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NUMERICAL SOLUTION OF TWO-DIMENSIONAL FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS BY CHEBYSHEV INTEGRAL OPERATIONAL MATRIX METHOD

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Abstract. This paper presents the Chebyshev Integral Operational Matrix Method (CIOMM) for the numerical solution of two-dimensional Fredholm Integro-Differential Equations (2D-FIDEs). The process of the method is obtaining the operational matrix of integration by evaluating a 2D integral of 2D Chebyshev polynomial basis functions and assuming approximate solutions of the 2D-FIDEs as a truncated 2D Chebyshev series. This leads to a system of linear algebraic equations which are solved to obtain the values of the unknown constants using Maple 18. Some numerical problems are solved to illustrate the practicability of the method.

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1. Introduction

Many problems in science, engineering, modelling and other disciplines are presented as integral and integro-differential equations. It is difficult to obtain the analytical solutions of most of these equations because of the complexity involved, therefore it is important to develop numerical methods to obtain the approximate solutions. Over the past fifty years, substantial progress in developing analytical and numerical solutions of integral equations of different kinds; linear and nonlinear cases have been considered [1]. Although several numerical methods for approximating the solutions of one-dimensional (1D) integral and integro-differential equations are known, only a few of them have been applied to two-dimensional (2D) problems. Methods for treating two-dimensional (2D) integral and integro-differential equations also deserve more consideration, as these equations have many applications in physics, mechanics, modelling, engineering and other applied sciences. [1] and [2], investigated the numerical solution of two-dimensional Fredholm integral equations by the Galerkin method using approximating subspace, a special space of spline functions. [3] and [4] developed the Regularization Method and the Differential Transform Method respectively for two-dimensional Fredholm integral equations of the first kind. applied 2D Chebyshev polynomials to solve 2D integral equations using a collocation scheme. [6] used integral collocation approximation methods for the numerical solution of linear 1D integro-differential equations. Integral collocation approximation methods were also employed by [7] for the numerical solution of high-orders linear Fredholm-Volterra integro-differential equations. [8] employed the Legendre Galerkin method for solving fractional integro-differential equations of the Fredholm type.

Recently, most works on 2D integral and integro-differential equations focused on the use of an operational matrix of integration and differentiation via orthogonal functions which also resulted into systems of algebraic equations and produced better results and faster in computation. For instance, [9] derived the operational matrices of integration, differentiation and product of Bernstein polynomials to solve problems involving calculus of variations, differential equations, optimal control and integral equations. [10] solved a class of non-linear Volterra integral equations by using operational matrices of two-dimensional triangular orthogonal functions. [11] presented a numerical solution of non-linear 2D Volterra-Fredholm integro-differential equations using operational matrices of integration and differentiation generated from a two-dimensional triangular function. Also, [12] presented a method based on operational matrices of Taylor polynomials to solve 1D linear Fredholm-Volterra integro-differential equations for which operational matrices obtained converted integro-differential equations to systems of algebraic equations without the use of the collocation scheme.

According to [13], a typical way to solve functional equations is to express the solution as a linear combination of the so-called basis functions. These basis functions can, for instance, be either orthogonal or non-orthogonal bases. Approximation by the orthogonal family of basis functions has found wide application in science and engineering. The most frequently used orthogonal functions are sine-cosine functions, block pulse functions, Legendre, Chebyshev and Laguerre polynomials. The main idea of using an orthogonal basis is that the problem under consideration reduces to a system of linear or nonlinear algebraic equations. Much work has been done with the application of 1D Chebyshev polynomials for the solutions of integral and integro-differential equations. For instance, [14] used a shifted 1D Chebyshev polynomial for the solution of 3D Volterra integral equations of the second kind. Various mathematical models, such as [15-20] and most of the references referenced therein, have been redefined in the context of fractional calculus and in epidemiology. In other to improve on the existing methods in the literature, this paper therefore presents a Chebyshev integral operational matrix method (CIOMM) for the numerical solutions of 2D Fredholm integro-differential equations. The computational time is reduced as compared to other methods in the literature. The general form of the class of problem considered in this work is given as:

