong SG_{k,k}.$$

For all $x, y \in \{1, 2, \ldots, k\}$ with $x - y \leq 1$, let $A^{(1)}_{x,y}$ be the $\gamma_{pr}(P_{4k-2}(v_1 : v_{4k-2}))$-set at the position (x, y) in this stepgrid $SG_{k,k}$, and let

$$D^{(1)}_{x,y} = A^{(1)}_{x,y} \cup \{v_{4k}, v_{4k+1}\}.$$

Thus, $D^{(1)}_{x,y}$'s are the only $\gamma_{pr}(C_{4k+2})$-sets containing the pair $\{v_{4k}, v_{4k+1}\}$, but not v_{4k-1}, v_0, and they form a stepgrid $SG_{k,k}$ in $PD_\gamma(C_{4k+2})$. By Lemma 2.3, without loss of generality, we assume that $A_{x,k}$ contains the pair $\{v_{4k-3}, v_{4k-2}\}$ for each $x \in \{1, 2, \ldots, k\}$. By Corollary 2.8 (A1.1), we have $A^{(1)}_{k,k}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}, \{v_{4k-3}, v_{4k-2}\}$. Let

$$D^{(1)}_{k+1,k} = (D^{(1)}_{k,k} \setminus \{v_{4k-3}\}) \cup \{v_{4k-1}\}.$$

By Lemma 3.4(2), the set $D^{(1)}_{k+1,k}$ is the only $\gamma_{pr}(C_{4k+2})$-set containing the pairs $\{v_{4k-2}, v_{4k-1}\}$ and $\{v_{4k}, v_{4k+1}\}$. By Corollary 2.8 (A2.1), we get $A^{(1)}_{1,1}$ contains the pairs $\{v_1, v_2\}, \{v_4, v_5\}$. Let

$$D^{(1)}_{1,0} = (D^{(1)}_{1,1} \setminus \{v_2\}) \cup \{v_0\}.$$

By Lemma 3.4(2), the set $D^{(1)}_{1,0}$ is the only $\gamma_{pr}(C_{4k+2})$-set containing the pairs $\{v_{4k}, v_{4k+1}\}$ and $\{v_0, v_1\}$. Therefore, all $D^{(1)}_{x,y}$'s form the graph, named $D^{(1)}$, in $PD_\gamma(C_{4k+2})$ as shown in Figure 11.

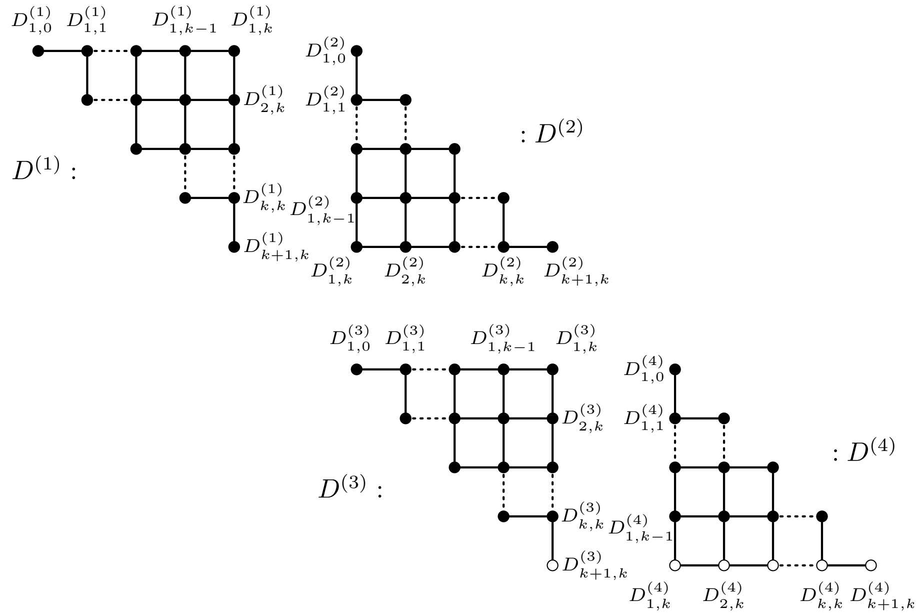

Fig. 11. The graphs $D^{(1)}$, $D^{(2)}$, $D^{(3)}$, and $D^{(4)}$ in $PD_\gamma(C_{4k+2})$

Similarly, we can construct all $\gamma_{pr}(C_{4k+2})$-sets as follows (the subscripts of all vertices are modulo $4k+2$): for all $x, y \in \{1, 2, \ldots, k\}$ with $x - y \leq 0$, and for each $i \in \{1, 2, 3, 4\}$,

$$D^{(i)}_{x,y} = A^{(i)}_{x,y} \cup \{v_{4k-1+i}, v_{4k+i}\}, \text{ where } A^{(i)}_{x,y} \text{ is a } \gamma_{pr}(P_{4k-2}(v_i : v_{4k-3+i}))\text{-set},$$

$$D^{(i)}_{k+1,k} = (D^{(i)}_{k,k} \setminus \{v_{4k-4+i}\}) \cup \{v_{4k-2+i}\},$$

and

$$D^{(i)}_{1,0} = (D^{(i)}_{1,1} \setminus \{v_{i+1}\}) \cup \{v_{i-1}\}.$$

These $D^{(i)}_{x,y}$'s are the only $\gamma_{pr}(C_{4k+2})$-sets containing the pair $\{v_{4k-1+i}, v_{4k+i}\}$, and they form the graph $D^{(i)}$ in $PD_\gamma(C_{4k+2})$ (see Figure 11). By Lemma 2.3, without loss of generality, we assume $A^{(i)}_{x,k}$ contains the pair $\{v_{4k-4+i}, v_{4k-3+i}\}$. For all $x, y \in \{1, 2, \ldots, k\}$ with $x - y \leq 1$, we get the following properties.

(A′1) If $y = k$, then $D^{(i)}_{x,y}$ contains the pairs $\{v_{4k-4+i}, v_{4k-3+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$; otherwise, it contains the pairs $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$.

(A′1.1) for all $x \in \{1, 2, \ldots, k-1\}$, $D^{(i)}_{x,k}$ contains the pairs $\{v_{4k-8+i}, v_{4k-7+i}\}$, $\{v_{4k-4+i}, v_{4k-3+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$, and $D^{(i)}_{k,k}$ contains the pairs $\{v_{4k-7+i}, v_{4k-6+i}\}$, $\{v_{4k-4+i}, v_{4k-3+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$.

(A′2) If $x = 1$, then $D^{(i)}_{x,y}$ contains the pairs $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_i, v_{i+1}\}$; otherwise, it contains the pairs $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_{i+1}, v_{i+2}\}$.

(A′2.1) $D^{(i)}_{1,1}$ contains the pairs $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_i, v_{i+1}\}$, $\{v_{i+3}, v_{i+4}\}$, and $D^{(i)}_{1,y}$ contains the pairs $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_i, v_{i+1}\}$, $\{v_{i+4}, v_{i+5}\}$ for all $y \in \{2, 3, \ldots, k\}$.

(A′3) $D^{(i)}_{k+1,k}$ is the only $\gamma_{pr}(C_{4k+2})$-set in $D^{(i)}$ containing the pairs $\{v_{4k-7+i}, v_{4k-6+i}\}$, $\{v_{4k-3+i}, v_{4k-2+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_{i+1}, v_{i+2}\}$.

(A′4) $D^{(i)}_{1,0}$ is the only $\gamma_{pr}(C_{4k+2})$-set in $D^{(i)}$ containing the pairs $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_{i-1}, v_i\}$, $\{v_{i+3}, v_{i+4}\}$.

