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An interactive compromise programming for portfolio investment problem*

by

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Abstract: This paper addresses an approach for solving multicriteria portfolio investment problem. The original Markowitz mean-variance model is formulated as a problem of bi-objective optimization with linear and quadratic objective functions. In the current work, this model is extended by introducing a new objective, reflecting asset properties that are useful for the portfolio allocation process. A method based on parameterized achievement scalarizing function is applied to produce Pareto optimal portfolios. A mathematical programming formulation that allows for solving the problem with conventional optimization methods is presented. In addition, a method of reflecting the decision maker's preferences by means of changing the weights in the achievement scalarizing functions is introduced. A decision making process is simulated for the three-objective portfolio optimization problem.

Keywords: modern portfolio theory, Markowitz model, mean-variance portfolio optimization, interactive multicriteria optimization, parameterized achievement scalarizing functions

1. Introduction

The mean-variance model of portfolio investment, first studied by Harry Markowitz in the 1950s (see Markowitz, 1952, 1959) is formulated as a bi-objective optimization problem with linear (expected return) and quadratic (variance) objective functions. The classical approach to making decisions assumes deriving the two-dimensional Pareto front and letting the investor choose the most preferred combination of the objective function values. In real world application, knowing only portfolio's risk-return characteristics might be not enough for portfolio selection. There are many other factors and measures that impact portfolio performance, such as transaction costs and tax effects, specific investment guidelines and institutional features, hedging needs, market impact costs, estimation errors etc. (see, e.g., Alcada-Almeida et al., 2009; Utz, Wimmer and Steuer, 2015). Furthermore, based on the liquidity of different securities,

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their trading costs may be significantly different (Kolm, Tütüncü and Fabozzi, 2014). Out of two securities with similar expected return and risk profiles, one with higher liquidity is more likely to have higher post-transaction cost returns. Therefore, a portfolio construction framework that ignores transaction costs will lead to suboptimal portfolios. For this purpose, the current research work proposes to incorporate into the model an additional linear objective that reflects those characteristics of an asset. Thus, original bi-objective problem transforms into a problem with three and more objectives.

The traditional approach to solving multicriteria optimization problems with three and more objectives is by scalarization. It involves formulating a single objective problem that is related to the multicriteria problem by means of a real-valued scalarizing function typically being a function of the individualized objective functions of the multicriteria problem, auxiliary scalar or vector variables, and/or scalar or vector parameters. Sometimes, the feasible set of the multicriteria optimization problem is additionally restricted by new constraint functions, related to the objective functions of the multicriteria problem and/or the new variables introduced. Two major requirements are set for a scalarizing function in order to provide method completeness (see Sawaragi, Nakayama and Tanino, 1985):

- it should be able to cover the entire set of Pareto optimal solutions, and
- every solution found by means of scalarization should be (weakly) Pareto optimal.

One of the widely spread approaches of dealing with multiple conflicting objectives involves constructing and optimizing a so-called achievement scalarizing function (ASF). This method was introduced in Wierzbicki (1980) and is based on a reference point of aspiration levels. The ASF can be interpreted as minimizing the distance from a reference point, specified by the decision maker (DM), to the feasible region, if the reference point is unattainable, or, alternatively, maximizing the distance. The distance is defined by some appropriate metric, introduced in the objective space. Sometimes, the DM may require more advanced scalarization mechanisms. In Nikulin, Miettinen and Mäkelä (2012) a parameterized version of an ASF was proposed. Those authors introduced an integer parameter in order to control the degree of metric flexibility varying from L_1 to L_∞. It was proven that the parameterized ASF is able to detect any Pareto optimal solutions. Moreover, conditions under which the Pareto optimality of each solution produced by the parameterized ASF is guaranteed was also obtained in Nikulin, Miettinen and Mäkelä (2012). In Wilppu, Mäkelä and Nikulin (2017) a new family of two-slope parameterized ASFs that generalizes both parameterized ASF and two-slope ASF were presented. These two-slope parameterized ASFs guarantee (weak) Pareto optimality of the solutions produced, and can generate every (weakly) Pareto optimal solution.

In interactive methods, the DM works together with an analyst of an interactive computer program. (By an *analyst* here is meant a person or a computer program responsible for the mathematical side of the solution process, Miettinen, 1999). One can say that the analyst tries to determine the preference structure of the DM in an interactive way. A solution pattern is formed and repeated several times. After every iteration, some information is given to the DM and (s)he is asked to answer some questions or provide some other type of information. The working order in these methods is: analyst, DM, analyst, DM, etc.

After a reasonable (finite) number of iterations every interactive method should yield a solution that the DM can be satisfied with and/or convinced that no considerably better solution exists. The basic steps in interactive algorithms can be expressed as

- find an initial feasible solution,
- interact with the DM, and
- obtain a new solution (or a set of new solutions). If the new solution (or one of them) or one of the previous solutions is acceptable to the DM, stop. Otherwise, go to the previous step.

Interactive methods differ from each other by the form, in which information is given to the DM, by the form, in which information is provided by the DM, and how the problem is transformed into a single objective optimization problem (see, e.g., Miettinen, 1999; Miettenen and Mäkelä, 2006; Steuer and Choo,1983).

In Montonen, Ranta and Mäkelä (2019) authors introduce the interactive multi-objective optimization method using the two-slope parameterized achievement scalarizing function and adapt it for a nonsmooth multiobjective mixed-integer nonlinear optimization problem. The current work investigates the applicability of interactive optimization approach, based on the parameterized ASF, to multicriteria portfolio investment problem. A new way to manage an interactive process by changing weighing coefficients of scalarizing functions is introduced. Using custom scalarization technique, the three objective portfolio investment problem is converted into a single objective problem of mathematical programming and solved by the optimization solver software. Numerical experiments illustrate how synchronous usage (Miettinen and Mäkelä, 2006) of the scalarizing functions may be potentially advantageous for obtaining different Pareto optimal portfolios.