$$u(x,t)+u_{xt}(x,t)+\int_{-1}^{1}\int_{-1}^{1}k(x,t,y,z)u(y,z)dydz=f(x,t);\ x,t\in[-1,1], \tag{1}$$

such that:

$$u(-1,-1)=u_0;\ u(-1,t)=g(t);\ u(x,-1)=h(x). \tag{2}$$

Where $(x,t)\in[-1,1]\times[-1,1]$, $u(x,t)$ is an unknown scalar-valued function, $f(x,t)$ and $k(x,t,y,z)$ are continuous functions on $[-1,1]^2$ and $[-1,1]^4$ respectively, u_{xt} is the derivative of $u(x,t)$ with respect to x and t, $h(x)$ and $g(t)$ are known functions and u_0 is a given number.

Equations like (1) above are usually solved by expressing the solution as a linear combination of basis functions which are either orthogonal or non-orthogonal. Some definitions and properties of Chebyshev polynomials and Chebyshev series expansion can be found in [13].

2. Construction of integral operational matrix of 2D Chebyshev polynomials

Chebyshev polynomials are a well-known family of orthogonal polynomials on the interval $[-1,1]$ and have many applications. They are widely used because of their good properties in the approximation of functions [21]. The problem of approximating a function is a central problem in numerical analysis due to its importance in the development of software for digital computers [22]. Chebyshev polynomials are employed as basis functions to approximate the solution of several numerical problems involving integral and integro-differential equations. According to [23], Chebyshev polynomials of the first kind valid in $[-1;1]$ are defined as

$$T_n(x)=cos(ncos^{-1}x), \tag{3}$$

where,

$$\begin{aligned} T_0(x)&=1,\\ T_1(x)&=x, \end{aligned} \tag{4}$$

and the recurrence relation is given as

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x);\qquad n\geq 1. \tag{5}$$

The orthogonality property of 1D Chebyshev polynomials $(T_i(x);\ i = 0,1,...N)$ with respect to the weight function:

$$w(x) = \frac{1}{\sqrt{1-x^2}}, \tag{6}$$

on the interval $[-1,1]$ is given by

$$(T_i(x),T_j(x))_{w(x)} = \int_{-1}^{1} T_i(x)T_j(x)w(x)dx = \begin{cases} 0, i \neq j \\ \frac{\pi}{2}, i = j \neq 0 \\ \pi, i = j = 0. \end{cases} \tag{7}$$

Given that the basis vector of 1D Chebyshev polynomials is $[T_0(x)\ T_1(x)\ ...\ T_N(x)]^T$, then the basis vector of 2D Chebyshev polynomials denoted by $(T_{ij}(x,t) = T_i(x)T_j(t);\ i,j = 0,...,N)$ is given as follows:

$$[T_0(x)T_0(t)\ \dots\ T_0(x)T_N(t)\ T_1(x)T_0(t)\ \dots\ T_N(x)T_N(t)]^T = (C_N \otimes B_N)^T \tag{8}$$

where $C_N = [T_0(x)\ T_1(x)\ ...\ T_N(x)]^T$ and $B_N = [T_0(t)\ T_1(t)\ ...\ T_N(t)]^T$ are both 1D Chebyshev vectors.

The orthogonality properties for the 2D Chebyshev polynomials with respect to the weight function:

$$w(x,t) = \frac{1}{\sqrt{1-x^2}\sqrt{1-t^2}}, \tag{9}$$

on the interval $[-1,1]\times[-1,1]$ is expressed as

$$(T_{ij}(x,t),T_{kl}(x,t))_{w(x,t)} = \int_{-1}^{1}\int_{-1}^{1} T_{ij}(x,t)T_{kl}(x,t)w(x,t)dxdt = \begin{cases} \frac{\pi^2}{4}, i = k \neq 0, j = l \neq 0 \\ \frac{\pi^2}{2}, i = k = 0, j = l \neq 0 \\ \frac{\pi^2}{2}, i = k \neq 0, j = l = 0 \\ \pi^2, i = k = 0, j = l = 0 \\ 0, else. \end{cases} \tag{10}$$

