Hough Transform for Lines with Slope Defined by a Pair of Co-primes

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Abstract. Data structure and Hough-type algorithm suitable for finding the lines having the slope exactly specified by the ratio of small co-prime numbers is proposed. It is suitable for analysing images like the Ulam square in which the points representing prime numbers form such structures as contiguous lines and other regular sequences of points. This analysis is different from that in the case of images in which the real-world objects are represented approximately. Until now in the Ulam square the horizontal and vertical sequences, and those inclined by 45 deg were typically analysed. With the proposed method the sequences having slopes represented by such tangents like 1/3, 2/3, 3/4 etc. can be looked for.

Key words: Hough transform, slope, co-prime, coprime, prime, Ulam, square, spiral.

1. Introduction

The Hough transform has its long history coming back to the first conference paper of P.V.C. Hough [1] and his frequently cited patent [2]. Since the paper by Duda and Hart [4] recapitulated the concept of the Hough transform, it gained large interest and was further developed by numerous authors, as described in the review paper [6]. After the next review [7] was published, a long break in publishing the state-of-the-art reviews in this domain seemed to appear, excluding some internal reports like for example [11].

The problem of our interest is the case of exact digital lines like those present in the Ulam spiral [3]. The central part of the Ulam spiral embedded in the Ulam square is shown in Fig. 1. The points in the square have coordinates (p, q) having the origin in the starting number 1 of the spiral. the image of an $U \times U = 51 \times 51$ square with the embedded Ulam spiral containing $P = 378$ primes, where the largest one is 2593, is shown in Fig. 2. In this picture some sequences of pixels which form lines can be seen, for example, the lines marked green and red. Please note that these are not the digital representation of continuous lines, but rather the regular sequences of pixels in which the increments of the two coordinates are co-primes. The term *line* will be considered here as equivalent to such type of a *sequence*. The meaning of the lines visible in the Ulam spiral were widely discussed (see for example [8], with the references to more literature).

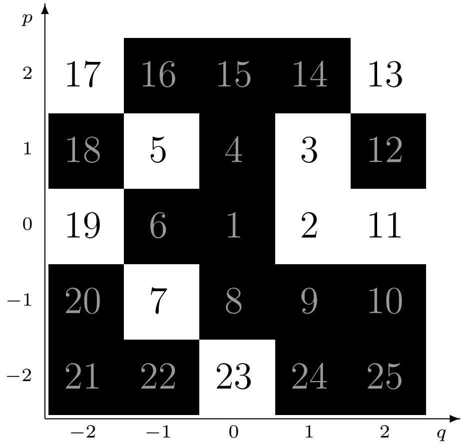

Fig. 1. The central part of the Ulam spiral for dimensions 5×5. Here, (p, q) represent coordinates in the square. Primes: black on white background, other numbers: grey on black.

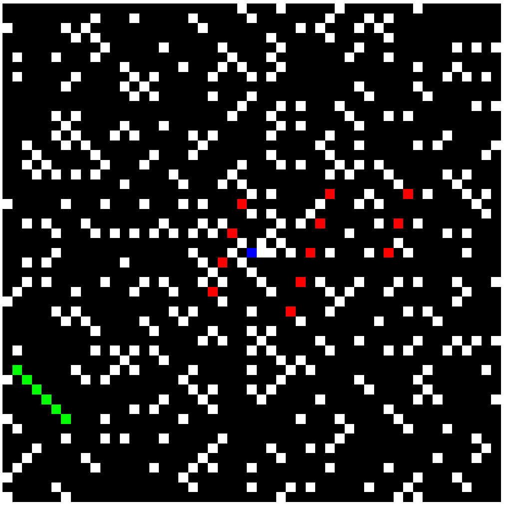

Fig. 2. Ulam spiral restricted to 51×51 pixels. Blue pixel corresponds to the spiral origin corresponding to number 1. White, red and green pixels are primes. Green pixels correspond to a line segment which form a line with slope $(\Delta p, \Delta q) = (1, -1)$ and red ones to three line segments inclined by $(3, 1)$, of different lengths.