2. Basic notations and definitions

Let X be an arbitrary set of feasible solutions or a set of decision vectors. Let a vector valued function $f : X \to \mathbb{R}^m$, consisting of $m \geq 2$ partial objective functions, be defined on the set of feasible solutions:

$$f(x) = (f_1(x), f_2(x), \dots, f_m(x)).$$

Without loss of generality we may assume that every objective function is to be minimized on the set of feasible solutions:

$$\min_{x \in X} f_i(x), \quad i \in N_m = \{1, 2, \dots, m\}. \tag{1}$$

Further, throughout the paper we will refer to $\mathbb{R}^m$ as an objective space and to vector $f(x)$ as an objective vector.

We also assume that

1. every objective function f_i is a lower semicontinuous function;
2. X is a nonempty compact set.

Let us denote by

$$M^i(X) = \arg\min_{x \in X} f_i(x), \;\; i \in N_m$$

a set of minima of the i^{th} objective function. Evidently, if

$$\bigcap_{i=1}^{m} M^i(X) \neq \emptyset,$$

then there exists at least one solution, which delivers a minimum for all objectives. Such a solution can be called an *ideal solution*. An optimization problem, which does not contain ideal solutions is called *non-degenerate* and objectives are at least partly conflicting (Nikulin, Miettinen and Mäkelä, 2012). Simultaneous optimization of several objectives for non-degenerate multiobjective optimization problems is not a straightforward task, and we need to define optimality for such problems. In what follows, we consider non-degenerate problems.

The traditional definition of the Pareto principle of optimality:

DEFINITION 1 *A decision vector* $x^* \in X$ *is* Pareto optimal *if there exists no* $x \in X$ *such that* $f_i(x) \leq f_i(x^*)$ *for all* $i \in N_m$ *and* $f_j(x) < f_j(x^*)$ *for at least one index* j.

We can denote the set of Pareto optimal decision vectors as $P^m(X)$. Furthermore, the set $\{f(x) \in \mathbb{R}^m : x \in P^m(X)\}$ is called the *Pareto frontier*. For two vectors $a, b \in \mathbb{R}^m$, we write $a \leq b$ if $a_i \leq b_i$ for all $i \in N_m$. Then we say that one vector a *dominates* another vector b if $a_i \leq b_i$ and $a \neq b$. Thus, the set of Pareto optimal solutions is simply a subset of feasible solutions, whose images are non-dominated in the objective space.

The optimality in a multiobjective case can be introduced in different ways. The following definition was firstly proposed by Slater (1950):

DEFINITION 2 *A decision vector* $x^* \in X$ *is* weakly Pareto optimal *if there exists no* $x \in X$ *such that* $f_i(x) < f_i(x^*)$ *for all* $i \in N_m$.

We can denote the set of weakly Pareto optimal decision vectors or the Slater set as $Sl^m(X)$. For two vectors $a, b \in \mathbb{R}^m$, we write $a < b$ if $a_i < b_i$ for all $i \in N_m$. Then we say that one vector a *strictly dominates* another vector b if $a_i < b_i$ for all $i \in N_m$. Thus, the set of weakly Pareto optimal solutions is a subset of feasible solutions, whose images are strictly non-dominated in the objective space. An objective vector

$f(x)$ is (weakly) Pareto optimal if the corresponding decision vector is (weakly) Pareto optimal.

Under the assumptions 1–2 mentioned earlier in the problem formulation, we know that the set of Pareto optimal solution is non-empty, that is, there always exists at least one Pareto optimal solution (Sawaragi, Nakayama and Tanino, 1985). Obviously, the set of Pareto optimal solutions is a subset of weakly Pareto optimal solutions.

Lower and upper bounds on objective values of all Pareto optimal solutions are given by the *ideal* and *nadir* objective vectors f^I and f^N, respectively. The components f_i of the ideal (nadir) objective vector $f^I = (f_1^I, \ldots, f_m^I)$ ($f^N = (f_1^N, \ldots, f_m^N)$) are obtained by minimizing (maximizing) each of the objective functions individually, subject to the set of Pareto optimal solutions:

$$f_i^I = \min_{x \in P^m(X)} f_i(x), \;\; i \in N_m,$$

$$f_i^N = \max_{x \in P^m(X)} f_i(x), \;\; i \in N_m.^*$$

3. Achievement scalarizing functions

In reference point based methods (see, e.g., Wierzbicki, 1980, 1986, 1999), the DM specifies a reference point f^R consisting of desirable or reasonable aspiration levels f_i^R for each objective function $f_i, \; i \in N_m$. The reference point only indicates what levels of objective function values the DM considers satisfactory.

A certain class of real-valued functions $s_R : \mathbb{R}^m \to \mathbb{R}$, referred to as achievement scalarizing functions (ASFs), can be used to scalarize a multiobjective optimization problem. Achievement scalarizing functions have been introduced by Wierzbicki in (1980). The scalarized problem is given by

$$\min_{x \in X} s_R(f(x)). \tag{2}$$

Pareto optimal solutions can be characterized by achievement scalarizing functions if the functions satisfy certain requirements.

DEFINITION 3 *(Wierzbicki, 1986) An ASF $s_R : \mathbb{R}^m \to \mathbb{R}$ is said to be*

1. increasing, *if for any $y^1, y^2 \in \mathbb{R}^m, \; y_i^1 \le y_i^2$ for all $i \in N_m$, there is $s_R(y^1) \le s_R(y^2)$;*

* At this point of this paper, but, in fact, concerning the entire issue, a note is due from the Editors. Namely, it should be emphasized that it is almost unknown in the world literature (and, alas, also to a limited extent in the Polish literature of the subject) that the concept of comparing the multidimensional objects analysed (here: portfolios, characterized by the shares of individuals assets) to some kind of ideal and nadir points, formed out of the maximum and minimum values for the particular dimensions (criteria), was formulated, along with the entire respective procedure, first by Zdzisław Hellwig (1968), otherwise a prominent Polish statistician and econometrician, author of several methodological innovations in these domains (ed.).

