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MINIMAL–PHASE REALIZATIONS FOR POSITIVE LINEAR SYSTEMS

Abstract

The problem of minimal-phase realization for continuous-time and discrete-time linear systems is addressed. Necessary and sufficient conditions for the existence of minimal-phase realizations for the linear systems are established. A procedure for computation of the realizations is proposed and illustrated by numerical examples.

INTRODUCTION

Determination of the state space equations for given transfer matrices is a classical problem, called the realization problem, which has been addressed in many papers and books [1, 2, 9, 10, 11, 29-31]. An overview of the positive realization problem is given in [1, 2, 12, 29]. The realization problem for positive continuous-time and discrete-time linear system has been considered in [4-7, 13-16, 20, 21, 23, 27-30] and for linear systems with delays in [4, 8, 14, 18, 27-29]. The realization problem for fractional linear systems has been analyzed in [19, 22, 24, 25, 29, 30] and for positive 2D hydrid linear systems in [17, 18, 26]. A new modified state variable diagram method for determination of positive realizations with reduced number of delays for given proper transfer matrices has been proposed in [3].

In this paper a new approach to the minimal-phase realization problem for linear systems will be proposed. Necessary and sufficient conditions for the existence of the solution to the problem will be established and a procedure for computation of the realizations will be proposed.

The paper is organized as follows. In section 1 some preliminaries on continuous-time and discrete-time linear positive systems and minimal-phase-realizations are given. The necessary and sufficient conditions for the existence of the minimal-phase realizations and a procedure for computation of the realizations for positive continuous-time linear are proposed in section 2 and for positive discrete-time systems in section 3.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries, $\Re^{n\times m}(s)$ - the set of $n\times m$ rational matrices in s with real coefficients, Z_+ - the set of nonnegative integers, I_n - the $n\times n$ identity matrix

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1. PRELIMINARIES

Consider the continuous-time linear system

$$\dot{x} = Ax + Bu\,, \tag{1a}$$
$$y = Cx + Du\,, \tag{1b}$$

where $x\in\Re^n$, $u\in\Re^m$, $y\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Definition 1. [12] The system (1) is called (internally) positive if $x = x(t)\in\Re_+^n$ and $y = y(t)\in\Re_+^p$, $t\in[0,+\infty]$ for all $x_0 = x(0)\in\Re_+^n$ and $u = u(t)\in\Re_+^m$, $t\in[0,+\infty]$.

Theorem 1. [12] The system (1) is positive if and only if

$$A\in M_n\,,\; B\in\Re_+^{n\times m}\,,\; C\in\Re_+^{p\times n}\,,\; D\in\Re_+^{p\times m}\,, \tag{2}$$

where M_n is the set of $n\times n$ Metzler matrices, i.e. the matrices with nonnegative off-diagonal entries.

The transfer matrix of (1) is given by

$$T(s) = C[I_n s - A]^{-1}B + D = \frac{N(s)}{d(s)} \in \Re^{p\times m}(s)\,, \tag{3}$$

where $N(s)$ is the polynomial matrix and $d(s)$ is the polynomial.

For single-input single-output (SISO, $m = p = 1$) linear system the transfer function can be written in the form

$$T(s) = \frac{n(s)}{d(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}\,. \tag{4}$$

Definition 2. The roots s_1, s_2,…, s_n of the equation

$$\begin{aligned} d(s) &= s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \\ &= (s - s_1)(s - s_2)\ldots(s - s_n) = 0 \end{aligned} \tag{5}$$

are called the poles of the linear system.

Definition 3. The roots s_1^0, s_2^0,…, s_n^0 of the equation

$$\begin{aligned} n(s) &= b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0 \\ &= b_n(s - s_1^0)(s - s_2^0)\ldots(s - s_n^0) = 0 \end{aligned} \tag{6}$$

are called the zeros of the linear system.

The poles s_1, s_2,…, s_n and the zeros s_1^0, s_2^0,…, s_n^0 are called distinct if $s_i \neq s_j$ for $i\neq j$ and $s_i^0 \neq s_j^0$ for $i\neq j$, $i, j = 1,\ldots,n$, respectively.