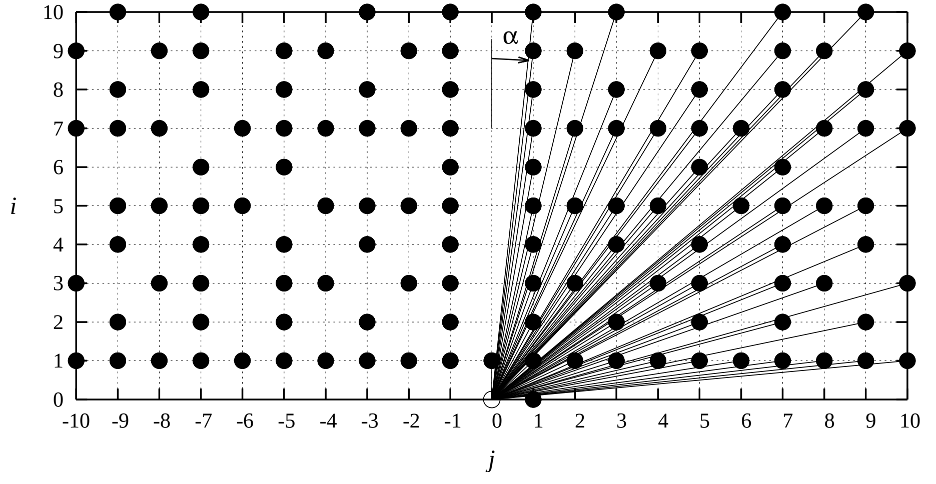

Fig. 3. Directional vectors represented with table of directions D_{ij} containing increments $(\Delta p, \Delta q) = (i, j)$. Each vector has the initial point at the empty circle $(0, 0)$ and the terminal point in one of the black circles (i, j), $\neg(i = 0 \wedge j = 0)$. α is the angle between the line segment and the vertical axis.

However, here we shall not consider this meaning. Rather, we shall concentrate solely on the data structure and organisation of the Hough-type algorithm which supports the detection of such lines, or sequences.

In the case considered the directions are restricted to those accurately represented by ratios of co-prime numbers. In the literature a data structure loosely related to such a case has been proposed by Wallace [5]. As described in [6], the line was parameterized by its two intersections with the perimeter of the image, and it was claimed that all the lines which can be drawn in the image can be represented in this way. This interesting idea is not adequate for our case, so a different structure will be proposed.

2. Structure of the accumulator

2.1. Representation of direction

The direction will be represented directly with two numbers which represent the increments of the coordinates between the subsequent points of the sequence, like for example those shown in Fig. 2 with red pixels, having the slope represented by $(\Delta p, \Delta q) = (3, 1)$. In Fig. 3 the direction table D_{ij} indexed by (i, j) which represent increments $(\Delta p, \Delta q)$ of coordinates p, q from Fig. 1. Only such pairs for which i, j are co-primes represent different slopes; other pairs are redundant. Each index belongs to the set $[-N, N] \cap \mathcal{C}$ and here it was assumed that $N = 10$. For the sake of equivocality $i \geq 0$.

It is evident that only a small number of angles can be represented. The relation of the coordinates in the direction table and the angles α formed by the directional vector and the vertical axis can be seen in Fig. 3.

2.2. Representation of offset

The offset of horizontal lines is represented by the intercept with the vertical axis p, that of other lines by the intercept with the horizontal axis q. For a given slope, the intercepts for different lines can differ only by a unit, so they can be easily represented by an integer index k. The choice of such an index is secondary. For example, it can be the smallest nonnegative index q of the point belonging to the line, or the intercept with the axis q rounded down.

2.3. Representation of a line and the accumulated value

A line in the square is represented by a triple of indexes (i, j, k). As shown in Fig. 4, from the array of directions only the co-prime pairs (i, j) have the data accumulated, so only for them the data along the axis k should be assigned. The set of data accumulated along k can be organized as a table or a list of elements. The table is more convenient when the number of data to be accumulated can be estimated before the calculations. Otherwise, the list is recommended, but it consumes more memory due to the need of storing the pointers. In our case, the number of data is equal to the number of different values of index k which can be easily determined from the dimensions of the Ulam square.

The last element to be determined is what data are accumulated. In the case of our concern, the points in the Ulam square corresponding to primes are considered, and the natural choice of the Hough transform type is the two-point transform, because the elemental subset (cf. [9, 10]), that is the set of points for which the line slope and the intercept can be determined, is a pair of points. Therefore, pairs of points are accumulated in the elements of the tables indexed by k. For the sake of further analysis, the pairs themselves should be stored also, which makes it necessary for each accumulator element to contain a list of such pairs. The memory requirements for the analysis are large, as it is usually the case for Hough transform. Pairs (i, j) belong to a small set with cardinality $|\{(i, j)\}| \ll N^2$ which can be considered constant. Therefore, the accumulator can be considered as three-dimensional.