Note that $D^{(1)}$ and $D^{(2)}$ cannot have any common vertices in $PD_\gamma(C_{4k+2})$ since otherwise there is a $\gamma_{pr}(C_{4k+2})$-set containing the pairs $\{v_{4k}, v_{4k+1}\}$ and $\{v_{4k+1}, v_0\}$, which is impossible. Similarly, $D^{(i)}$ and $D^{(i+1)}$ do not share any vertices in $PD_\gamma(C_{4k+2})$ for all $i \in \{2, 3\}$.

We then consider all $\gamma_{pr}(C_{4k+2})$-sets that are in both $D^{(1)}$ and $D^{(3)}$. Then these sets must contain the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_0, v_1\}$. By (A′4) and (A′3), $D^{(1)}_{1,0}$ and $D^{(3)}_{k+1,k}$ are the only $\gamma_{pr}(C_{4k+2})$-sets in $D^{(1)}$ and $D^{(3)}$, respectively, containing the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_0, v_1\}$. By Lemma 3.4(2), we get $D^{(1)}_{1,0} = D^{(3)}_{k+1,k}$. Similarly, $D^{(2)}_{1,0} = D^{(4)}_{k+1,k}$ is the only $\gamma_{pr}(C_{4k+2})$-set that is in both $D^{(2)}$ and $D^{(4)}$.

We next consider all $\gamma_{pr}(C_{4k+2})$-sets, which are in both $D^{(1)}$ and $D^{(4)}$. These sets must contain the pairs $\{v_{4k}, v_{4k+1}\}, \{v_1, v_2\}$. By (A′2), $D^{(1)}_{1,1}, D^{(1)}_{1,2}, \ldots, D^{(1)}_{1,k}$ are the only $\gamma_{pr}(C_{4k+2})$-sets in $D^{(1)}$ containing the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, and they form a path with k vertices. Then they also form a path in $D^{(4)}$. By (A′1), $D^{(4)}_{1,k}, D^{(4)}_{2,k}, \ldots, D^{(4)}_{k,k}$ are the only $\gamma_{pr}(C_{4k+2})$-sets in $D^{(4)}$ containing the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, and they form a path with k vertices. To show that $D^{(1)}_{1,y} = D^{(4)}_{y,k}$ for each $y \in \{1, 2, \ldots, k\}$, it suffices to show that $D^{(1)}_{1,1} = D^{(4)}_{1,k}$. By (A′2.1), $D^{(1)}_{1,1}$ contains the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, $\{v_4, v_5\}$. By (A′1) and (A′2), $D^{(4)}_{1,k}$ is the only $\gamma_{pr}(C_{4k+2})$-set in $D^{(4)}$ containing these three pairs, and hence, $D^{(1)}_{1,1} = D^{(4)}_{1,k}$.

Next, we consider all edges between a set in $D^{(1)}$ and a set in $D^{(2)}$. We first show that $D^{(1)}_{1,0}$ has no neighbors in $D^{(2)}$. By (A′4), $D^{(1)}_{1,0}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_{4k+1}\}$, $\{v_0, v_1\}$, $\{v_4, v_5\}$. Since each set in $D^{(2)}$ contains the pair $\{v_{4k+1}, v_0\}$, the set $D^{(1)}_{1,0}$ is adjacent to some set in $D^{(2)}$ if and only if $(D^{(1)}_{1,0} \setminus \{v_{4k}\}) \cup \{v_2\}$ or $(D^{(1)}_{1,0} \setminus \{v_1\}) \cup \{v_{4k-1}\}$ is a $\gamma_{pr}(C_{4k+2})$-set. It is easy to check that $D^{(1)}_{1,0}$ is not adjacent to any sets in $D^{(2)}$. By (A′3),

$D^{(1)}_{k+1,k}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_{4k+1}\}$, $\{v_2, v_3\}$, so $D^{(1)}_{k+1,k}$ is adjacent to some set in $D^{(2)}$ if and only if $(D^{(1)}_{k+1,k} \setminus \{v_{4k}\}) \cup \{v_0\}$ or $(D^{(1)}_{k+1,k} \setminus \{v_{4k-2}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+2})$-set. We have $(D^{(1)}_{k+1,k} \setminus \{v_{4k}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+2})$-set, but $(D^{(1)}_{k+1,k} \setminus \{v_{4k-2}\}) \cup \{v_0\}$ is not. We show that $D^{(1)}_{k+1,k}$ and $D^{(2)}_{1,k}$ are adjacent, i.e., $(D^{(2)}_{1,k} \setminus \{v_0\}) \cup \{v_{4k}\} = D^{(1)}_{k+1,k}$. By (A′1) and (A′2), $D^{(2)}_{1,k}$ contains the pairs $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k+1}, v_0\}$, $\{v_2, v_3\}$, so $(D^{(2)}_{1,k} \setminus \{v_0\}) \cup \{v_{4k}\}$ is a $\gamma_{pr}(C_{4k+2})$-set containing the pairs $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_{4k+1}\}$. Since $D^{(1)}_{k+1,k}$ also contains these two pairs, $(D^{(2)}_{1,k} \setminus \{v_0\}) \cup \{v_{4k}\} = D^{(1)}_{k+1,k}$ by Lemma 3.4(2).

We next find all neighbors in $D^{(2)}$ of the other $\gamma_{pr}(C_{4k+2})$-sets in $D^{(1)}$. We show that $D^{(1)}_{x,k}$ is adjacent to $D^{(2)}_{1,x-1}$ for all $x \in \{1, 2, \ldots, k\}$. Recall that for all $x, y \in \{1, 2, \ldots, k\}$ with $x-y \le 1$, $D^{(1)}_{x,y}$ contains the pair $\{v_{4k}, v_{4k+1}\}$ but not v_{4k-1}, v_0. Note that $D^{(1)}_{x,y}$ is adjacent to some set in $D^{(2)}$ if and only if $(D^{(1)}_{x,y} \setminus \{v_{4k}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+2})$-set. By (A′1), if $y \ne k$, then $D^{(1)}_{x,y}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_{4k+1}\}$, so $(D^{(1)}_{x,y} \setminus \{v_{4k}\}) \cup \{v_0\}$ is not a dominating set. By (A′1) and (A′2), $D^{(1)}_{1,k}$ contains the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, and $D^{(1)}_{2,k}, D^{(1)}_{3,k}, \ldots, D^{(1)}_{k,k}$ contain the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_{4k+1}\}$, $\{v_2, v_3\}$. For each $x \in \{1, 2, \ldots, k\}$, let $D_x = (D^{(1)}_{x,k} \setminus \{v_{4k}\}) \cup \{v_0\}$, so D_x is a $\gamma_{pr}(C_{4k+2})$-set, and these D_x's form a path with k vertices in $D^{(2)}$. Note that D_1 contains the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k+1}, v_0\}$, $\{v_1, v_2\}$, and $D_2, D_3, \ldots, D_k$ contain the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k+1}, v_0\}$, $\{v_2, v_3\}$. By (A′4), $D^{(2)}_{1,0}$ is the only $\gamma_{pr}(C_{4k+2})$-set in $D^{(2)}$ containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k+1}, v_0\}$, $\{v_1, v_2\}$, and by (A′1), (A′2), $D^{(2)}_{1,1}, D^{(2)}_{1,2}, \ldots, D^{(2)}_{1,k-1}$ are the only $\gamma_{pr}(C_{4k+2})$-sets in $D^{(2)}$ containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k+1}, v_0\}$, $\{v_2, v_3\}$, and they also form a path with k vertices in $D^{(2)}$. Then we can conclude that for all $x \in \{1, 2, \ldots, k\}$, $D_x = D^{(2)}_{1,x-1}$, implying that $D^{(1)}_{x,k}$ is adjacent to $D^{(2)}_{1,x-1}$. To sum up, $D^{(1)}_{x,k}$ is adjacent to $D^{(2)}_{1,x-1}$ for all $x \in \{1, 2, \ldots, k+1\}$. Similarly, for all $i \in \{2, 3\}$, we get $D^{(i)}_{x,k}$ is adjacent to $D^{(i+1)}_{1,x-1}$ for all $x \in \{1, 2, \ldots, k+1\}$.