Definition 4. The linear system is called minimal-phase if

$$\mathrm{Re}\, s_k < 0 \text{ and } \mathrm{Re}\, s_k^0 < 0 \text{ for } k = 1,\ldots,n\,, \tag{7}$$

where Re denotes the real part of the complex number.

Definition 5. [12] The positive system (1) is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \text{ for all } x_0 \in \Re_+^n\,. \tag{8}$$

Theorem 2. [12] The positive system (1) is asymptotically stable if and only if

$$\mathrm{Re}\lambda_k < 0 \text{ for } k = 1,\ldots,n, \quad (9)$$

where λ_k is the eigenvalue of the matrix $A \in M_n$ and

$$\det[I_n\lambda - A] = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n). \quad (10)$$

Note that the set of poles $\{s_1, s_2, \ldots, s_n\}$ in general case is the subset of the set of eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ [10].

Now let us consider the discrete-time linear system

$$x_{i+1} = \bar{A}x_i + \bar{B}u_i, \; x_i \in Z_+ = \{0,1,..\} \quad (11a)$$

$$y_i = \bar{C}x_i + \bar{D}u_i, \quad (11b)$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $\bar{A} \in \Re^{n\times n}$, $\bar{B} \in \Re^{n\times m}$, $\bar{C} \in \Re^{p\times n}$, $\bar{D} \in \Re^{p\times m}$.

Definition 6. [12] The system (11) is called (internally) positive if $x_i \in \Re_+^n$ and $y_i \in \Re_+^p$, $i \in Z_+$ for all $x_0 \in \Re_+^n$ and $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 3. [12] The system (11) is positive if and only if

$$\bar{A} \in \Re_+^{n\times n}, \; \bar{B} \in \Re_+^{n\times m}, \; \bar{C} \in \Re_+^{p\times n}, \; \bar{D} \in \Re_+^{p\times m}. \quad (12)$$

The transfer matrix of (11) is given by

$$T(z) = \bar{C}[I_n z - \bar{A}]^{-1}\bar{B} + \bar{D} = \frac{N(z)}{d(z)} \in \Re^{p\times m}(z), \quad (13)$$

where $N(z)$ is the polynomial matrix and $d(z)$ is the polynomial. For single-input single-output (SISO, $m = p = 1$) linear system the transfer function can be written in the form

$$T(z) = \frac{n(z)}{d(z)} = \frac{\bar{b}_n z^n + \bar{b}_{n-1}z^{n-1} + \ldots + \bar{b}_1 z + \bar{b}_0}{z^n + \bar{a}_{n-1}z^{n-1} + \ldots + \bar{a}_1 z + \bar{a}_0}. \quad (14)$$

Definition 7. The roots $z_1, z_2, \ldots, z_n$ of the equation

$$\begin{aligned} d(z) &= z^n + \bar{a}_{n-1}z^{n-1} + \ldots + \bar{a}_1 z + \bar{a}_0 \\ &= (z - z_1)(z - z_2)\ldots(z - z_n) = 0 \end{aligned} \quad (15)$$

are called the poles of the linear system.

Definition 8. The roots $z_1^0, z_2^0, \ldots, z_n^0$ of the equation

$$\begin{aligned} n(z) &= \bar{b}_n z^n + \bar{b}_{n-1}z^{n-1} + \ldots + \bar{b}_1 z + \bar{b}_0 \\ &= \bar{b}_n(z - z_1^0)(z - z_2^0)\ldots(z - z_n^0) = 0 \end{aligned} \quad (16)$$

are called the zeros of the linear system.

The poles $z_1, z_2, \ldots, z_n$ and the zeros $z_1^0, z_2^0, \ldots, z_n^0$ are called distinct if $z_i \neq z_j$ for $i \neq j$ and $z_i^0 \neq z_j^0$ for $i \neq j$, $i, j = 1,\ldots,n$, respectively.

Definition 9. The linear system (11) is called minimal-phase if

$$|z_k| < 1 \text{ and } |z_k^0| < 1 \text{ for } k = 1,\ldots,n, \quad (17)$$

where $|\;|$ denotes the module of the complex number.