2. strictly increasing, *if for any* $y^1, y^2 \in \mathbb{R}^m$, $y_i^1 < y_i^2$ *for all* $i \in N_m$, *there is* $s_R(y^1) < s_R(y^2)$*;*
3. strongly increasing, *if for any* $y^1, y^2 \in \mathbb{R}^m$, $y_i^1 \leq y_i^2$ *for all* $i \in N_m$ *and* $y^1 \neq y^2$, *there is* $s_R(y^1) < s_R(y^2)$.

Obviously, any strongly increasing ASF is also strictly increasing, and any strictly increasing ASF is also increasing. The following theorems define necessary and sufficient conditions for an optimal solution of (2) to be (weakly) Pareto optimal:

THEOREM 1 *(Wierzbicki, 1986)*

1. *Let* s_R *be strongly (strictly) increasing. If* $x^* \in X$ *is an optimal solution of problem* (2)*, then* x^* *is (weakly) Pareto optimal.*
2. *If* s_R *is increasing and the solution of* (2) $x^* \in X$ *is unique, then* x^* *is Pareto optimal.*

THEOREM 2 *(Miettinen, 1999) If* s_R *is strictly increasing and* $x^* \in X$ *is weakly Pareto optimal, then it is a solution of* (2) *with* $f^R = f(x^*)$ *and the optimal value of* s_R *is zero.*

The advantage of ASFs is that any (weakly) Pareto optimal solution can be obtained by moving the reference point only. It was shown in Wierzbicki (1986) that the solution of an ASF depends Lipschitz continuously on the reference point. In general, ASFs are conceptually very appealing to generate Pareto optimal solutions, and they overcome most of the difficulties arising with other methods, Miettinen (1999), in the class of methods for generating Pareto optimal solutions.

In the great majority of cases, the ASF is based on the Chebyshev distance or L_∞:

$$s_R^\infty(f(x), \lambda) = \max_{i \in N_m} \lambda_i (f_i(x) - f_i^R), \tag{3}$$

where λ is an m-vector of non-negative coefficients.

An achievement scalarizing function based on the linear distance L_1 is proposed in Ruiz et al. (2008). Given problem (1), a reference vector $f^R \in \mathbb{R}^m$ and a vector of strictly positive weights λ, the additive achievement scalarizing function is defined as follows:

$$s_R^1(f(x), \lambda) = \sum_{i \in N_m} \lambda_i |f_i(x) - f_i^R|. \tag{4}$$

The following properties of $s_R^1(f(x), \lambda)$ were proven in Ruiz et al. (2008).

THEOREM 3 *(Ruiz et al., 2008) Given problem* (2) *with ASF defined by* (4)*, let* f^R *be a reference point such that* f^R *is not dominated by an objective vector of any feasible solution of problem* (2)*. Also assume* $\lambda_i > 0$ *for all* $i \in N_m$*. Then, any optimal solution of problem* (2) *is a weakly Pareto optimal solution.*

THEOREM 4 *(Ruiz et al., 2008) Given problem* (2) *with ASF defined by* (4) *and any reference point* f^R*, assume* $\lambda_i > 0$ *for all* $i \in N_m$*. Then, among the optimal solutions of problem* (2) *there exists at least one Pareto optimal solution. If the optimal solution of problem* (2) *is unique, then it is Pareto optimal.*

In Nikulin, Miettinen and Mäkelä (2012) authors extend the ideas of Ruiz et al. (2008) by introducing parameterization based on the notion of embedded subsets. Here, an integer parameter $q \in N_m$ is used in order to control the degree of metric flexibility varying from L_1 to L_∞.

Let I^q be a subset of N_m of cardinality q. A parameterized ASF is defined as follows:

$$s_R^q(f(x),\lambda) = \max_{I^q \subseteq N_m : |I^q| = q} \left\{ \sum_{i \in I^q} \max\left[\lambda_i(f_i(x) - f_i^R), 0\right] \right\}, \tag{5}$$

where $q \in N_m$ and $\lambda = \{\lambda_1, \dots, \lambda_m\}, \lambda_i > 0, i \in N_m$. Notice that

$$for q \in N_m : s_R^q(f(x),\lambda) \geq 0;$$

$$q = 1 : s_R^1(f(x),\lambda) = \max_{i \in N_m} \max[\lambda_i(f_i(x) - f_i^R), 0] \cong s_R^\infty(f(x),\lambda);$$

$$q = m : s_R^m(f(x),\lambda) = \sum_{i \in N_m} \max[\lambda_i(f_i(x) - f_i^R), 0] = s_R^1(f(x),\lambda).$$

Here, "$\cong$" means equality in the case where there exist no feasible solutions $x \in X$, which strictly dominate the reference point, that is, $f_i(x) < f_i^R$ for all $i \in N_m$.

For any given $q \in N_m$, the problem to be solved is

$$\min_{x \in X} s_R^q(f(x),\lambda). \tag{6}$$

It is obvious that using problem (6), every feasible solution of the multiobjective problem (including Pareto optimal) is supported. Indeed, given any $x \in X$, the reference point $f^R = f(x)$ and a vector of weighting coefficients $\lambda > 0$, the optimal solution to problem (6) is x with the optimal value of $s_R^q(f(x),\lambda)$ equal zero. Thus, the first of the two requirements, mentioned in the introduction, is satisfied.

For any $x \in X$, denote $I_x = \{i \in N_m : f_i^R \leq f_i(x)\}$. The following two results, analogous to Theorems 3 and 4, describe the conditions, under which the second of the two requirements mentioned in introduction is satisfied.