Now, all $\gamma_{pr}(C_{4k+2})$-sets and edges form a loopgrid LG_{k+1} in $PD_\gamma(C_{4k+2})$. Then we only need to show that there is no more edge in $PD_\gamma(C_{4k+2})$. We first consider all edges between a set in

$$\widehat{D}^{(1)} = D^{(1)} - D^{(1)}_{1,0}$$

and a set in

$$\widehat{D}^{(3)} = D^{(3)} - D^{(3)}_{k+1,k}$$

since $D^{(1)}_{1,0} = D^{(3)}_{k+1,k}$. Note that each set in $\widehat{D}^{(1)}$ contains either the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, or the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_2, v_3\}$ while every set in $\widehat{D}^{(3)}$ contains the pair $\{v_0, v_1\}$ but not $\{v_{4k}, v_{4k+1}\}$. Hence, there is no edge between a set in $\widehat{D}^{(1)}$ and a set in $\widehat{D}^{(3)}$. Similarly, there is no edge between a set in $D^{(2)} - D^{(2)}_{1,0}$ and a set in

$D^{(4)} - D^{(4)}_{k+1,k}$. Recall that $D^{(1)}_{1,y} = D^{(4)}_{y,k}$ for all $y \in \{1, 2, \ldots, k\}$. Also, $D^{(1)}_{1,0} = D^{(3)}_{k+1,k}$, which has a neighbor in $D^{(4)}$. Thus, we consider all edges between a set in

$$\widetilde{D}^{(1)} = D^{(1)} - \{D^{(1)}_{1,y} : 0 \le y \le k\}$$

and a set in

$$\widetilde{D}^{(4)} = D^{(4)} - \{D^{(4)}_{y,k} : 1 \le y \le k\}.$$

Note that each set in $\widetilde{D}^{(1)}$ contains the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_2, v_3\}$ while each set in $\widetilde{D}^{(4)}$ contains the pair $\{v_1, v_2\}$ but not $\{v_{4k}, v_{4k+1}\}$. Hence, there is no edge between a set in $\widetilde{D}^{(1)}$ and a set in $\widetilde{D}^{(4)}$. This completes the proof. □

For any positive integer k, let $G_1 : u_1u_2u_3 \cdots u_{2k}$, $G_2 : v_1v_2v_3 \cdots v_{2k}$, and $G_3 : w_1w_2w_3 \cdots w_{2k+1}$ be three paths with $2k$, $2k$, and $2k+1$ vertices, respectively. We define a *loopbox* of size k, denoted by LB_k, as the graph with the vertex set

$$\begin{aligned} V(LB_k) =& \{(u_x, v_y, w_z) \in V(G_1 \Box G_2 \Box G_3) : 1 \le x, y \le 2k, 1 \le z \le 2k+1, \\ & 0 \le y - x \le k, -1 \le y - z \le k-1, 0 \le z - x \le k\}, \end{aligned}$$

and the edge set

$$\begin{aligned} E(LB_k) = E(G_1 \Box G_2 \Box G_3) &\cup \{(u_x, v_{x+k-1}, w_x)(u_x, v_{x+k}, w_{x+1}) : 1 \le x \le k\} \\ &\cup \{(u_1, v_1, w_1)(u_{k+1}, v_{2k}, w_{k+1})\} \\ &\cup \{(u_x, v_x, w_{x+1})(u_{x+1}, v_{x+1}, w_{x+1}) : 1 \le x \le 2k-1\} \\ &\cup \{(u_1, v_k, w_{k+1})(u_{2k}, v_{2k}, w_{2k+1})\} \\ &\cup \{(u_x, v_{x+k}, w_{x+k})(u_{x+1}, v_{x+k}, w_{x+k+1}) : 1 \le x \le k\} \\ &\cup \{(u_1, v_y, w_z)(u_{z+k}, v_{2k}, w_{y+k+1}) : 1 \le y, z \le k, -1 \le y - z \le k-1\}. \end{aligned}$$

For example, the loopboxes of size 1, 2, and 3 are shown in Figures 12, 13, and 14, respectively, where we write (x, y, z) as (u_x, v_y, w_z).

Lemma 3.6. *Let $k \ge 2$ be an integer.*

(1) *Each $\gamma_{pr}(C_{4k+1})$-set cannot contain any six or more consecutive vertices.*
(2) *For any fixed four consecutive vertices in C_{4k+1}, there are k $\gamma_{pr}(C_{4k+1})$-sets containing them, and each set is a union of a $\gamma_{pr}(P_{4k-5})$-set and a set of these four vertices.*

Proof. Similar to 3.4(1), we can easily prove the first claim. Next, without loss of generality, we assume the four vertices are v_1, v_2, v_3, v_4. Then these four vertices dominate six vertices in C_{4k+1}. Note that $\gamma_{pr}(C_{4k+1}) = 2k+2$, so the other $2k-2$ vertices must dominate all vertices in $P_{4k-5}(v_6 : v_{4k})$. Since

$$\gamma_{pr}(P_{4k-5}(v_6 : v_{4k})) = 2k - 2,$$

these $2k-2$ vertices form a $\gamma_{pr}(P_{4k-5}(v_6 : v_{4k}))$-set. Hence, each such $\gamma_{pr}(C_{4k+1})$-set is a union of a $\gamma_{pr}(P_{4k-5}(v_6 : v_{4k}))$-set and $\{v_1, v_2, v_3, v_4\}$. By Theorem 2.6, there are k $\gamma_{pr}(P_{4k-5}(v_6 : v_{4k}))$-sets, so the claim follows. □