Definition 10. [12] The positive system (11) is called asymptotically stable if

$$\lim_{i\to\infty} x_i = 0 \text{ for all } x_0 \in \Re_+^n. \quad (18)$$

Theorem 4. [12] The positive system (11) is asymptotically stable if and only if

$$|\bar{\lambda}_k| < 1 \text{ for } k = 1,\ldots,n, \quad (19)$$

where $\bar{\lambda}_k$ is the eigenvalue of the matrix $\bar{A} \in \Re_+^{n\times n}$ and

$$\det[I_n\bar{\lambda} - \bar{A}] = (\bar{\lambda} - \bar{\lambda}_1)(\bar{\lambda} - \bar{\lambda}_2)\ldots(\bar{\lambda} - \bar{\lambda}_n). \quad (20)$$

Definition 11. The matrices A, B, C, D satisfying (2) ($\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ satisfying (12)) are called a positive realization of a given transfer matrix $T(s)$ ($T(z)$) if they fulfill the equality (3) ((13)).

2. POSITIVE MINIMAL-PHASE REALIZATIONS OF CONTINUOUS-TIME LINEAR SYSTEMS

First let us consider the SISO continuous-time linear system with the transfer function (4). From (4) we have

$$D = \lim_{s\to\infty} T(s) = b_n \quad (21)$$

and the strictly proper transfer function has the form

$$\begin{aligned} T_{sp}(s) &= T(s) - D = C[I_n s - A]^{-1}B \\ &= \frac{\hat{b}_{n-1}s^{n-1} + \ldots + \hat{b}_1 s + \hat{b}_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = \frac{\hat{n}(s)}{d(s)}, \end{aligned} \quad (22a)$$

where

$$\hat{b}_k = b_k - b_n a_k, \; k = 0,1,\ldots,n-1, \quad (22b)$$

$$\hat{n}(s) = \hat{b}_{n-1}s^{n-1} + \ldots + \hat{b}_1 s + \hat{b}_0. \quad (22c)$$

It is assumed that the poles $s_1, s_2, \ldots, s_n$ and the zeros s_1^0, $s_2^0, \ldots, s_{n-1}^0$ of (22) are real, negative, distinct and satisfy the conditions

$$s_k \le s_k^0 \le s_{k+1} \text{ for } k = 1,\ldots,n-1. \quad (23)$$

It is well-known [29] that the strictly proper transfer function (22) can be written in the form

$$T_{sp}(s) = \sum_{k=1}^{n} \frac{T_k}{s - s_k}, \quad (24a)$$

where

$$T_k = \lim_{s\to s_k}(s - s_k)T_{sp}(s) = \frac{\hat{n}(s_k)}{\prod_{\substack{j=1 \\ j\neq k}}^{n}(s_k - s_j)}. \quad (24b)$$

Note that $T_k > 0$ for $k = 1,\ldots,n$ if and only if the poles and zeros are distinct and satisfy the condition (23). In this case we can choose $c_k > 0$, $b_k > 0$ so that

$$T_k = c_k b_k, \; k = 1,\ldots,n \quad (25)$$

and the matrices

$$A = \mathrm{diag}[s_1 \;\; s_2 \;\; \cdots \;\; s_n] \in M_n, \; B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \Re_+^{n\times 1},$$

$$C = [c_1 \;\; c_2 \;\; \cdots \;\; c_n] \in \Re_+^{1\times n} \quad (26)$$

are a positive realization of the transfer function (22).

Using (26) it is easy to check that

$$T_{sp}(s)=C[I_n s-A]^{-1}B$$

$$=[c_1 \quad c_2 \quad \cdots \quad c_n]\,\mathrm{diag}[(s-s_1)^{-1} \quad (s-s_2)^{-1} \quad \cdots \quad (s-s_n)^{-1}]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$=\sum_{k=1}^{n}\frac{c_k b_k}{s-s_k}=\sum_{k=1}^{n}\frac{T_k}{s-s_k}. \tag{27}$$

By Definition 4 the realization is minimal-phase if the conditions

$$\mathrm{Re}\,s_k<0 \text{ for } k=1,\dots,n \text{ and } \mathrm{Re}\,s_k^0<0 \text{ for } k=1,\dots,n-1, \tag{28}$$

are satisfied. Therefore, the following theorem has been proved.