THEOREM 5 *(Nikulin, Miettinen and Mäkelä, 2012) Given problem* (6)*, let* f^R *be a reference point such that there exists no feasible solution, whose image strictly dominates* f^R*. Also assume* $\lambda_i > 0$ *for all* $i \in N_m$*. Then, any optimal solution of problem* (6) *is a weakly Pareto optimal solution.*

THEOREM 6 *(Nikulin, Miettinen and Mäkelä, 2012) Given problem* (6)*, let* f^R *be a reference point. Also assume* $\lambda_i > 0$ *for all* $i \in N_m$*. Then, among the optimal solutions of problem* (6) *there exists at least one Pareto optimal solution.*

Theorem 6 implies that the uniqueness of the optimal solution guarantees its Pareto optimality. Notice that the facts stated above about the solutions of parameterized ASFs also implicitly follow from the results of Theorem 1. To show this, it is sufficient to prove that $s_R^q(f(x), \lambda)$ is increasing. Moreover, the parameterized ASF is strictly increasing if there are no feasible solutions dominating f^R.

4. Scalarization technique and mathematical programming problem formulation for 3-objective portfolio investment

Let us first consider multicriteria minimization problem (1). In order to derive its single objective version, all m objectives are converted into the following constraints:

$$\begin{aligned} & f_i(x) \le y_i,\ i \in N_m, \\ & y \in \mathbb{R}^m. \end{aligned} \tag{7}$$

For reference point based scalarizations, constraints (7) are transformed into the following inequalities:

$$\lambda_i(f_i(x) - f_i^R) \le y_i,\ i \in N_m, \tag{8}$$

where y_i are dummy variables, f_i^R are aspiration levels and λ_i are weights.

Let vector $x = (x_1, x_2, \ldots, x_n)$ denote portfolios, i.e. vectors of fractions of the investment capital spent on individual assets, n be the number of assets, which are denoted by natural numbers $\{1, \ldots, n\}$, Q be a positive semidefinite covariance matrix, $\mu \in \mathbb{R}^n$ is a vector of mean returns and $v = (v_1, \ldots, v_n)$, where v_i is a liquidity characteristic of asset i. For simplicity, we assume that the liquidity of the portfolio is additive, and the characteristic of asset liquidity is subject to minimization. Both the mean returns and the covariance coefficients are estimated using statistical methods over time series of historical asset prices.

Using the notations above, the extended mean-variance-liquidity portfolio investment problem is defined as follows:

$$\begin{aligned} & \min\ f_1(x) := x^T Q x, \\ & \min\ f_2(x) := -\mu^T x, \\ & \min\ f_3(x) := v^T x, \\ \text{s.t.} \quad & \left\{ x \in \mathbb{R}_+^n : \sum_{i \in N_n} x_i = 1 \right\}. \end{aligned} \tag{9}$$

Here the first quadratic objective minimizes variance, the second linear objective of maximizing the mean return is replaced with minimizing the negative mean return for uniformity and the third linear objective minimizes the liquidity level of portfolios.

Then, in the case of parameterized ASF (5), the scalarized problem for three objectives, that is $m = 3$, is formulated as follows:

$$\begin{array}{ll} & \min\ z, \\ \text{s.t.} & \sum\limits_{i \in I^q} y_i \leq z \text{ for any } I^q \subseteq N_m : |I^q| = q, \\ & y_i \geq 0,\ i \in N_m, \\ & \lambda_i(f_i(x) - f_i^R) \leq y_i,\ i \in N_m, \\ & x \in \mathbb{R}^n_+ : \sum\limits_{i \in N_n} x_i = 1. \end{array} \tag{10}$$

5. Numerical experiments

When mathematical programming model as above is formulated for the problem, it can be solved with standard optimization methods. To illustrate the use of the proposed method for solving the three-objective portfolio investment problem (10) it was implemented in Python language and computed using Gurobi Optimizer software.

The problem is generated for 300 randomly selected company shares from the US stock market. Rates of return were calculated for each business day relative to the same day one week before. The time period was from 01.01.2014 to 16.09.2019. As a liquidity characteristic of assets we considered the inverse of their median trading volumes.

However this approach may seem primitive, it reflects the definition of liquidity as a minimization objective, which suffices for a proof-of-concept example.

Solutions are obtained for four sets of weights, which are scaled based on ideal and nadir points. One set consists of equal weights for all objectives and three other sets emphasize each one particular objective. Ideal point is chosen as the vector of aspiration levels.

Calculation results for each objective, different sets of weighting coefficients and values of q are presented in Tables 1 through 3. Below, the objective function values in conventional forms suitable for investor are presented: the first objective, related to standard deviation of returns and the second objective, related to expected return (in contrast to the negative expected return in the model). Graphical representation of these data is displayed in the Figs. 1 through 3.

Despite the fact that experiments were conducted for one data set and results are preliminary, some conclusions can already be made. First of all, calculation are practically relevant, the number of assets used is comparable with portfolio sizes that investors deal with in the real world.