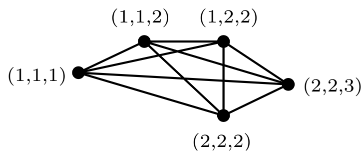

Fig. 12. The loopbox of size 1

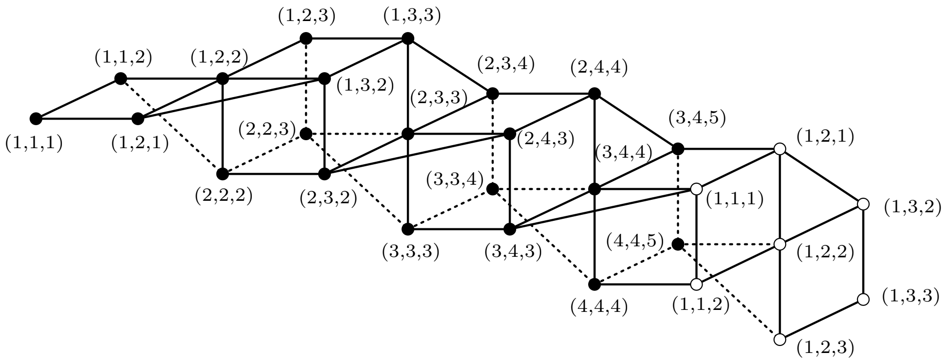

Fig. 13. The loopbox of size 2

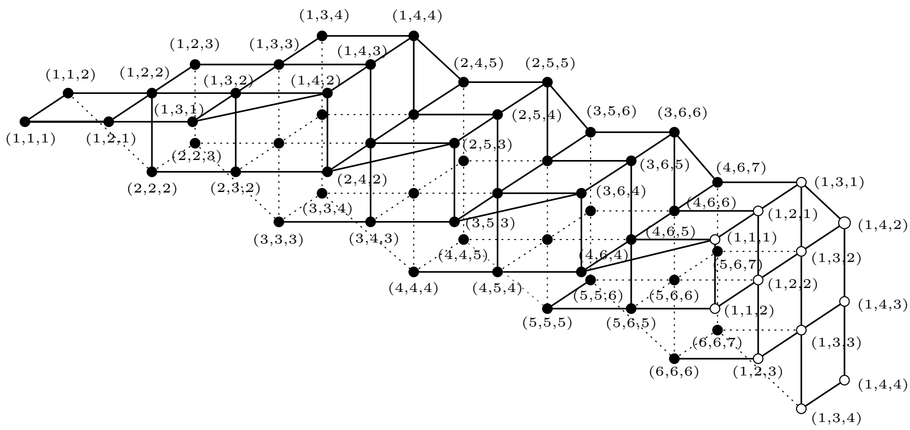

Fig. 14. The loopbox of size 3

Theorem 3.7. *Let $k \geq 1$ be an integer. Then $PD_\gamma(C_{4k+1}) \cong LB_k$.*

Proof. Figure 1 shows that

$$PD_\gamma(C_5) \cong K_5 \cong LB_1.$$

For $k = 2$, we have

$$PD_\gamma(C_9) \cong LB_2$$

(see Figure 13), where

$$\begin{aligned}
&(1,1,1) = \{v_0, v_1, v_2, v_3, v_5, v_6\}, \\
&(1,2,1) = \{v_0, v_1, v_2, v_3, v_6, v_7\}, \quad (1,1,2) = \{v_0, v_1, v_3, v_4, v_5, v_6\}, \\
&(1,2,2) = \{v_0, v_1, v_3, v_4, v_6, v_7\}, \quad (1,2,3) = \{v_0, v_1, v_3, v_4, v_7, v_8\}, \\
&(2,2,2) = \{v_0, v_1, v_4, v_5, v_6, v_7\}, \quad (2,2,3) = \{v_0, v_1, v_4, v_5, v_7, v_8\}, \\
&(1,3,2) = \{v_1, v_2, v_3, v_4, v_6, v_7\}, \quad (1,3,3) = \{v_1, v_2, v_3, v_4, v_7, v_8\}, \\
&(2,3,2) = \{v_1, v_2, v_4, v_5, v_6, v_7\}, \quad (2,3,3) = \{v_1, v_2, v_4, v_5, v_7, v_8\}, \\
&(3,3,3) = \{v_1, v_2, v_4, v_5, v_8, v_0\}, \quad (2,4,3) = \{v_1, v_2, v_5, v_6, v_7, v_8\}, \\
&(3,4,3) = \{v_1, v_2, v_5, v_6, v_8, v_0\}, \quad (2,3,4) = \{v_2, v_3, v_4, v_5, v_7, v_8\}, \\
&(3,3,4) = \{v_2, v_3, v_4, v_5, v_8, v_0\}, \quad (2,4,4) = \{v_2, v_3, v_5, v_6, v_7, v_8\}, \\
&(3,4,4) = \{v_2, v_3, v_5, v_6, v_8, v_0\}, \quad (3,4,5) = \{v_2, v_3, v_6, v_7, v_8, v_0\}, \\
&(4,4,4) = \{v_3, v_4, v_5, v_6, v_8, v_0\}, \quad (4,4,5) = \{v_3, v_4, v_6, v_7, v_8, v_0\}.
\end{aligned}$$

Let $k \geq 3$. Since each $\gamma_{pr}(C_{4k+1})$-set must dominate the vertex v_0, it contains either the pair $\{v_{4k-1}, v_{4k}\}$, $\{v_{4k}, v_0\}$, $\{v_0, v_1\}$, or $\{v_1, v_2\}$. We first find all $\gamma_{pr}(C_{4k+1})$-sets containing the pair $\{v_{4k-1}, v_{4k}\}$. By Lemma 3.6(1), such a $\gamma_{pr}(C_{4k+1})$-set must satisfy one of the following:

(i) it contains the pair $\{v_{4k-1}, v_{4k}\}$ but not v_{4k-2}, v_0,
(ii) it contains the pairs $\{v_{4k-3}, v_{4k-2}\}$ and $\{v_{4k-1}, v_{4k}\}$,
(iii) it contains the pairs $\{v_{4k-1}, v_{4k}\}$ and $\{v_0, v_1\}$.

Note that each $\gamma_{pr}(C_{4k+1})$-set containing the pair $\{v_{4k-1}, v_{4k}\}$ but not v_{4k-2}, v_0 is a union of a $\gamma_{pr}(P_{4k-3}(v_1 : v_{4k-3}))$-set and $\{v_{4k-1}, v_{4k}\}$. By Theorem 2.9, we have

$$PD_\gamma(P_{4k-3}(v_1 : v_{4k-3})) \cong SG_{k,k,k-1}.$$

For all $x, y \in \{1, 2, \ldots, k\}$ and $z \in \{1, 2, \ldots, k-1\}$ with $x - y \leq 0$, $x - z \leq 1$, $y - z \geq 0$, let $B^{(1)}_{x,y,z}$ be the $\gamma_{pr}(P_{4k-3}(v_1 : v_{4k-3}))$-set at the position (x, y, z) in $SG_{k,k,k-1}$, and let

$$D^{(1)}_{x,y,z} = B^{(1)}_{x,y,z} \cup \{v_{4k-1}, v_{4k}\}.$$

Thus $D^{(1)}_{x,y,z}$'s are the only $\gamma_{pr}(C_{4k+1})$-sets containing the pair $\{v_{4k-1}, v_{4k}\}$ but not v_{4k-2}, v_0, and they also form a stepgrid $SG_{k,k,k-1}$ in $PD_\gamma(C_{4k+1})$. By Lemma 2.4, without loss of generality, we may assume that $B^{(1)}_{x,k,z}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$.

By Corollary 2.10 (B1.1), the set $B^{(1)}_{x,k,k-1}$ contains the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-4}, v_{4k-3}\}$ for all $x \in \{1, 2, \dots, k-1\}$, and $B^{(1)}_{k,k,k-1}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-4}, v_{4k-3}\}$. For each $x \in \{1, 2, \dots, k\}$, let

$$D^{(1)}_{x,k,k} = (D^{(1)}_{x,k,k-1} \setminus \{v_{4k-4}\}) \cup \{v_{4k-2}\}.$$

By Lemma 3.6(2), these $D^{(1)}_{x,k,k}$'s are the only $\gamma_{pr}(C_{4k+1})$-sets containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$. By Corollary 2.10 (B2.1), the set $B^{(1)}_{1,1,1}$ contains the pairs $\{v_1, v_2\}$, $\{v_3, v_4\}$, and $B^{(1)}_{1,y,1}$ contains the pairs $\{v_1, v_2\}$, $\{v_4, v_5\}$ for all $y \in \{2, 3, \dots, k\}$. For each $y \in \{1, 2, \dots, k\}$, let

$$D^{(1)}_{1,y,0} = (D^{(1)}_{1,y,1} \setminus \{v_2\}) \cup \{v_0\}.$$

By Lemma 3.6(2), these $D^{(1)}_{1,y,0}$'s are the only $\gamma_{pr}(C_{4k+1})$-sets containing the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$. Therefore, all $D^{(1)}_{x,y,z}$'s form the graph, named $D^{(1)}$, in $PD_\gamma(C_{4k+1})$ as shown in Figure 15.