Theorem 5. There exists minimal-phase realization (26), (21) of the transfer function (4) if and only if the poles and zeros of (22) are distinct, real, negative and the conditions (23) are satisfied.

For computation of the minimal-phase realization of the transfer function (4) we have the following procedure.

Procedure 1.

Step 1. Using (21) and (22) compute D and the strictly proper transfer function $T_{sp}(s)$.

Step 2. Compute the poles s_1, $s_2, \dots, s_n$ and the zeros s_1^0, $s_2^0, \dots, s_{n-1}^0$ of $T_{sp}(s)$.

Step 3. Using (4) and (25) compute T_k, $k=1,\dots,n$ and choose b_k, c_k for $k=1,\dots,n$.

Step 4. Using (26) find the matrices A, B, C.

Example 1. Find a minimal-phase realization of the transfer function

$$T(s)=\frac{2s^3+19s^2+52s+38}{s^3+9s^2+23s+15}. \tag{29}$$

Using Procedure 1 and (29) we obtain the following:

Step 1. Using (21), (22) and (29) we obtain

$$D=\lim_{s\to\infty}T(s)=\lim_{s\to\infty}\frac{2s^3+19s^2+52s+38}{s^3+9s^2+23s+15}=2 \tag{30}$$

and

$$T_{sp}(s)=T(s)-D=\frac{s^2+6s+8}{s^3+9s^2+23s+15}. \tag{31}$$

Step 2. The poles and zeros of (31) are

$$s_1=-1,\ s_2=-3,\ s_3=-5 \text{ and } s_1^0=-2,\ s_2^0=-4, \tag{32}$$

since $(s+1)(s+3)(s+5)=s^3+9s^2+23s+15$ and $(s+2)(s+4)=s^2+6s+8$.

Step 3. Using (24b), (25) and (31) we compute

$$T_1=\lim_{s\to s_1}(s-s_1)T_{sp}(s)=\left.\frac{(s+2)(s+4)}{(s+3)(s+5)}\right|_{s=-1}=\frac{3}{8}=c_1b_1,$$

$$T_2=\lim_{s\to s_2}(s-s_2)T_{sp}(s)=\left.\frac{(s+2)(s+4)}{(s+1)(s+5)}\right|_{s=-3}=\frac{1}{4}=c_2b_2, \tag{33a}$$

$$T_3=\lim_{s\to s_3}(s-s_3)T_{sp}(s)=\left.\frac{(s+2)(s+4)}{(s+1)(s+3)}\right|_{s=-5}=\frac{3}{8}=c_3b_3$$

and we choose

$$b_1=1,\ b_2=\frac{1}{2},\ b_3=\frac{1}{4},\ c_1=\frac{3}{8},\ c_2=\frac{1}{2},\ c_3=\frac{3}{2}. \tag{33b}$$

Step 4. Using (26), (32) and (33) we obtain

$$A=\begin{bmatrix}-1 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -5\end{bmatrix},\ B=\begin{bmatrix}1\\ \frac{1}{2}\\ \frac{1}{4}\end{bmatrix},\ C=\begin{bmatrix}\frac{3}{8} & \frac{1}{2} & \frac{3}{2}\end{bmatrix}. \tag{34}$$

The desired positive minimal-phase realization is given by (34) and (30).

Remark 1. For different choice of the entries of matrices B and C for given $T_k>0$, $k=1,\dots,n$ we obtain different positive minimal-phase realizations of the transfer function (4).