The matrix form obtained by the evaluation of 2D integral of Chebyshev polynomial basis functions is given by:

$$\int_{-1}^{x}\int_{-1}^{t} T_i(s)T_j(r)dsdr = QT_{ij}(x,t)\ ;\ i,j = 0,1,2, \tag{11}$$

where Q is the $(N+1)^2 \times (N+1)^2$ operational matrix of integration of the 2D Chebyshev basis functions.

3. Description of the proposed method

This section describes the proposed method mentioned earlier.
Chebyshev Integral Operational Matrix Method (CIOMM)
This work considers the 2D Fredholm integro-differential problem given in (1) and (2).
Any function $\phi(x,t)$ on $[-1,1]\times[-1,1]$ can be expanded by the 2D Chebyshev polynomials approximation and truncated as follows:

$$\phi(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{ij}T_{ij}(x,t)\approx\sum_{i=0}^{N}\sum_{j=0}^{N}\alpha_{ij}T_i(x)T_j(t), \tag{12}$$

where,

$$\alpha_{ij}=\frac{<\phi(x,t),T_{ij}(x,t)>_{w(x,t)}}{<T_{ij}(x,t),T_{ij}(x,t)>_{w(x,t)}}. \tag{13}$$

The first process of the method is to obtain the approximate solution of problem considered as a truncated 2D Chebyshev series defined by

$$u(x,t)\approx u_N(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{N}a_{ij}T_i(x)T_j(t), \tag{14}$$

and

$$u_{xt}(x,t)\approx\psi_N(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{N}e_{ij}T_i(x)T_j(t). \tag{15}$$

The matrix forms of (14) and (15) are given by:

$$u_N(x,t)=T^T(x,t)A, \tag{16}$$

$$\psi_N(x,t)=T^T(x,t)E, \tag{17}$$

where

$$A=[a_{00},\dots,a_{0N},\dots,a_{N0},\dots,a_{NN}]^T, \tag{18}$$

$$E=[e_{00},\dots,e_{0N},\dots,e_{N0},\dots,e_{NN}]^T, \tag{19}$$

are unknown vectors, and N is any natural number.
The other functions in (1) and (2) can also be expanded in terms of the 2D Chebyshev basis as

$$f(x,t)\approx f_N(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{N}f_{ij}T_i(x)T_j(t), \tag{20}$$

$$u_0 \approx \mu_N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} b_{ij} T_i(x) T_j(t), \tag{21}$$

$$h(x,t) \approx h_N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} h_{ij} T_i(x) T_j(t), \tag{22}$$

$$g(x,t) \approx g_N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} g_{ij} T_i(x) T_j(t), \tag{23}$$

$$k(x,t,y,z) \approx k_N(x,t,y,z) = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{l=0}^{N}\sum_{m=0}^{N} k_{ijlm} T_i(x) T_j(t) T_l(y) T_m(z). \tag{24}$$

Having their respective matrix forms:

$$f_N(x,t) = T^T(x,t)F, \tag{25}$$

$$\mu_N(x,t) = T^T(x,t)B, \tag{26}$$

$$h_N(x) = T^T(x,t)H, \tag{27}$$

$$g_N(t) = T^T(x,t)G, \tag{28}$$

$$k_N(x,t,y,z) = T^T(x,t)KT(y,z). \tag{29}$$

Where

$$F = [f_{00} \ \dots \ f_{0N} \ \dots \ f_{N0} \ \dots \ f_{NN}]^T, \tag{30}$$

$$B = [b_{00} \ \dots \ b_{0N} \ \dots \ b_{N0} \ \dots \ b_{NN}]^T, \tag{31}$$

$$H = [h_{00} \ \dots \ h_{0N} \ \dots \ h_{N0} \ \dots \ h_{NN}]^T, \tag{32}$$

$$G = [g_{00} \ \dots \ g_{0N} \ \dots \ g_{N0} \ \dots \ g_{NN}]^T. \tag{33}$$

The components of (30)-(33) are derived from (13); K is an $(N+1)^2 \times (N+1)^2$ matrix whose elements are obtained by:

$$k_{ijlm} = \frac{\Big(T_{ij}(x,t), \left(k(x,t,y,z), T_{lm}(y,z)\right)_{w(y,z)}\Big)_{w(x,t)}}{\left(T_{ij}(x,t), T_{ij}(x,t)\right)_{w(x,t)} \left(T_{lm}(y,z), T_{lm}(y,z)\right)_{w(y,z)}} \ ; \ i,j,l,m = 0,1,...,N. \tag{34}$$

We now substitute (16), (17), (25) and (29) into (1) to obtain:

$$T^T(x,t)A + T^T(x,t)E + T^T(x,t)K\left(\int_{-1}^{1}\int_{-1}^{1} T^T(y,z)T(y,z)dydz\right)A = T^T(x,t)F. \tag{35}$$

(35) is simplified to obtain

$$A + E + K\Omega A = F, \tag{36}$$

where

$$\Omega = \int_{-1}^{1}\int_{-1}^{1} T^T(y,z)T(y,z)dydz. \tag{37}$$

The conditions in (2) and the unknown function $u(x,t)$ are expressed as polynomials to establish a relation between the unknown function and its derivatives in terms of the operational matrix of integration in (11)

$$\begin{aligned} u(x,t) + u(-1,-1) - u(-1,t) - u(x,-1) &= \int_{-1}^{x}\int_{-1}^{t} u_{sr} dsdr \\ &= \int_{-1}^{x}\int_{-1}^{t} T^T(s,r)E\, dsdr. \end{aligned} \tag{38}$$

By the integral operational matrix of (11), the RHS of (38) becomes

$$\int_{-1}^{x}\int_{-1}^{t} T^T(s,r)E\, dsdr = Q^T E\, T^T(x,t), \tag{39}$$

substitute (16), (26), (27), (28) and (39) into (38) to obtain

$$T^T(x,t)A + T^T(x,t)B - T^T(x,t)G - T^T(x,t)H = Q^T E\, T^T(x,t). \tag{40}$$

Simplifying further we obtain

$$A + B - G - H = Q^T E, \tag{41}$$

substituting (36) into (41) to obtain

$$A + B - G - H = Q^T(F - A - K\Omega A), \tag{42}$$

(42) gives a system of linear algebraic equations and are solved by Maple18 to obtain the unknown vector A. The results obtained are compared with the exact solutions of the given 2D FIDEs.

4. Numerical examples

In this section, examples of 2D FIDE are given to illustrate the method described in the previous sections.

Example 1. Consider the Fredholm integro-differential equation:

$$u(x,t)+u_{xt}(x,t)+\int_{-1}^{1}\int_{-1}^{1}(x-yz)u(y,z)dydz = xt-sin(t)+1, \quad (43)$$

where

$$\begin{aligned} u(-1,-1) &= 1+sin(1), \\ u(-1,t) &= -t-sin(t) \\ u(x,-1) &= -x+sin(1). \end{aligned} \quad (44)$$

The exact solution is $u(x,t) = xt - sin(t)$ for $x,t \in [-1,1]$.
Following the procedure described in section 4, the approximate solution for example 1 at $N = 2$ is given as:
$u_2(x,t) = 0.0012982147 + 0.09898364t - 0.09148294t^2 - 0.29826123x - 0.099745312xt - 0.121315t^2x + 0.799512434x^2 - 0.4994562tx^2 + 0.00042637184t^2x^2$

The approximate solutions of the 2D integro-differential problems under consideration for N = 2 as well as N = 4 and N = 6, using the integral operational matrix of 2D Chebyshev polynomials are compared with their exact solutions $u(x,t)$ and the results are presented in Tables 1-3.
Figures 1-3 illustrate the exact solutions and the approximate solutions obtained by CIOMM when N = 2. The absolute error function is given by $Er(x,t) = |u(x,t) - u_N(x,t)|$.
Comparisons are made between the approximate solution and the exact solution to illustrate the validity of the method.