Similarly, we can construct all $\gamma_{pr}(C_{4k+1})$-sets as follows (the subscripts of all vertices are modulo $4k+1$): for all $x, y \in \{1, 2, \dots, k\}$ and $z \in \{1, 2, \dots, k-1\}$ with $x - y \le 0$, $x - z \le 1$, $y - z \ge 0$, and for each $i \in \{1, 2, 3, 4\}$,

$$D^{(i)}_{x,y,z} = B^{(i)}_{x,y,z} \cup \{v_{4k-2+i}, v_{4k-1+i}\},$$

where $B^{(i)}_{x,y,z}$ is a $\gamma_{pr}(P_{4k-3}(v_i : v_{4k-4+i}))$-set,

$$D^{(i)}_{x,k,k} = (D^{(i)}_{x,k,k-1} \setminus \{v_{4k-5+i}\}) \cup \{v_{4k-3+i}\},$$
$$D^{(i)}_{1,y,0} = (D^{(i)}_{1,y,1} \setminus \{v_{i+1}\}) \cup \{v_{i-1}\}.$$

We get that these $D^{(i)}_{x,y,z}$'s are the only $\gamma_{pr}(C_{4k+1})$-sets containing the pair $\{v_{4k-2+i}, v_{4k-1+i}\}$, and they form the graph $D^{(i)}$ (see Figure 15) in $PD_\gamma(C_{4k+1})$. By Lemma 2.4, without loss of generality, we may assume that $B^{(i)}_{x,k,z}$ contains the pair $\{v_{4k-5+i}, v_{4k-4+i}\}$, and then we get the following properties.

(B′1) Let $x \in \{1, 2, \dots, k\}$ and $z \in \{0, 1, \dots, k-1\}$. If $y = k$, then $D^{(i)}_{x,y,z}$ contains the pairs $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$; otherwise, it contains the pairs $\{v_{4k-6+i}, v_{4k-5+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$.
- (B′1.1) $D^{(i)}_{x,k,k-1}$ contains the pairs $\{v_{4k-8+i}, v_{4k-7+i}\}$, $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$ for all $x \in \{1, 2, \dots, k-1\}$, and $D^{(i)}_{k,k,k-1}$ contains the pairs $\{v_{4k-7+i}, v_{4k-6+i}\}$, $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$.
- (B′1.2) if $z \neq k-1$, then $D^{(i)}_{x,k,z}$ contains the pairs $\{v_{4k-9+i}, v_{4k-8+i}\}$, $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$.

(B′2) Let $y \in \{1, 2, \dots, k\}$ and $z \in \{1, 2, \dots, k\}$. If $x = 1$, then $D^{(i)}_{x,y,z}$ contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_i, v_{i+1}\}$; otherwise, it contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_{i+1}, v_{i+2}\}$.

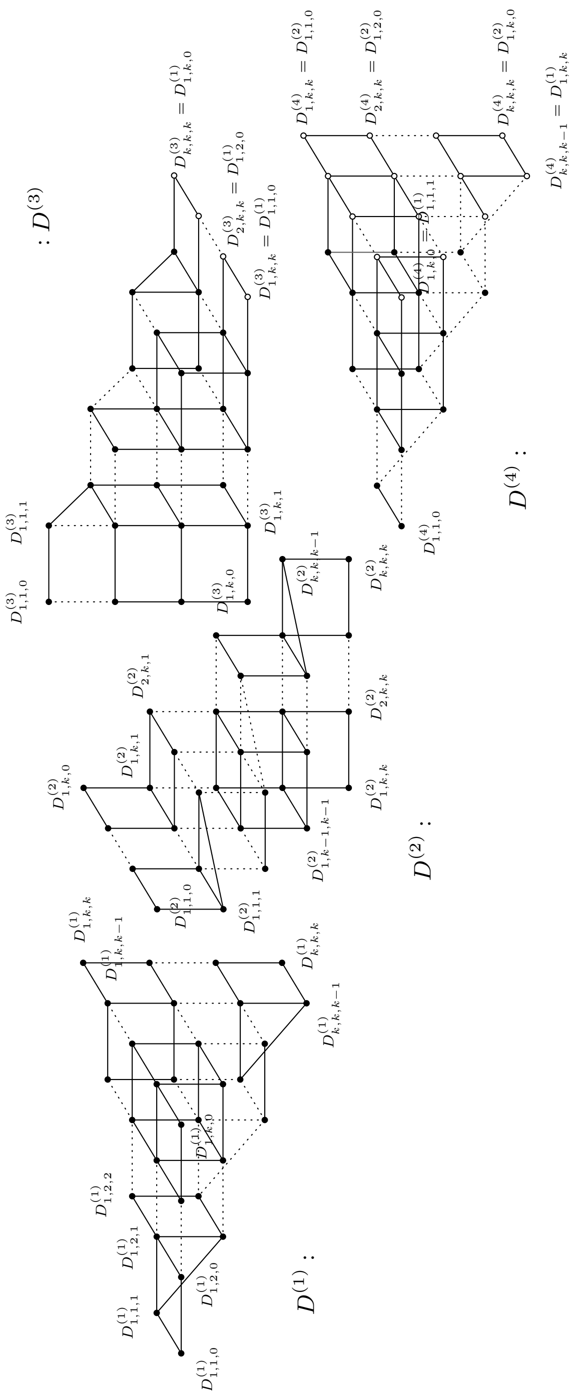

Fig. 15. The graphs $D^{(1)}$, $D^{(2)}$, $D^{(3)}$, and $D^{(4)}$ in $PD_\gamma(C_{4k+1})$

(B′2.1) $D^{(i)}_{1,1,1}$ contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_i, v_{i+1}\}$, $\{v_{i+2}, v_{i+3}\}$, $D^{(i)}_{1,y,1}$ contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_i, v_{i+1}\}$, $\{v_{i+3}, v_{i+4}\}$ for all $y \in \{2, 3, \ldots, k\}$.

(B′2.2) If $z \neq 1$, then $D^{(i)}_{1,y,z}$ contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_i, v_{i+1}\}$, $\{v_{i+4}, v_{i+5}\}$.

(B′3) $D^{(i)}_{1,k,k}, D^{(i)}_{2,k,k}, \ldots, D^{(i)}_{k,k,k}$ are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(i)}$ containing the pairs $\{v_{4k-4+i}, v_{4k-3+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$.

(B′3.1) $D^{(i)}_{1,k,k}$ contains the pair $\{v_i, v_{i+1}\}$, and the others contain the pair $\{v_{i+1}, v_{i+2}\}$.

(B′3.2) $D^{(i)}_{k,k,k}$ contains the pair $\{v_{4k-7+i}, v_{4k-6+i}\}$, and the others contain the pair $\{v_{4k-8+i}, v_{4k-7+i}\}$.

(B′4) $D^{(i)}_{1,1,0}, D^{(i)}_{1,2,0}, \ldots, D^{(i)}_{1,k,0}$ are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(i)}$ containing the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_{i-1}, v_i\}$.

(B′4.1) $D^{(i)}_{1,1,0}$ contains the pair $\{v_{i+2}, v_{i+3}\}$, and the others contain the pair $\{v_{i+3}, v_{i+4}\}$.

(B′4.2) $D^{(i)}_{1,k,0}$ contains the pair $\{v_{4k-5+i}, v_{4k-4+i}\}$, and the others contain the pair $\{v_{4k-6+i}, v_{4k-5+i}\}$.