Now let us consider the m-inputs and p-outputs (MIMO) continuous-time linear system with the strictly proper transfer matrix

$$T_{sp}(s)=\frac{N(s)}{d(s)}\in\Re^{p\times m}(s), \tag{35a}$$

where

$$d(s)=(s-s_1)(s-s_2)\dots(s-s_n), \tag{35b}$$

$$N(s)=\begin{bmatrix}(s-s_{11}^{0,1})\dots(s-s_{11}^{0,n_{11}}) & \cdots & (s-s_{1m}^{0,1})\dots(s-s_{1m}^{0,n_{1m}})\\ \vdots & \ddots & \vdots\\ (s-s_{p1}^{0,1})\dots(s-s_{p1}^{0,n_{p1}}) & \cdots & (s-s_{pm}^{0,1})\dots(s-s_{pm}^{0,n_{pm}})\end{bmatrix} \tag{35c}$$

with distinct real negative poles s_1, $s_2, \dots, s_n$ and distinct real negative zeros $s_{11}^{0,1},\dots,s_{11}^{0,n_{11}}$, $s_{1m}^{0,1},\dots,s_{pm}^{0,n_{pm}}$.

The transfer matrix (35) can be written in the form

$$T_{sp}(s)=\sum_{k=1}^{n}\frac{T_k}{s-s_k}, \tag{36a}$$

where

$$T_k=\lim_{s\to s_k}(s-s_k)T_{sp}(s)=\frac{N(s_k)}{\prod_{\substack{j=1\\ j\neq k}}^{n}(s_k-s_j)} \tag{36b}$$

and

$$\mathrm{rank}\,T_k=r_k\le\min(m,p). \tag{37}$$

It is easy to check that if the conditions

$$s_k\le s_{ij}^{0,k}\le s_{k+1} \text{ for } i=1,\dots,p,\ j=1,\dots,m,\ k=1,\dots,n_{ij} \tag{38}$$

are satisfied then $T_k\in\Re_+^{p\times m}$ for $k=1,\dots,n$ and it can be written as the product

$$T_k=C_kB_k, \tag{39a}$$

where

$$C_k\in\Re_+^{p\times r_k},\ B_k\in\Re_+^{r_k\times m} \text{ and}$$
$$\mathrm{rank}\,C_k=\mathrm{rank}\,B_k=r_k,\ k=1,\dots,n. \tag{39b}$$

In a similar way as for SISO systems it can be shown that the matrices

$$A = \mathrm{blockdiag}[I_{r_1}s_1 \quad \cdots \quad I_{r_n}s_n] \in M_r,$$

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} \in \Re_+^{r\times m},\ C = [C_1 \quad \cdots \quad C_n] \in \Re_+^{p\times r},\ r = \sum_{i=1}^{n} r_i \tag{40}$$

are a positive realization of the matrix (35).

Therefore, the following theorem has been proved.

Theorem 6. There exists a minimal-phase realization (40) of the strictly proper transfer matrix (35) if and only if the poles and zeros are distinct, real, negative and the conditions (38) are satisfied.

For computation of the minimal-phase realization (40) of the transfer matrix (35) Procedure 1 with some evident modifications can be applied.

Example 2. Find the minimal-phase realization of the transfer matrix

$$T_{sp}(s) = \frac{1}{s^3 + 9s^2 + 23s + 15}\begin{bmatrix} s^2+6s+8 & s^2+5s+4 \\ s^2+7s+10 & s^2+6s+8 \end{bmatrix}. \tag{41}$$

Using Procedure 1 we obtain the following:

Step 1. In this case the matrix $D = 0$.

Step 2. The poles of (3.21) are $s_1 = -1$, $s_2 = -3$, $s_3 = -5$ and the zeros are $s_{11}^{01} = -2$, $s_{11}^{02} = -4$, $s_{12}^{01} = -1$, $s_{12}^{02} = -4$, $s_{21}^{01} = -2$, $s_{21}^{02} = -5$, $s_{22}^{01} = -2$, $s_{22}^{02} = -4$.