Table 1. Numerical results and Absolute Errors for example 1

(x,t)	$u(x,t)$	$u_2(x,t)$	Absolute Error N = 2	Absolute Error N = 4	Absolute Error N = 6
(–1,–1)	1.017452406	2.181028788	0.11636E-1	7.1835E-4	3.4174E-7
(–0.8,–0.8)	1.357356091	1.548068246	1.9071E-1	3.5528E-4	7.2269E-7
(–0.6,–0.6)	0.924642473	1.019902141	9.5260E-2	3.1992E-5	6.3275E-9
(–0.4,–0.4)	0.549418342	0.590243757	4.0825E-2	7.1962E-6	6.0369E-9
(–0.2,–0.2)	0.238669331	0.252805362	1.4136E-2	3.7042E-6	2.4872E-10
(0,0)	0.000000000	0.001298215	1.2982E-3	3.4117E-6	4.1716E-10
(0.2,0.2)	–0.158669331	–0.170567441	1.1898E-2	1.7850E-4	3.5296E-9
(0.4,0.4)	–0.229418342	–0.269082373	3.9664E-2	9.9921E-4	7.2251E-8
(0.6,0.6)	–0.204642473	–0.300538362	9.5896E-2	4.2166E-4	5.9573E-8
(0.8,0.8)	–0.077356091	–0.271228202	1.9387E-1	3.1274E-4	1.2737E-7
(1,1)	0.982547594	–0.187445699	0.1170E-1	6.7352E-4	3.7935E-7

Example 2. Consider the Fredholm integro-differential equation:

$$u(x,t)+u_{xt}(x,t)+\int_{-1}^{1}\int_{-1}^{1}(x+t)u(y,z)dydz = 2x+x^2t, \quad (45)$$

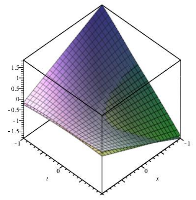

Fig. 1. 2D graph for exact solution and approximate solution of example 1

where

$$\begin{aligned} u(-1,-1) &= -1, \\ u(-1,t) &= t \\ u(x,-1) &= -x^2, \end{aligned} \tag{46}$$

for $x,t \in [-1,1]$ with exact solution $u(x,t) = x^2 t$.
Following the same procedure as in section 4, the approximate solution obtained is:
$u_2(x,t) = 0.0008794363 + 0.0004833378t + 0.0001003148t^2 + 0.0001027962x + 0.0009424387xt + 0.0002515847xt^2 + 0.0053774376x^2 + 0.8996554104x^2t + 0.004857426t^2x^2$
The approximate solution is compared with the exact solution in Table 2.

Table 2. Numerical results and Absolute Errors for example 2

(x,t)	$u(x,t)$	$u_2(x,t)$	Absolute Error N = 2	Absolute Error N = 4	Absolute Error N = 6
(–1,–1)	–1.000000000	–0.888336076	1.1166E-1	7.9271E-3	7.2922E-6
(–0.8,–0.8)	–0.512000000	–0.454243329	5.7757E-2	3.7234E-4	4.7665E-7
(–0.6,–0.6)	–0.216000000	–0.190911364	2.5089E-2	3.2174E-4	2.9221E-7
(–0.4,–0.4)	–0.064000000	–0.055797484	8.2025E-3	6.2309E-5	1.6913E-7
(–0.2,–0.2)	–0.008000000	–0.006172467	1.8275E-3	6.9794E-5	1.0205E-8
(0,0)	0.000000000	0.000879436	8.7943E-4	2.1584E-6	4.1383E-8
(0.2,0.2)	0.008000000	0.008460499	4.6050E-4	2.5352E-6	1.2535E-7
(0.4,0.4)	0.064000000	0.059859518	4.1405E-3	6.6147E-5	2.1476E-7
(0.6,0.6)	0.216000000	0.198551819	1.7448E-2	2.1958E-5	3.5853E-7
(0.8,0.8)	0.512000000	0.468199249	4.3801E-2	5.1761E-5	5.6086E-6
(1,1)	1.000000000	0.912650183	8.7350E-2	1.0018E-4	8.2785E-6