Note that $D^{(1)}$ and $D^{(2)}$ cannot have any common vertices in $PD_\gamma(C_{4k+1})$ since otherwise there is a $\gamma_{pr}(C_{4k+1})$-set containing the pairs $\{v_{4k-1}, v_{4k}\}$ and $\{v_{4k}, v_0\}$, which is impossible. Similarly, $D^{(i)}$ and $D^{(i+1)}$ do not share any vertices in $PD_\gamma(C_{4k+1})$ for all $i \in \{2, 3\}$.

Then we consider all $\gamma_{pr}(C_{4k+1})$-sets that are in both $D^{(1)}$ and $D^{(3)}$. Then these sets must contain the pairs $\{v_{4k-1}, v_{4k}\}, \{v_0, v_1\}$. By (B′4) and (B′4.2), $D^{(1)}_{1,1,0}, D^{(1)}_{1,2,0}, \ldots, D^{(1)}_{1,k,0}$ are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(1)}$ containing the pairs $\{v_{4k-1}, v_{4k}\}, \{v_0, v_1\}$, and $D^{(1)}_{1,k,0}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$. By (B′3) and (B′3.2), $D^{(3)}_{1,k,k}, D^{(3)}_{2,k,k}, \ldots, D^{(3)}_{k,k,k}$ are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(3)}$ containing the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$, and $D^{(3)}_{k,k,k}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$. By Lemma 3.6(2), for each $y \in \{1, 2, \ldots, k\}$, we have

$$D^{(1)}_{1,y,0} = T^{(1)}_{1,y,0} \cup \{v_{4k-1}, v_{4k}, v_0, v_1\},$$

where $T^{(1)}_{1,y,0}$ is a $\gamma_{pr}(P_{4k-5}(v_3 : v_{4k-3}))$-set, and

$$D^{(3)}_{y,k,k} = T^{(3)}_{y,k,k} \cup \{v_{4k-1}, v_{4k}, v_0, v_1\},$$

where $T^{(3)}_{y,k,k}$ is a $\gamma_{pr}(P_{4k-5}(v_3 : v_{4k-3}))$-set. Since $D^{(1)}_{1,k,0}$ and $D^{(3)}_{k,k,k}$ contain the pair $\{v_{4k-4}, v_{4k-3}\}$, so do $T^{(1)}_{1,k,0}$ and $T^{(3)}_{k,k,k}$. By Lemma 2.2, $T^{(1)}_{1,k,0} = T^{(3)}_{k,k,k}$. By Theorem 2.6, for each $y \in \{1, 2, \ldots, k\}$, we have $T^{(1)}_{1,y,0} = T^{(3)}_{y,k,k}$, and hence $D^{(1)}_{1,y,0} = D^{(3)}_{y,k,k}$. Similarly, we get $D^{(2)}_{1,y,0} = D^{(4)}_{y,k,k}$ for all $y \in \{1, 2, \ldots, k\}$.

We next consider all $\gamma_{pr}(C_{4k+1})$-sets, which are in both $D^{(1)}$ and $D^{(4)}$. Then these sets must contain the pairs $\{v_{4k-1}, v_{4k}\}, \{v_1, v_2\}$. By (B′2), for all $y, z \in \{1, 2, \ldots, k\}$, all $D^{(1)}_{1,y,z}$'s are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(1)}$ containing the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$, and they form the left graph in Figure 16. By (B′1), for all $x \in \{1, 2, \ldots, k\}$ and $z \in \{0, 1, \ldots, k-1\}$, all $D^{(4)}_{x,k,z}$'s are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(4)}$ containing the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$, and they form the right graph in Figure 16. To show that $D^{(1)}_{1,y,z} = D^{(4)}_{z,k,y-1}$ for all $y, z \in \{1, 2, \ldots, k\}$ with $y - z \geq 0$, it suffices to show that $D^{(1)}_{1,k,k} = D^{(4)}_{k,k,k-1}$. By (B′3.1), $D^{(1)}_{1,k,k}$ contains the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$. By (B′1.1), $D^{(4)}_{k,k,k-1}$ contains these three pairs as well. By Lemma 3.6(2), we have

$$D^{(1)}_{1,k,k} = T^{(1)}_{1,k,k} \cup \{v_{4k-3}, v_{4k-2}, v_{4k-1}, v_{4k}\}$$

and

$$D^{(4)}_{k,k,k-1} = T^{(4)}_{k,k,k-1} \cup \{v_{4k-3}, v_{4k-2}, v_{4k-1}, v_{4k}\},$$

where $T^{(1)}_{1,k,k}$ and $T^{(4)}_{k,k,k-1}$ are $\gamma_{pr}(P_{4k-5}(v_1 : v_{4k-5}))$-sets containing the pair $\{v_1, v_2\}$. By Lemma 2.2, we get $T^{(1)}_{1,k,k} = T^{(4)}_{k,k,k-1}$, and thus $D^{(1)}_{1,k,k} = D^{(4)}_{k,k,k-1}$.

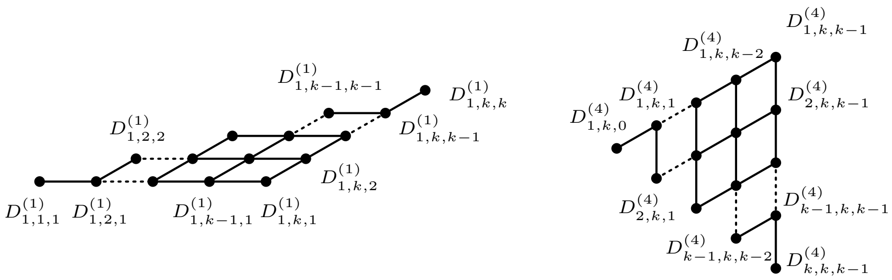

Fig. 16. The $\gamma_{pr}(C_{4k+1})$-sets containing the pairs $\{v_{4k-1}, v_{4k}\}$ and $\{v_1, v_2\}$

Next, we consider all edges between a set in $D^{(1)}$ and a set in $D^{(2)}$. We first find all neighbors of $D^{(1)}_{1,y,0}$ in $D^{(2)}$ for each $y \in \{1, 2, \ldots, k\}$. We show that $D^{(1)}_{1,1,0}$ is adjacent to $D^{(2)}_{k,k,k}$, and $D^{(1)}_{1,k,0}$ is adjacent to $D^{(2)}_{1,1,0}$. By (B′4), (B′4.1), (B′4.2), $D^{(1)}_{1,1,0}$ contains the pairs $\{v_{4k-5}, v_{4k-4}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$, $\{v_3, v_4\}$, the set $D^{(1)}_{1,y,0}$ contains the pairs $\{v_{4k-5}, v_{4k-4}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$, $\{v_4, v_5\}$ for each $y \in \{2, 3, \ldots, k-1\}$, and $D^{(1)}_{1,k,0}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$, $\{v_4, v_5\}$. Since each set in $D^{(2)}$ contains the pair $\{v_{4k}, v_0\}$, the set $D^{(1)}_{1,y,0}$ is adjacent to some set in $D^{(2)}$ if and only if $(D^{(1)}_{1,y,0} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ or $(D^{(1)}_{1,y,0} \setminus \{v_{4k-1}\}) \cup \{v_2\}$ is a $\gamma_{pr}(C_{4k+1})$-set. We have $(D^{(1)}_{1,1,0} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ is a $\gamma_{pr}(C_{4k+1})$-set, but $(D^{(1)}_{1,y,0} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ is not if $y \neq 1$. Note that $(D^{(1)}_{1,1,0} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ contains the pairs $\{v_{4k-5}, v_{4k-4}\}$, $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_0\}$. By (B′3.2), $D^{(2)}_{k,k,k}$ also contains these three pairs. By Lemma 2.2 and 3.6(2), we get