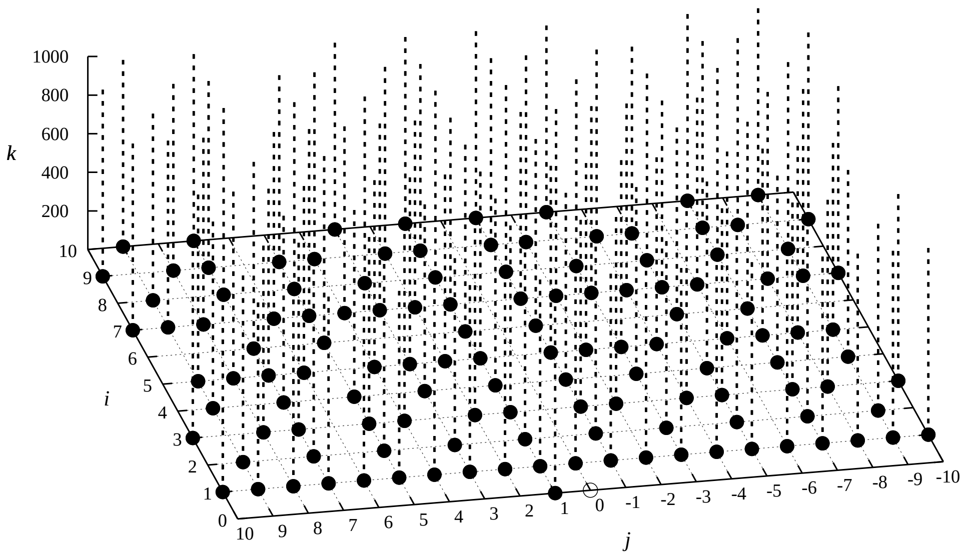

Fig. 4. Directions (i, j) together with offset index k represented in the accumulator. Along the coordinate k the accumulator can be organized as a table or as a list.

3. Accumulation and analysis of results

The data points in the Ulam square are the prime numbers which form the Ulam spiral. As it has already been mentioned, the voting subset of the set of prime points is a pair of such points. At least two strategies of forming such pairs are possible.

The local strategy is that due to the limit on directions determined by the size of the direction array D specified by N, for each prime point, its neighbours in the square at Manhattan distance up to N should be checked for the existence of a second prime point. Due to symmetry in the pair, only a half (roughly) of this neighbourhood should be checked, for example, if the square is analysed from left to right and from top to bottom, then the second points in the same verse to the right of the first point, and all those under the first point. The number of these points is proportional to N^2, so the number of pars is proportional to PN^2. In this way, if the sequence or line containing prime points which are mutually farther from each other than N can not be detected. Consequently, the prime points which are isolated on a line in the sense of this distance are never accumulated. The complexity of the accumulation algorithm is $O(PN^2)$.

The global strategy is, for each point in the sequence of the Ulam spiral, being the first point of the voting subset, to consider each other point located further in the Ulam spiral. For this, it is necessary to have the spiral stored as an ordered set of prime points, each with its index, prime value and coordinates (p, q). Then, the number of voting pairs is $P(P-1)/2$ and the complexity of the accumulation algorithm is $O(P^2)$. The lines, or sequences of prime points at arbitrary mutual distance inside the Ulam square, but with slopes limited to those represented by D, can be detected in this way.

After the accumulation, each accumulator element (i, j, k) contains a set of pairs of prime points which all belong to a line inclined by (i, j) and with offset determined by k. The pairs can be easily transformed into the ordered set of prime points, each with its index, prime value and coordinates (p, q). The number of prime points and the number of pairs for the current line is stored. Both in the case of the global and local strategy of accumulation, the continuity condition of the subsequent points in the ordered set of points can be tested to find the contiguous subsets of points in the lines, or the subset forming the other conditions of choice.

4. Example of results for a test case: square 5×5

To make it possible to understand the results shown further, let us consider a simple example for which the results can be explicitly presented in detail. To this end let us consider a small Ulam square 5×5, shown in Fig. 1. It seems to be the smallest square in which the results are not too trivial to exemplify the action of the algorithm. A file with results for this spiral is shown in detail in the listing in Fig. 5. The results shown in this listing can be visually presented in the graphs shown in Fig. 6. Some of the known features of the Ulam spiral can be seen even in such a small square. For example, at the directions $\alpha = -45°, 0°, 45°, 90°$ the numbers of lines, primes on these lines and primes per line are nearly always larger than those for the other angles.