Table 1. Standard deviation

Emphasis / q	No emphasis	Standard deviation	Expected return	Trade volume index
1	0.0789773	0.0599774	0.0935434	0.0668605
2	0.0510246	0.0412636	0.0935434	0.0474164
3	0.0388386	0.0348017	0.0530248	0.0423503

Table 2. Expected return

Emphasis / q	No emphasis	Standard deviation	Expected return	Trade volume index
1	0.0087615	0.0083066	0.0096741	0.007894
2	0.0072291	0.0070765	0.0096741	0.006576
3	0.0061713	0.0060213	0.0078653	0.005823

Table 3. Trade volume index

Emphasis / q	No emphasis	Standard deviation	Expected return	Trade volume index
1	8.57E-06	7.72E-06	1.202E-05	5.514E-06
2	3.58E-06	4.447E-06	1.202E-05	1.57E-06
3	1.14E-06	1.294E-06	6.057E-06	2.115E-07

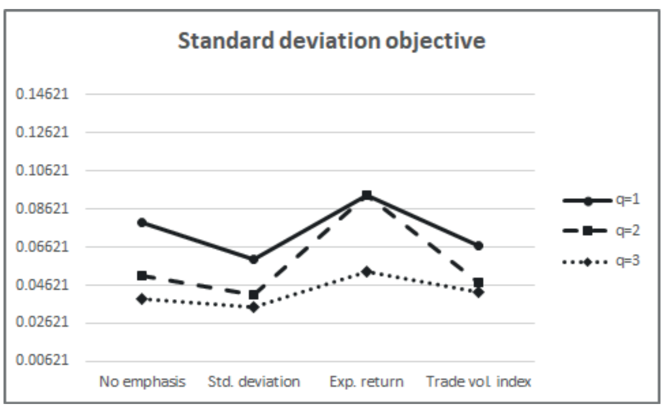

Figure 1. Standard deviation

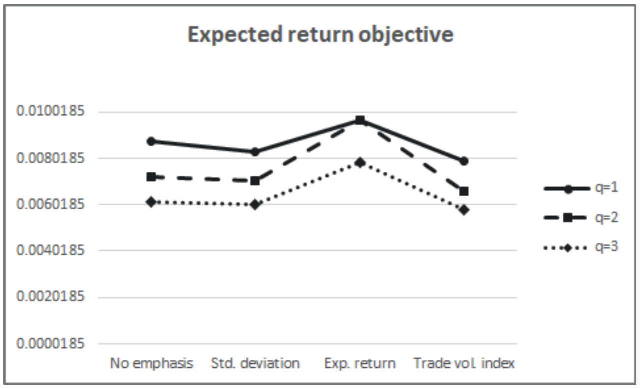

Figure 2. Expected return

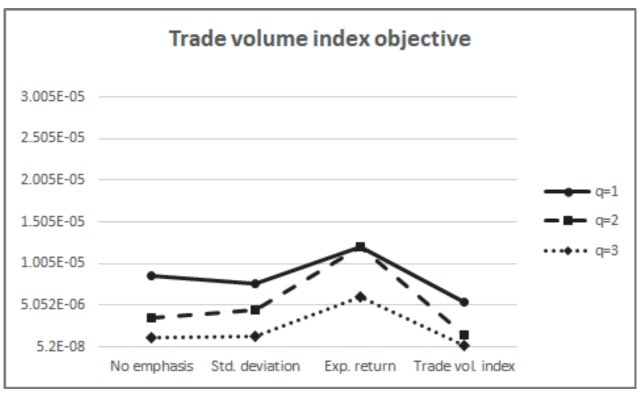

Figure 3. Trade volume index

Second, different Pareto solutions are obtained for different values of parameter q. The values of objectives for $q = 2$ always lie between the corresponding values for $q = 1$ and $q = 3$. In the case of $q = 3$, the scalarizing function resulted in lower values of all objectives (which is better for standard deviation and liquidity characteristic, and worse for expected return). Thus, solution in the case of $q = 2$ might be more preferable for the DM, this showing the benefits from using different types of scalarization.

6. Assisting the decision maker in guiding interactive solution process

In the numerical experiments, reported in the previous section, weights were chosen with emphases on different objectives and they were fixed. Pareto optimal solutions are generated for these weights and presented to the DM. Alternatively, the DM may take actively part in the solution process and specify preference information interactively by classifying each of the objective functions into different classes (Miettinen and Mäkelä, 2006). In the proposed approach two classes are considered:

- f_i values that are desired to be improved (i.e. decreased),
- f_i values that may be impaired (i.e. increased).

In comparison to other classification-based approach, we ask for binary preference information for each objective, which is the least possible amount of information.

Further on, this section describes how the preferences of the DM can be transformed into the weighing coefficients based on the properties of the corresponding scalarization type. For illustrative purpose, the interactive procedure is considered for the case of three objectives, that is, $m = 3$. Then, the parameterized ASF (5) has the following form:

$$s_R^q(f(x),\lambda) = \max_{I^q \subseteq \{1,2,3\}:|I^q|=q} \left\{ \sum_{i \in I^q} \max\left[\lambda_i(f_i(x) - f_i^R), 0\right] \right\}, \tag{11}$$

where $q = 1, 2, 3$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3),\ \lambda_i > 0,\ i \in N_3$.

Taking into account the fact that an ideal vector $f^I = (f_1^I, f_2^I, f_3^I)$ is chosen as a reference point f^R, we have:

for $q = 1$

$$\begin{aligned} s_R^1(f(x),\lambda) &= \max\Big\{ \max[\lambda_1(f_1(x) - f_1^R), 0], \max[\lambda_2(f_2(x) - f_2^R), 0], \\ &\qquad \max[\lambda_3(f_3(x) - f_3^R), 0]\Big\} \\ &= \max\Big\{ \lambda_1(f_1(x) - f_1^I), \lambda_2(f_2(x) - f_2^I), \lambda_3(f_3(x) - f_3^I)\Big\}; \end{aligned}$$

for $q = 2$

$$\begin{aligned} s_R^2(f(x),\lambda) &= \max\Big\{ \max[\lambda_1(f_1(x) - f_1^R), 0] + \max[\lambda_2(f_2(x) - f_2^R), 0], \\ &\qquad \max[\lambda_1(f_1(x) - f_1^R), 0] + \max[\lambda_3(f_3(x) - f_3^R), 0], \\ &\qquad \max[\lambda_2(f_2(x) - f_2^R), 0] + \max[\lambda_3(f_3(x) - f_3^R), 0]\Big\} \\ &= \max\Big\{ \lambda_1(f_1(x) - f_1^I) + \lambda_2(f_2(x) - f_2^I), \\ &\lambda_1(f_1(x) - f_1^I) + \lambda_3(f_3(x) - f_3^I), \\ &\lambda_2(f_2(x) - f_2^I) + \lambda_3(f_3(x) - f_3^I)\Big\}; \end{aligned}$$

for $q = 3$

$$s_R^3(f(x),\lambda) = \max\Big\{ \max[\lambda_1(f_1(x) - f_1^R), 0] + \max[\lambda_2(f_2(x) - f_2^R), 0] + \max[\lambda_3(f_3(x) - f_3^R), 0]\Big\} = \lambda_1(f_1(x) - f_1^I) + \lambda_2(f_2(x) - f_2^I) + \lambda_3(f_3(x) - f_3^I).$$