$(D^{(1)}_{1,1,0} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ and $D^{(2)}_{k,k,k}$ are unions of a unique $\gamma_{pr}(P_{4k-5}(v_2 : v_{4k-4}))$-set containing the pair $\{v_{4k-5}, v_{4k-4}\}$ and $\{v_{4k-2}, v_{4k-1}, v_{4k}, v_0\}$. Hence,

$$(D^{(1)}_{1,1,0} \setminus \{v_1\}) \cup \{v_{4k-2}\} = D^{(2)}_{k,k,k},$$

that is, $D^{(1)}_{1,1,0}$ is adjacent to $D^{(2)}_{k,k,k}$. Moreover, we see that $(D^{(1)}_{1,k,0} \setminus \{v_{4k-1}\}) \cup \{v_2\}$ is a $\gamma_{pr}(C_{4k+1})$-set, but $(D^{(1)}_{1,y,0} \setminus \{v_{4k-1}\}) \cup \{v_2\}$ is not if $y \neq k$. Note that $(D^{(1)}_{1,k,0} \setminus \{v_{4k-1}\}) \cup \{v_2\}$ contains the pairs $\{v_{4k}, v_0\}$, $\{v_1, v_2\}$, $\{v_4, v_5\}$. By (B′4), $D^{(2)}_{1,1,0}$ also contains these three pairs. By Lemmas 3.6(2) and 2.2, we have $(D^{(1)}_{1,k,0} \setminus \{v_{4k-1}\}) \cup \{v_2\} = D^{(2)}_{1,1,0}$, that is, $D^{(1)}_{1,k,0}$ is adjacent to $D^{(2)}_{1,1,0}$.

We next find all neighbors of $D^{(1)}_{x,k,k}$ in $D^{(2)}$ for each $x \in \{1, 2, \ldots, k\}$. We claim that $D^{(1)}_{x,k,k}$ is adjacent to $D^{(2)}_{1,k,x-1}$ for each $x \in \{1, 2, \ldots, k\}$, and $D^{(1)}_{k,k,k}$ is adjacent to $D^{(2)}_{1,k,k}$. By (B′3), (B′3.1), (B′3.2), $D^{(1)}_{1,k,k}$ contains the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$, the set $D^{(1)}_{x,k,k}$ contains the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_2, v_3\}$ for each $x \in \{2, 3, \ldots, k-1\}$, and $D^{(1)}_{k,k,k}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_2, v_3\}$. Note that $D^{(1)}_{x,k,k}$ is adjacent to some set in $D^{(2)}$ if and only if $(D^{(1)}_{x,k,k} \setminus \{v_{4k-1}\}) \cup \{v_0\}$ or $(D^{(1)}_{x,k,k} \setminus \{v_{4k-3}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+1})$-set. We have $(D^{(1)}_{x,k,k} \setminus \{v_{4k-1}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+1})$-set for each $x \in \{1, 2, \ldots, k\}$, and then we let $N_x = (D^{(1)}_{x,k,k} \setminus \{v_{4k-1}\}) \cup \{v_0\}$. Note that N_1 contains the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_0\}$, $\{v_1, v_2\}$, and $N_2, N_3, \ldots, N_k$ contain the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_0\}$, $\{v_2, v_3\}$, and they form a path with k vertices in $D^{(2)}$. By (B′4.2), we have $D^{(2)}_{1,k,0}$ is the only $\gamma_{pr}(C_{4k+1})$-set in $D^{(2)}$ containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_0\}$, $\{v_1, v_2\}$, and by (B′1) and (B′2), we have that $D^{(2)}_{1,k,1}, D^{(2)}_{1,k,2}, \ldots, D^{(2)}_{1,k,k-1}$ are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(2)}$ containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_0\}$, $\{v_2, v_3\}$, and they form a path with k vertices in $D^{(2)}$. Then we can conclude that for each $x \in \{1, 2, \ldots, k\}$, $N_x = D^{(2)}_{1,k,x-1}$, which means $D^{(1)}_{x,k,k}$ is adjacent to $D^{(2)}_{1,k,x-1}$. Moreover, $(D^{(1)}_{k,k,k} \setminus \{v_{4k-3}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+1})$-set, but $(D^{(1)}_{x,k,k} \setminus \{v_{4k-3}\}) \cup \{v_0\}$ is not if $x \neq k$. Note that $(D^{(1)}_{k,k,k} \setminus \{v_{4k-3}\}) \cup \{v_0\}$ contains the pairs $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_0\}$, $\{v_2, v_3\}$. By (B′3.1), $D^{(2)}_{1,k,k}$ also contains these three pairs. By Lemmas 3.6(2) and 2.2, we get $(D^{(1)}_{k,k,k} \setminus \{v_{4k-3}\}) \cup \{v_0\} = D^{(2)}_{1,k,k}$, that is, $D^{(1)}_{k,k,k}$ is adjacent to $D^{(2)}_{1,k,k}$.

Last but not least, we find all neighbors in $D^{(2)}$ of the other $\gamma_{pr}(C_{4k+1})$-sets in $D^{(1)}$. We prove that $D^{(1)}_{x,k,z}$ is adjacent to $D^{(2)}_{1,z,x-1}$ for all $x \in \{1, 2, \ldots, k\}$, $z \in \{1, 2, \ldots, k-1\}$. Recall that for all $x, y \in \{1, 2, \ldots, k\}$, $z \in \{1, 2, \ldots, k-1\}$, $D^{(1)}_{x,y,z}$ contains the pair $\{v_{4k-1}, v_{4k}\}$ but not v_{4k-2}, v_0. Then $D^{(1)}_{x,y,z}$ is adjacent to some set in $D^{(2)}$ if and only if $(D^{(1)}_{x,y,z} \setminus \{v_{4k-1}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+1})$-set. By (B′1), $D^{(1)}_{x,y,z}$ contains the pairs $\{v_{4k-5}, v_{4k-4}\}$, $\{v_{4k-1}, v_{4k}\}$ for all $y \neq k$, so $(D^{(1)}_{x,y,z} \setminus \{v_{4k-1}\}) \cup \{v_0\}$ is not a dominating set. By (B′1) and (B′2), for all $z \in \{1, 2, \ldots, k-1\}$, we have

$D^{(1)}_{1,k,z}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$, and $D^{(1)}_{x,k,z}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_2, v_3\}$ for all $x \neq 1$. For all $x \in \{1, 2, \ldots, k\}$ and $z \in \{1, 2, \ldots, k-1\}$, let $D_{x,z} = (D^{(1)}_{x,k,z} \setminus \{v_{4k-1}\}) \cup \{v_0\}$, so $D_{x,z}$ is a $\gamma_{pr}(C_{4k+1})$-set in $D^{(2)}$, and these $D_{x,z}$'s form the graph shown in Figure 17. Note that for all $z \in \{1, 2, \ldots, k-1\}$, $D_{1,z}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_0\}$, $\{v_1, v_2\}$, and $D_{x,z}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_0\}$, $\{v_2, v_3\}$ for all $x \neq 1$. By (B′4) and (B′4.2), $D^{(2)}_{1,1,0}, D^{(2)}_{1,2,0}, \ldots, D^{(2)}_{1,k-1,0}$ are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(2)}$ containing the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_0\}$, $\{v_1, v_2\}$, and by (B′1) and (B′2), for all $y, z \in \{1, 2, \ldots, k-1\}$, $D^{(2)}_{1,y,z}$'s are the only $\gamma_{pr}(C_{4k+1})$-sets in $D^{(2)}$ containing the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_0\}$, $\{v_2, v_3\}$, and they form the graph shown in Figure 18. Then the graphs in Figures 17 and 18 are the same, so we can conclude that $D_{x,z} = D^{(2)}_{1,z,x-1}$ for all $x \in \{1, 2, \ldots, k\}$, $z \in \{1, 2, \ldots, k-1\}$, that is, $D^{(1)}_{x,k,z}$ is adjacent to $D^{(2)}_{1,z,x-1}$.