The matrix (41) can be written in the form

$$T_{sp}(s) = \frac{1}{(s+1)(s+3)(s+5)}\begin{bmatrix} (s+2)(s+4) & (s+1)(s+4) \\ (s+2)(s+5) & (s+2)(s+4) \end{bmatrix}. \tag{42}$$

Step 3. Using (36) and (42) we obtain

$$\begin{aligned}
T_1 &= \lim_{s\to s_1}(s-s_1)T_{sp}(s) \\
&= \frac{1}{(s+3)(s+5)}\begin{bmatrix} (s+2)(s+4) & (s+1)(s+4) \\ (s+2)(s+5) & (s+2)(s+4) \end{bmatrix}_{s=-1} = \frac{1}{8}\begin{bmatrix} 3 & 0 \\ 4 & 3 \end{bmatrix}, \\
T_2 &= \lim_{s\to s_2}(s-s_2)T_{sp}(s) \\
&= \frac{1}{(s+1)(s+5)}\begin{bmatrix} (s+2)(s+4) & (s+1)(s+4) \\ (s+2)(s+5) & (s+2)(s+4) \end{bmatrix}_{s=-3} = \frac{1}{4}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \\
T_3 &= \lim_{s\to s_3}(s-s_3)T_{sp}(s) \\
&= \frac{1}{(s+1)(s+3)}\begin{bmatrix} (s+2)(s+4) & (s+1)(s+4) \\ (s+2)(s+5) & (s+2)(s+4) \end{bmatrix}_{s=-5} = \frac{1}{8}\begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix}.
\end{aligned} \tag{43}$$

In this case we choose

$$\begin{aligned}
&\mathrm{rank}\, T_1 = 2,\ T_1 = C_1B_1,\ C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ B_1 = \frac{1}{8}\begin{bmatrix} 3 & 0 \\ 4 & 3 \end{bmatrix}, \\
&\mathrm{rank}\, T_2 = 2,\ T_2 = C_2B_2,\ C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ B_2 = \frac{1}{4}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \\
&\mathrm{rank}\, T_3 = 2,\ T_3 = C_3B_3,\ C_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ B_3 = \frac{1}{8}\begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix}.
\end{aligned} \tag{44}$$

Step 4. Using (40) and (44) we obtain the desired realization in the form

$$A = \mathrm{diag}[-1 \quad -1 \quad -3 \quad -3 \quad -5 \quad -5],\ B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 3 & 0 \\ 4 & 3 \\ 2 & 4 \\ 4 & 2 \\ 3 & 4 \\ 0 & 3 \end{bmatrix},$$

$$C = [C_1 \quad C_2 \quad C_3] = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}. \tag{45}$$

3. POSITIVE MINIMAL-PHASE REALIZATIONS OF DISCRETE-TIME LINEAR SYSTEMS

First let us consider the SISO discrete-time linear system with the transfer function (24). From (24) we have

$$\overline{D} = \lim_{z\to\infty} T(z) = \overline{b}_n \tag{46}$$

and the strictly proper transfer function has the form

$$\begin{aligned}
T_{sp}(z) &= T(z) - \overline{D} = \overline{C}[I_n z - \overline{A}]^{-1}\overline{B} \\
&= \frac{\tilde{b}_{n-1}z^{n-1} + \ldots + \tilde{b}_1 z + \tilde{b}_0}{z^n + \overline{a}_{n-1}z^{n-1} + \ldots + \overline{a}_1 z + \overline{a}_0} = \frac{\tilde{n}(z)}{d(z)},
\end{aligned} \tag{47a}$$

where

$$\tilde{b}_k = \overline{b}_k - \overline{b}_n \overline{a}_k,\ k = 0,1,\ldots,n-1, \tag{47b}$$

$$\tilde{n}(z) = \tilde{b}_{n-1}z^{n-1} + \ldots + \tilde{b}_1 z + \tilde{b}_0. \tag{47c}$$

It is assumed that the poles z_1, $z_2, \ldots, z_n$ and the zeros z_1^0, $z_2^0, \ldots, z_{n-1}^0$ of (47) are real, positive, distinct and satisfy the conditions

$$z_k < 1,\ k = 1,\ldots,n \text{ and } z_k \le z_k^0 \le z_{k+1},\ k = 1,\ldots,n-1. \tag{48}$$

In a similar way as for continuous-time linear systems the strictly proper transfer function (4.2) can be written in the form

$$T_{sp}(z) = \sum_{k=1}^{n} \frac{\overline{T}_k}{z - z_k}, \tag{49a}$$

where

$$\overline{T}_k = \lim_{z\to z_k}(z - z_k)T_{sp}(z) = \frac{\tilde{n}(z_k)}{\prod_{\substack{j=1 \\ j\ne k}}^{n}(z_k - z_j)}. \tag{49b}$$