In order to show how coefficients λ_i are used to direct the interactive procedure, suppose that the problem (6) is solved for $q = 1,\ q = 2$ and $q = 3$. Let us first consider the case of $q = 1$:

$$\max\Big\{\lambda_1(f_1(x) - f_1^I), \lambda_2(f_2(x) - f_2^I), \lambda_3(f_3(x) - f_3^I)\Big\} = \alpha. \tag{12}$$

In other words, the minimal distance from the ideal point to the Pareto frontier in terms of parameterized ASF with $q = 1$ is α. The set of points, for which the distance from the reference point is equal to α is called the α-level set. The image of α-level set for (12) looks similar to what we always have for the Chebyshev type ASF, that is, it has cubic shape. Now, we determine the direction in which this set intersects the Pareto frontier, that is, the coordinates of the corner of the cube.

Here, three cases are possible:
Case 1:

$$\{f_1(x) = \alpha/\lambda_1 + f_1^I, f_2(x) \in [f_2^I, \alpha/\lambda_2 + f_2^I], f_3(x) \in [f_3^I, \alpha/\lambda_3 + f_3^I]\};$$

Case 2:

$$\{f_1(x) \in [f_1^I, \alpha/\lambda_1 + f_1^I], f_2(x) = \alpha/\lambda_2 + f_2^I, f_3(x) \in [f_3^I, \alpha/\lambda_3 + f_3^I]\};$$

Case 3:

$$\{f_1(x) \in [f_1^I, \alpha/\lambda_1 + f_1^I], f_2(x) \in [f_2^I, \alpha/\lambda_2 + f_2^I], f_3(x) = \alpha/\lambda_3 + f_3^I\}.$$

From here we obtain that the corner coordinates are $(\alpha/\lambda_1+f_1^I, \alpha/\lambda_2+f_2^I, \alpha/\lambda_3+f_3^I)$. Thus, we can change coordinates of the ideal point projection onto Pareto front by moving this corner.

Now let us consider the corresponding functions for $q = 2$ and $q = 3$:

$$\max\Big\{\lambda_1(f_1(x) - f_1^I) + \lambda_2(f_2(x) - f_2^I), \lambda_1(f_1(x) - f_1^I) + \lambda_3(f_3(x) - f_3^I), \lambda_2(f_2(x) - f_2^I) + \lambda_3(f_3(x) - f_3^I)\Big\} = \alpha. \tag{13}$$

$$\lambda_1(f_1(x) - f_1^I) + \lambda_2(f_2(x) - f_2^I) + \lambda_3(f_3(x) - f_3^I) = \alpha. \tag{14}$$

Here, α is the minimal distance from the ideal point to the Pareto frontier in terms of the parameterized ASF for $q = 2$ and $q = 3$, respectively. Detailed graphical constructions of α-level sets for (13) and (14) are demonstrated in Nikulin, Miettinen and Mäkelä (2012). For controlling the interactive process we are interested only in the case where all the three sums in (13) and the sum in (14) are equal to α. For (14), this forms a flat triangular face, which is contained in a plane with the normal vector $(\lambda_1, \lambda_2, \lambda_3)$. For (13), the flat triangle transforms into a triangle pyramid with top vertex $(\alpha/2\lambda_1 + f_1^I, \alpha/2\lambda_2 + f_2^I, \alpha/2\lambda_3 + f_3^I)$. From here it follows that we can control the place of potential contact of α-level set for (13) and α-level set for (14) with the image of the feasible set by changing the normal vector or the top vertex.

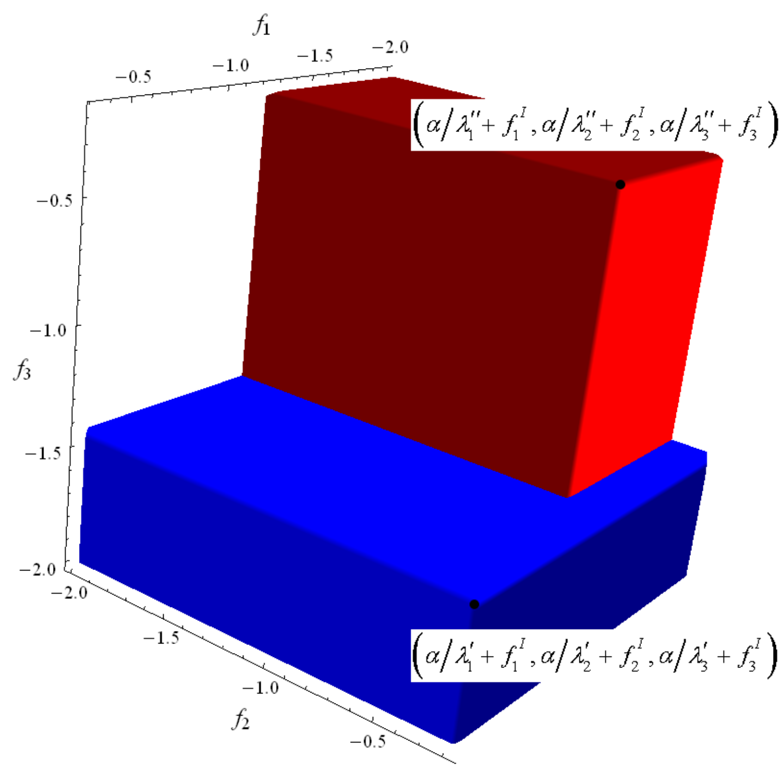

Figure 4. Two α-level sets of parameterized ASFs for $q = 1$ corresponding to different values of weighing coefficients $(\lambda_1', \lambda_2', \lambda_3')$ and $(\lambda_1'', \lambda_2'', \lambda_3'')$