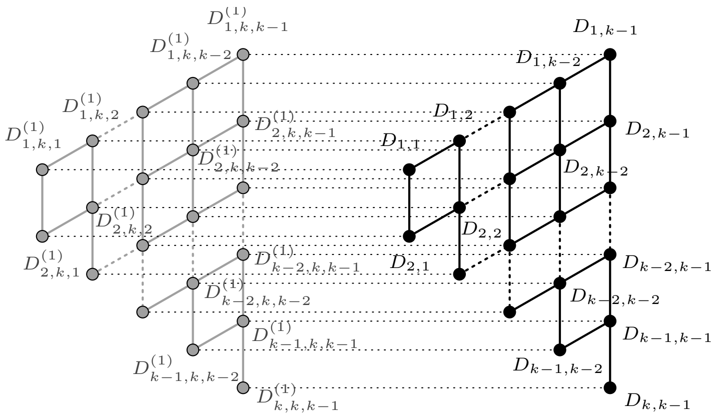

Fig. 17. The graph in $D^{(2)}$ induced by $D_{x,z}$'s

The results about the edges between a set in $D^{(i)}$ and a set in $D^{(i+1)}$ for all $i \in \{2, 3\}$ are the same as the edges between a set in $D^{(1)}$ and a set in $D^{(2)}$. Since $D^{(1)}_{1,1,0} = D^{(3)}_{1,k,k}$, the edges $D^{(1)}_{1,1,0}D^{(2)}_{k,k,k}$ and $D^{(2)}_{k,k,k}D^{(3)}_{1,k,k}$ are the same. Similarly, $D^{(1)}_{1,k,0}D^{(2)}_{1,1,0} = D^{(2)}_{1,1,0}D^{(3)}_{k,k,k}$ and $D^{(2)}_{1,k,0}D^{(3)}_{1,1,0} = D^{(3)}_{1,1,0}D^{(4)}_{k,k,k}$. Now, all $\gamma_{pr}(C_{4k+1})$-sets and edges form a loopbox LB_k in $PD_\gamma(C_{4k+1})$. Then we need to show that there is no more edge in $PD_\gamma(C_{4k+1})$. Recall that $D^{(1)}_{1,y,0} = D^{(3)}_{y,k,k}$ for all $y \in \{1, 2, \ldots, k\}$, so we consider all edges between a set in

$$\widehat{D}^{(1)} = D^{(1)} - \{D^{(1)}_{1,y,0} : 1 \leq y \leq k\}$$

and a set in

$$\widehat{D}^{(3)} = D^{(3)} - \{D^{(3)}_{y,k,k} : 1 \leq y \leq k\}.$$

Note that a set in $\widehat{D}^{(1)}$ contains either the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$, or the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_2, v_3\}$ while a set in $\widehat{D}^{(3)}$ contains the pair $\{v_0, v_1\}$ but not $\{v_{4k-1}, v_{4k}\}$.

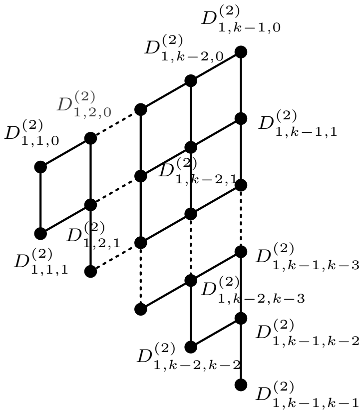

Fig. 18. The graph in $D^{(2)}$ induced by $D^{(2)}_{1,y,z}$'s

Thus, there is no edge between a set in $\widehat{D}^{(1)}$ and a set in $\widehat{D}^{(3)}$. Similarly, there is no edge between a set in $D^{(2)} - \{D^{(2)}_{1,y,0} : 1 \leq y \leq k\}$ and a set in $D^{(4)} - \{D^{(4)}_{y,k,k} : 1 \leq y \leq k\}$. Recall that $D^{(1)}_{1,y,z} = D^{(4)}_{z,k,y-1}$ for all $y, z \in \{1, 2, \ldots, k\}$. Also, for all $y \in \{1, 2, \ldots, k\}$, $D^{(1)}_{1,y,0} = D^{(3)}_{y,k,k}$, which has a neighbor in $D^{(4)}$. Hence, we consider all edges between a set in

$$\widetilde{D}^{(1)} = D^{(1)} - \{D^{(1)}_{1,y,z}, D^{(1)}_{1,y,0} : 1 \leq y, z \leq k\}$$

and a set in

$$\widetilde{D}^{(4)} = D^{(4)} - \{D^{(4)}_{z,k,y-1} : 1 \leq y, z \leq k\}.$$

Note that a set in $\widetilde{D}^{(1)}$ contains the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_2, v_3\}$ while a set in $\widetilde{D}^{(4)}$ contains the pair $\{v_1, v_2\}$ but not $\{v_{4k-1}, v_{4k}\}$. Thus, there is no edge between a set in $\widetilde{D}^{(1)}$ and a set in $\widetilde{D}^{(4)}$. This completes the proof. $\square$

Acknowledgements
This research was supported by Ph.D. scholarship from Thammasat University, 1/2018.

REFERENCES

[1] P. Eakawinrujee, N. Trakultraipruk, *γ-paired dominating graphs of paths*, Int. J. Math. Comput. Sci. **17** (2022), no. 2, 739–752.

[2] G.H. Fricke, S.M. Hedetniemi, S.T. Hedetniemi, K.R. Hutson, *γ-graphs of graphs*, Discuss. Math. Graph Theory **31** (2011), 517–531.

[3] R. Haas, K. Seyffarth, *The k-dominating graph*, Graphs Combin. **30** (2014), 609–617.

[4] T.W. Haynes, P.J. Slater, *Paired-domination in graphs*, Networks **32** (1998), 199–206.

[5] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[6] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[7] S.A. Lakshmanan, A. Vijayakumar, *The gamma graph of a graph*, Graphs Combin. **7** (2010), 53–59.

[8] R. Samanmoo, N. Trakultraipruk, N. Ananchuen, *γ-independent dominating graphs of paths and cycles*, Maejo Int. J. Sci. Technol. **13(03)** (2019), 245–256.

[9] D.B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, 2001.

[10] A. Wongsriya, N. Trakultraipruk, *γ-total dominating graphs of paths and cycles*, ScienceAsia **43** (2017), 326–333.

Pannawat Eakawinrujee
pannawattong@gmail.com
https://orcid.org/0000-0003-1336-1019

Department of Mathematics and Statistics
Faculty of Science and Technology
Thammasat University, Pathum Thani 12120, Thailand

Nantapath Trakultraipruk (corresponding author)
n.trakultraipruk@yahoo.com
https://orcid.org/0000-0002-2826-0176

Department of Mathematics and Statistics
Faculty of Science and Technology
Thammasat University, Pathum Thani 12120, Thailand

Received: November 20, 2020.
Revised: August 30, 2021.
Accepted: December 1, 2021.