Example 3. Consider the Fredholm integro-differential equation:

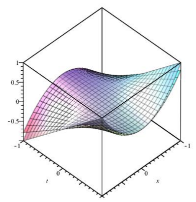

Fig. 2. 2D graph for exact solution and approximate solution of example 2

$$u(x,t)+u_{xt}(x,t)+\int_{-1}^{1}\int_{-1}^{1}(y^2-xz\sin(t))u(y,z)dydz=x^2-x\cos(t)+\sin(t)+\frac{4}{5}, \tag{47}$$

where

$$\begin{aligned} u(-1,-1)&=1+\cos(1),\\ u(-1,t)&=1+\cos(t),\\ u(x,-1)&=x^2-x\cos(1). \end{aligned} \tag{48}$$

Following the same procedure we obtain the approximate solution:
$u_2(x,t)=0.0098942423+0.09898364t-0.009148294t^2-0.799626123x-0.499745312xt-0.121315xt^2+0.999512434x^2-0.04994562x^2t+0.00042637184x^2t^2$.
Table 3 shows the numerical comparisons with the exact solution $u(x,t)=x^2-xcos(t)$ for $N=2,4,6$.

Table 3. Numerical results and Absolute Errors for example 3

(x,t)	$u(x,t)$	$u_2(x,t)$	Absolute Error N = 2	Absolute Error N = 4	Absolute Error N = 6
(–1,–1)	1.999847695	2.616766388	6.1692E-1	1.5041E-3	5.3385E-6
(–0.8,–0.8)	1.439922019	1.745112800	3.0519E-1	3.1158E-3	2.1228E-7
(–0.6,–0.6)	0.959967102	1.081324360	1.2136E-1	6.5722E-4	4.1384E-7
(–0.4,–0.4)	0.559990252	0.591815334	3.1825E-2	4.1994E-5	5.3487E-7
(–0.2,–0.2)	0.239997319	0.242989936	2.9926E-3	7.1273E-5	3.8186E-8
(0,0)	0.000000000	0.001242328	1.2423E-3	5.0237E-6	7.9119E-9
(0.2,0.2)	–0.159998782	–0.167043384	7.0446E-3	7.0001E-5	6.1743E-8
(0.4,0.4)	–0.239990252	–0.295493145	5.5503E-2	8.1724E-5	1.3338E-8
(0.6,0.6)	–0.239967102	–0.417742955	1.7778E-1	2.1162E-4	3.9156E-7
(0.8,0.8)	–0.159922019	–0.567438867	4.0752E-1	4.7344E-4	4.3477E-6
(1,1)	0.000152395	–0.778236989	7.7809E-1	6.0185E-4	8.5104E-6

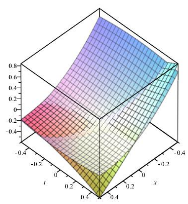

Fig. 3. 2D graph for exact solution and approximate solution of example 3

5. Conclusion

In this work, an operational matrix of integration of 2D Chebyshev polynomials basis functions was constructed and used to convert 2D Fredholm integro-differential problems to systems of algebraic equations without the use of collocation. A major advantage of the CIOMM is the establishment of a relationship between the unknown function and its derivatives through an operational matrix of integration. Numerical results and graphs show that the method yields good approximations even for a small value of N for the problems considered. On the whole, it can be concluded that the numerical scheme presented in this study is easy-to-implement and suitable for solving problems of similar type with a reasonable level of accuracy. The operational matrices derived in this paper can be used to numerically solve problems involving the fractional order left-sided mixed Riemann-Liouville integral operator.

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