Thus, the weighing coefficients vector λ indicates the relative importance of the deviations of the objectives values $f_i(x), i = 1, 2, 3$, from the ideal vector (f_1^I, f_2^I, f_3^I) and can be used to reflect the DM preferences. It follows from the above discussion that for all values of parameter q by decreasing λ_i, we increase the value of $f_i(x)$ and by increasing λ_i, we decrease the value of $f_i(x)$.

Figures 4 through 6 illustrate how different values of λ_1, λ_2 and λ_3 affect the directions, in which α-level sets of ASFs will intersect the Pareto frontier.

Using the information obtained above about the relation between objective functions and weighing coefficients, the basic steps of the interactive procedure can be described as follows:

1. Initially, an ideal vector is chosen as a reference point and weighing coefficients are inverse with respect to the corresponding differences between the nadir and ideal vector to provide objective functions normalization.
2. At each iteration c, objective function values are calculated at the current Pareto optimal decision vector $x_c \in P^m(X)$ and presented to the DM.
3. (S)he then expresses what kind of changes would be desirable to her/him by classifying each of the objective functions into two classes, i.e. whether the objective is desired to be decreased or increased.
4. Then, weighing coefficients are changed according to the DM answer and relation between objectives and weights described above.
5. Process is stopped when the DM is satisfied with the obtained solution.

The advantage of this procedure is that the investor has a better view of the solutions set and may direct solution process towards the potentially more desirable efficient portfolios. These portfolios may be skipped when weights are generated by the method programmatically. Moreover, the procedure does not require any additional mathematical knowledge for the investor and s/he may express preferences in a convenient way, based on the information known about objective functions.

7. Conclusion

In this paper an approach, based on parameterized ASFs is proposed for applying in solving the multiobjective portfolio investment problem. Quadratic programming formulation of the problem with three objectives is presented. A new way of directing interactive process by changing the weighing coefficients depending on the metric used in ASF is introduced. The DM is asked to provide simple information, which can be considered an advantage of this approach.

The numerical experiments were performed to illustrate the fact that different ASFs allow to detect more versatile Pareto optimal points, and synchronous use (Miettenen and Mäkelä, 2006) of these ASFs, as a result, may reduce the number of iterations needed for interactive procedure to converge. Thus, the proposed method provides the DM with a better view of the potential compromises and gives more flexible tools for detecting Pareto optimal solutions.

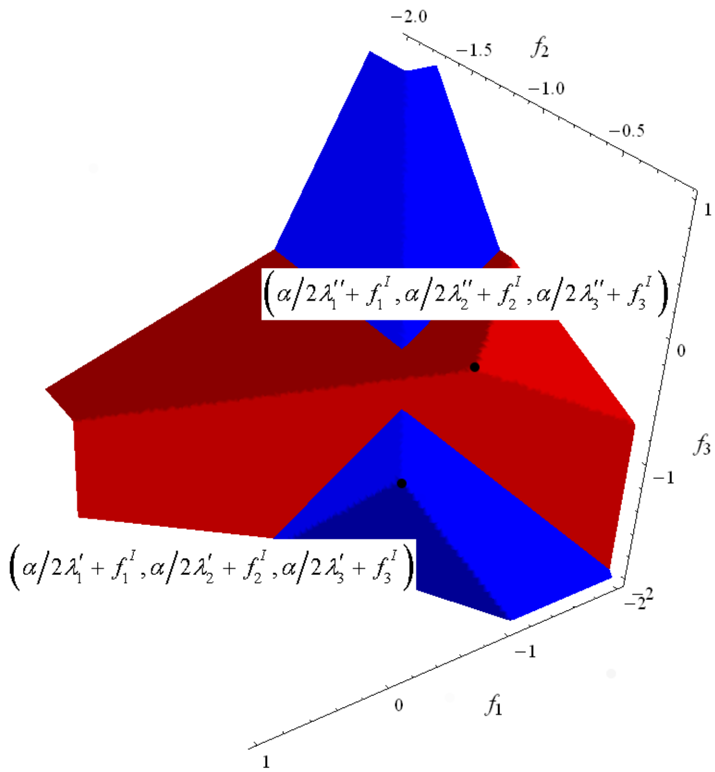

Figure 5. Two α-level sets of parameterized ASFs for $q = 2$ corresponding to different values of weighing coefficients $(\lambda_1', \lambda_2', \lambda_3')$ and $(\lambda_1'', \lambda_2'', \lambda_3'')$

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References

ALCADA-ALMEIDA, L., TRALHAO, L., SANTOS, L. AND COUTINHO-RODRIGUES, J. (2009) A multiobjective approach to locate emergency shelters and identify evacuation routes in urban area. *Geographical Analysis*, 41, 9–29.

AOUNI, B., DOUMPOS, M., PÉREZ-GLADISH, B. AND STEUER, R. E. (2018) On the increasing importance of multiple criteria decision aid methods for portfolio selection. *Journal of the Operational Research Society*, 69, 1525–1542.