Note that $\overline{T}_k > 0$ for $k = 1,\ldots,n$ if and only if the conditions (48) are satisfied. In this case we can choose $\overline{c}_k > 0$, $\overline{b}_k > 0$ so that

$$\overline{T}_k = \overline{c}_k \overline{b}_k,\ k = 1,\ldots,n \tag{50}$$

and the matrices

$$\overline{A} = \mathrm{diag}[z_1 \quad z_2 \quad \cdots \quad z_n] \in \Re_+^{n\times n},\ \overline{B} = \begin{bmatrix} \overline{b}_1 \\ \overline{b}_2 \\ \vdots \\ \overline{b}_n \end{bmatrix} \in \Re_+^{n\times 1},$$

$$\overline{C} = [\overline{c}_1 \quad \overline{c}_2 \quad \cdots \quad \overline{c}_n] \in \Re_+^{1\times n} \tag{51}$$

are a positive realization of the transfer function (47).

By Definition 9 the realization (51) is minimal-phase if the conditions (48) are satisfied. Therefore, the following theorem has been proved.

Theorem 7. There exists a minimal-phase realization (51), (46) of the transfer function (47) if and only if the poles and zeros are real, positive, distinct and the conditions (48) are satisfied.

For computation of the minimal-phase realization of the transfer function (14) Procedure 1 with some evident modifications can be also used.

Example 3. Find the minimal-phase realization of the transfer function

$$T(z)=\frac{z^2+0.6z-0.17}{z^2-0.4z+0.03}. \tag{52}$$

Using Procedure 1 we obtain the following:

Step 1. Using (46) and (52) we obtain

$$\bar{D}=\lim_{z\to\infty}T(z)=\lim_{z\to\infty}\frac{z^2+0.6z-0.17}{z^2-0.4z+0.03}=1 \tag{53}$$

and

$$T_{sp}(z)=T(z)-\bar{D}=\frac{z-0.2}{z^2-0.4z+0.03}. \tag{54}$$

Step 2. The poles of (54) are $z_1=0.1$, $z_2=0.3$ and the zero is $z_2=0.2$.

The transfer function (54) can be written in the form

$$T_{sp}(z)=\frac{z-0.2}{(z-0.1)(z-0.3)}. \tag{55}$$

Step 3. Using (49) and (50) we obtain

$$\begin{aligned}\bar{T}_1&=\lim_{z\to z_1}(z-z_1)T_{sp}(z)=\left.\frac{z-0.2}{z-0.3}\right|_{z=0.1}=0.5=\bar{c}_1\bar{b}_1,\\ \bar{T}_2&=\lim_{z\to z_2}(z-z_2)T_{sp}(z)=\left.\frac{z-0.2}{z-0.1}\right|_{z=0.3}=0.5=\bar{c}_2\bar{b}_2\end{aligned} \tag{56}$$

and we choose $\bar{b}_1=1$, $\bar{b}_2=1$ and $\bar{c}_1=0.5$, $\bar{c}_2=0.5$.

Step 4. Using (51) and (56) we obtain

$$\bar{A}=\begin{bmatrix}0.1 & 0\\ 0 & 0.3\end{bmatrix},\ \bar{B}=\begin{bmatrix}\bar{b}_1\\ \bar{b}_2\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix},\ \bar{C}=[\bar{c}_1\quad \bar{c}_2]=\frac{1}{2}[1\quad 1]. \tag{57}$$

The desired minimal-phase realization of (52) is given by (53) and (57).