5. Conclusion and future prospects

The data structure and the algorithm based on the Hough transform principle suitable for finding the lines, or the sequences of points, having the slope specified exactly by the ratio of co-prime numbers, was proposed. This data structure and algorithm make it possible to analyse the images like the Ulam square, in which the points which represent prime numbers form such structures as contiguous lines and other regular sequences of points. Such sequences should be analysed in the exact way. The exactness of this analysis makes it different from that for images in which the real-world objects are represented in an approximate way.

Until now the sequences with slope described $(0, 1)$ or $(1, 1)$ in the Ulam square were investigated. With the proposed method, more complex sequences like those described with slopes $(1, 3)$, $(2, 3)$, $(3, 4)$ etc. can be looked for. Configurations of points other than continuous point sequences can be studied.

```
# Unsorted lines through 5*5 spiral square
# 25 numbers, among them 9 primes
5  25  9

# list of all 25 numbers; among them the 9 primes have at least one nonzero coordinate
 1[0,0] 2[0,1] 3[1,1] 4[0,0] 5[1,-1] 6[0,0] 7[-1,-1] 8[0,0] 9[0,0] 10[0,0] 11[0,2] 12[0,0] 13[2,2]
 14[0,0] 15[0,0] 16[0,0] 17[2,-2] 18[0,0] 19[0,-2] 20[0,0] 21[0,0] 22[0,0] 23[-2,0] 24[0,0] 25[0,0]

#(<direction>) <number of its lines> <number of its prime points>:
#  <number of primes in a line>: <list of primes in a line>
(  0,1  )              3              7:
   2: 17 13
   2: 3 5
   3: 11 19 2
(  1,-3 )              2              4:
   2: 3 17
   2: 11 5
(  1,-2 )              2              4:
   2: 11 17
   2: 2 5
(  1,-1 )              3              7:
   2: 3 11
   2: 5 17
   3: 7 19 23
(  1,0  )              4              8:
   2: 11 13
   2: 2 3
   2: 5 7
   2: 17 19
(  1,1  )              3              7:
   2: 11 23
   3: 7 3 13
   2: 19 5
(  1,2  )              2              4:
   2: 2 7
   2: 13 19
(  1,3  )              3              6:
   2: 11 7
   2: 3 19
   2: 13 5
(  2,-3 )              1              2:
   2: 17 2
(  2,-1 )              1              2:
   2: 23 17
(  2,1  )              1              3:
   3: 23 2 13
(  3,-1 )              2              4:
   2: 5 23
   2: 17 7
(  3,1  )              1              2:
   2: 23 3

# Found 28 lines.
# Maximum number of primes 3 found in line at direction (0,1) through prime 2[0,1].
# Maximum number of primes 8 found at direction (1,0).
```

Fig. 5. Listing of the results file for 5 × 5 Ulam square.

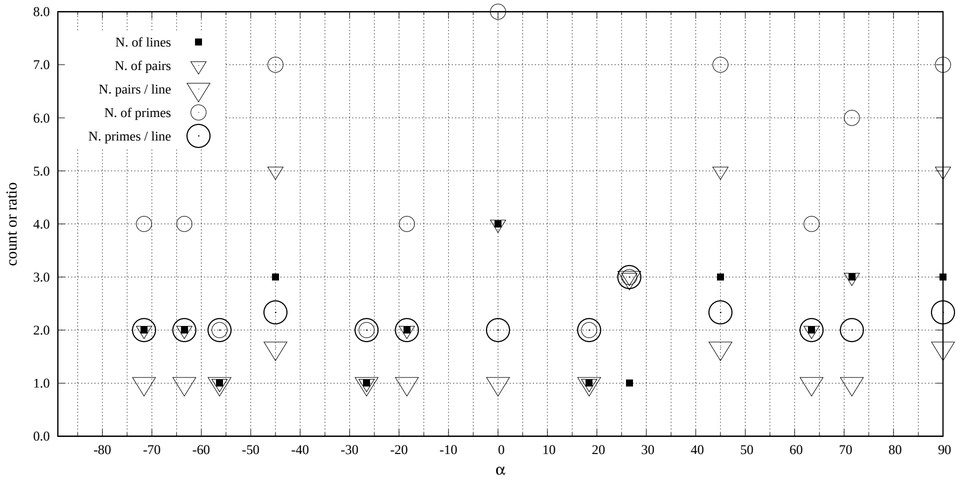

Fig. 6. Numbers of the lines, voting pairs and primes, absolute and per line, versus line direction α for the square of size 5×5 plotted from the data contained in the listing shown in Fig. 5.

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