CHRISTOFIDES, N. (1975) *Graph Theory. An Algorithmic Approach.* Academic

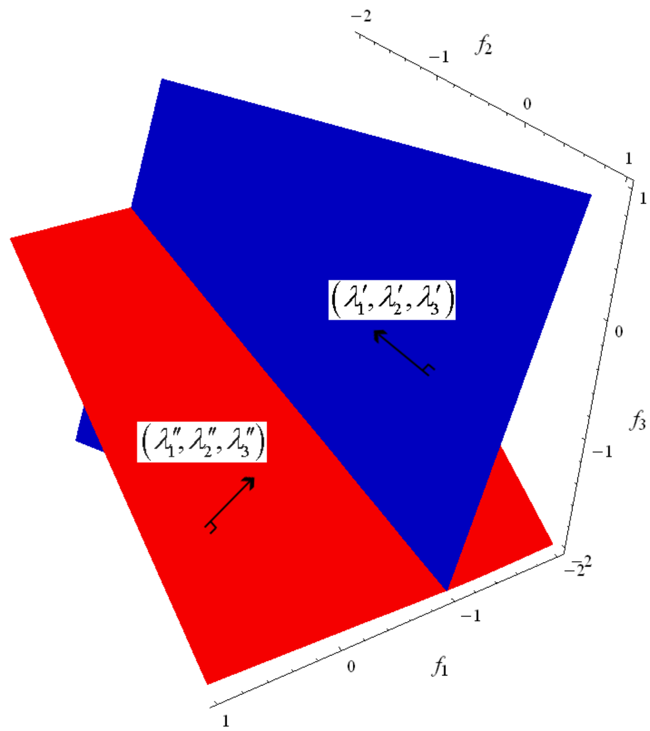

Figure 6. Two α-level sets of parameterized ASFs for $q = 3$ corresponding to different values of weighing coefficients $(\lambda'_1, \lambda'_2, \lambda'_3)$ and $(\lambda''_1, \lambda''_2, \lambda''_3)$

Press, New York.

EHRGOTT, M. (2000) *Multicriteria Optimization*. Springer, Berlin.

HELLWIG, Z. (1968) Zastosowanie metody taksonomicznej do typologicznego podziału krajów ze względu na poziom ich rozwoju oraz zasoby i strukture wykwalifikowanych kadr. *Przegląd Statystyczny* **15** (4), 307–326. (in Polish)

KOLM, P. N., TÜTÜNCÜ, R. AND FABOZZI, F.J. (2014) 60 Years of portfolio optimization: Practical challenges and current trends. *European Journal of Operational Research*, 234, 356–371.

MARKOWITZ, H.M. (1952) Portfolio selection. *The Journal of Finance*, **7** (1), 77–91.

MARKOWITZ, H.M. (1959) *Portfolio Selection: Efficient Diversification of Investments*. John Willey and Sons, New York.

MIETTINEN, K. (1999) *Nonlinear Multiobjective Optimization*. Kluwer Academic Publeshers, Boston.

MIETTINEN, K. AND MÄKELÄ, M. M. (2006) Synchronous approach in interactive multiobjective optimization. *European Journal of Operational Research*, 170, 909–922.

MIETTINEN, K., RUIZ, F. AND WIERZBICKI, A. P. (2009) Incorporating preference information in interactive reference point methods for multiobjective optimization. *Omega, The International Journal of Management Science*, 37, 450–462.

MONTONEN, O., RANTA, T. AND MÄKELÄ, M. M. (2019) Planning the schedule for the disposal of the spent nuclear fuel with interactive multiobjective optimization. *Algorithms*, **2** (12), 252.

NIKULIN, YU., MIETTINEN, K. AND MÄKELÄ, M. M. (2012) A new achievement scalarizing function based on parameterization in multiobjective optimization. *OR Spectrum*, **34** (1), 69–87.

RUIZ, F., LUQUE, M., MIGUEL, F. AND DEL MAR MUÑOZ, M. (2008) An additive achievement scalarizing function for multiobjective programming problems. *European Journal of Operational Research*, **188** (3), 683–694.

SAWARAGI, Y., NAKAYAMA, H. AND TANINO, T. (1985) *Theory of Multiobjective Optimization*. Academic Press, Orlando.

SLATER, M. (1950) Lagrange multipliers revisited. *Cowles Commision Discussion Paper: Mathematics*, 403.

STEUER, R. E. AND CHOO, E.-U. (1983) An interactive weighted Tchebyshev procedure for multiple objective programming. *Mathematical Programming*, 26, 326–344.

UTZ, S., WIMMER, M. AND STEUER, R. E. (2015) Tri-criterion modeling for constructing more-sustainable mutual funds. *European Journal of Operational Research*, 246, 331–338.

WIERZBICKI, A. P. (1980) The use of reference objectives in multiobjective optimization. In: G. Fandel and T. Gal, eds., *Multiple Criteria Decision Making Theory and Applications. MCDM Theory and Applications Proceedings. Lecture Notes in Economics and Mathematical Systems* **177**. Springer, Berlin, 468–486.

WIERZBICKI, A. P. (1986) A methodological approach to comparing parametric characterizations of efficient solutions. In: G. Fandel et al., eds., *Large-Scale Modelling and Interactive Decision Analysis. Lecture Notes in Economics and Mathematical Systems* **273**. Springer, Berlin, 27–45.

WIERZBICKI, A. P. (1999) Reference point approaches. In: T. Gal et al., eds., *Multicriteria Decision Making: Advances in MCDM Models, Theory and Applications*. Kluwer, Boston, 1–39.

WILPPU, O., MÄKELÄ, M. M. AND NIKULIN, Y. (2017) New two-slope parameterized achievement scalarizing functions for nonlinear multiobjective optimization. In: N.J. Daras and T.M. Rassias, eds., *Operations Research, Engineering, and Cyber Security*. Springer, Berlin/Heidelberg, **113**, 403–422.