Now let us consider the MIMO discrete-time linear system with the strictly proper transfer matrix

$$T_{sp}(z)=\frac{N(z)}{d(z)}\in\Re^{p\times m}(z), \tag{58a}$$

where

$$d(z)=(z-z_1)(z-z_2)...(z-z_n), \tag{58b}$$

$$N(s)=\begin{bmatrix}(z-z_{11}^{0,1})...(z-z_{11}^{0,n_{11}}) & \cdots & (z-z_{1m}^{0,1})...(s-z_{1m}^{0,n_{1m}})\\ \vdots & \ddots & \vdots\\ (z-z_{p1}^{0,1})...(z-z_{p1}^{0,n_{p1}}) & \cdots & (z-z_{pm}^{0,1})...(z-z_{pm}^{0,n_{pm}})\end{bmatrix} \tag{58c}$$

with distinct real positive poles z_1, z_2,..., z_n and distinct real positive zeros $z_{11}^{0,1}$,..., $z_{11}^{0,n_{11}}$, $z_{1m}^{0,1}$,..., $z_{pm}^{0,n_{pm}}$.

In a similar way as for continuous-time systems the transfer matrix (58) can be written in the form

$$T_{sp}(z)=\sum_{k=1}^{n}\frac{\bar{T}_k}{z-z_k}, \tag{59a}$$

where

$$\bar{T}_k=\lim_{z\to z_k}(z-z_k)T_{sp}(z)=\frac{N(z_k)}{\prod_{j=1, j\neq k}^{n}(z_k-z_j)} \tag{59b}$$

and

$$\operatorname{rank}\bar{T}_k=\bar{r}_k. \tag{60}$$

In a similar way as for continuous-time systems it can be shown that $\bar{T}_k\in\Re_+^{p\times m}$ for $k=1,\dots,n$ if the conditions

$$z_k\le z_{ij}^{0,k}\le z_{k+1}\ \text{for}\ i=1,\dots,p,\ j=1,\dots,m,\ k=1,\dots,n_{ij} \tag{61}$$

are satisfied. In this case there exist the matrices $\bar{C}_k\in\Re_+^{p\times\bar{r}_k}$, $\bar{B}_k\in\Re_+^{\bar{r}_k\times m}$, $\operatorname{rank}\bar{C}_k=\operatorname{rank}\bar{B}_k=\bar{r}_k$, $k=1,\dots,n$ such that

$$\bar{T}_k=\bar{C}_k\bar{B}_k. \tag{62}$$

In a similar way as for continuous-time systems it can be shown that the matrices

$$\bar{A}=\operatorname{blockdiag}[I_{\bar{r}_1}z_1\ \cdots\ I_{\bar{r}_n}z_n]\in\Re_+^{\bar{r}\times\bar{r}},$$
$$\bar{B}=\begin{bmatrix}\bar{B}_1\\ \vdots\\ \bar{B}_n\end{bmatrix}\in\Re_+^{\bar{r}\times m},\ \bar{C}=[\bar{C}_1\ \cdots\ \bar{C}_n]\in\Re_+^{p\times\bar{r}},\ \bar{r}=\sum_{k=1}^{n}\bar{r}_k \tag{63}$$

are a positive realization of the matrix (58).

Therefore, the following theorem has been proved.

Theorem 8. There exists a minimal-phase realization (63) of the strictly proper transfer matrix (58) if and only if the poles and zeros are distinct, real, positive and the conditions (61) are satisfied.

With same evident modifications Procedure 1 can be also used to compute the positive minimal-phase realization (63) of the transfer matrix (58).

CONCLUDING REMARKS

The problem of minimal-phase realization for continuous-time and discrete-time linear systems has been formulated and solved. Necessary and sufficient conditions for the existence of minimal-phase realizations for the linear systems have been established (Theorems 5, 6 and 7, 8). A procedure for computation of the realizations has been proposed and illustrated by numerical examples. The considerations can be extended to fractional positive linear systems without and with delays.

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MINIMALNO-FAZOWE REALIZACJE DLA DODATNICH UKŁADÓW LINIOWYCH

Streszczenie

Pracy zaproponowano nową metodę wyznaczanie minimalno-fazowych realizacji dla dodatnich ciągłych i dyskretnych układów liniowych. Podano warunki konieczne i wystarczające na istnienie minimalno-fazowych realizacji dla tych klas układów liniowych. Podano procedurę wyznaczania tych realizacji minimalno-fazowych oraz zlustrowano efektywność tej procedury na przykładach liczbowych.

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