Practical aspects of heat conduction simulation in household heating optimization tasks

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Abstract
Mathematical model selection for simulation heat conduction processes in household heating optimization task is considered. The essence of the matter is that the heat transfer dynamics properties are very diversified, so simulation procedure formulae and parameters should be properly selected to avoid excessive modeling errors with reasonable calculation time being held. The typical state-space model and analytical formulae for step response of the heat conduction across a homogeneous wall are presented and compared in terms of modeling errors. Formal and numerical problems of heat losses simulation are discussed. Semi-analytical step-response formulae for multilayer walls are derived and their accuracy is compared with effects of simulation based on the state-space model. Some recommendations for time and space discretization parameters are given.

Keywords: thermal diffusivity, cheat convection, analytical solutions of diffusion process, household heating

Introduction

Household heating optimization needs mathematical models to make possible fast simulation of such processes as heat exchange, air mixing and thermal conduction across the walls. A specific feature of the processes to be considered in this task is large diversification of their dynamics. There are very fast cheat exchange phenomena in cheating devices (gas boilers and radiators), where transient response is of seconds, a bit slower air mixing with time constants of minutes, and cheat transfer across the walls which is of fast initial response (more than 50% of final level in several seconds) but going very slowly (in tens of hours) to its steady state.

Typical calculations of buildings thermal properties are in general very simplified and they are limited to steady states. Appropriate recommendations and data are widely available in literature, e.g. [1, 2, 3]. In turn, to the process dynamics calculations one may exploit typical discretized state space models [4], producing time responses of temperature and heat streams at any point. However, elaboration of effective algorithms (e.g. modern MPC controllers [4, 5]) to minimise the energy consumption needs more accurate analysis of the process dynamics. In particular, to avoid excessive simulation errors the state space model accuracy should be evaluated, depending on the space and time discretization parameters, especially cheat losses at the initial transient response interval. For homogeneous walls it may be done by using analytical solutions of the thermal diffusion process. But for typical multilayer building partitions there are no so simple formulae, hence a semi-analytical approach is proposed in the paper.

To this aim numerical properties of the analytical formulae are taken under considerations, and a semi-analytical model of the thermal diffusion process is elaborated, which makes possible calculation for multilayer (heterogeneous) walls more precisely than with the state-space model. In effect a convolution model is proposed to simulate the household heating process as an alternative to the state-space model. Recommendations for the time and space discretization and for reduction of calculation time are given.

Mathematical modeling of household heating processes

Let us take a household heating system in a typical single-family house consisting of a gas boiler heating the circulating water and radiators placed in particular rooms. The heat transfer is by the conduction from hot gases to water in the boiler and from hot water to air in radiators, the convection outside of the building with air due to natural or forced ventilation, mixing of air inside the rooms, and conduction by different building partitions (in particular heat losses across the external walls). Rules and appropriate formulae for heat demand evaluation are specified by European and polish standards (e.g. [2, 3]), with respect to comfort needs, climate conditions and building properties, which affect averaged heat losses and so – heat consumption depending mainly on the surface of building partition and ventilation

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Table 1. Heat losses distribution in a typical single-family house (source [1])

Ventilation	Windows and doors	External walls	Floors	Ceilings and roof
30-40 %	10-15%	20-30%	5-10%	10-25%

intensity. Distribution of heat losses is roughly characterized in Table 1.

Application of typical automated heating control systems make possible to reduce the energy consumption by 5–15% (in older buildings better isolation of external wall may yield energy savings by 10–25%) [1]. However, implementation of modern control algorithms, especially predictive controllers [4, 5], employing precisely specified time profiles of required and external temperature to optimize in real time the heat production, make possible reach noticeably higher energy savings. Nevertheless it needs elaboration of mathematical models enabling for fast simulation of the heat transfer dynamics.

The heat convection dynamics in a room space (including heat losses due to ventilation) may be described by the mixing dynamics formula [4]. When assuming an ideal mixing of air and negligeable changes of its density r_a≈1.12 kg/m^3, the energy balance in a room of volume V_a may be written as:

$$\frac{dT_a}{dt} = \frac{F_a}{V_a}(T_e - T_a) + \frac{1}{V_a \rho_a c_a}\left(q_r - \sum_i q_i\right) \quad (1)$$

where T_a denotes averaged temperature in the room, c_a – specific heat capacity of air at constant pressure (c_a≈1 kJ/kg/K); F_a – ventilation air flow (F_a≈ 25 m^3/h, F_{amin}=17.5 m^3/h [2]), T_e.- temperature of air going into the room, q_{si} – the i-th stream of heat losses due to conduction across the i-th building partition [kWh] and q_r stands for the heat inflow from radiators [kWh]. Parameters for the model (1) are widely available in literature (eq. [1]).

The transient response of the process (1) disappears in minutes (V_a/F_a≈120 min, but first reaction to q_r and q_i changes is much faster). The ideal mixing assumption is rather serious simplification, but it is necessary to avoid tremendously complicated calculations of fluid dynamics [4]. Thus the errors due to this simplification are a reference measure for demanded accuracy of remaining processes modeling.

The heat conduction across the radiator and boiler walls (producing q_r) is very fast. Hence a transient response of this process may me omitted, and the steady state model of this process (involving water transportation lags) [4] may be used in simulations. It has the following form:

$$q_r = F_w c_w (T_{wh} - T_{wc}); \quad F_w = \frac{C_{Fr}}{\eta_{wh}}(T_{wh} - T_{wc});$$
$$(T_{wh} - T_{wc}) = (T_{wh} - T_a)\left(1 - \exp\left(-\frac{K_r}{F_w c_w}\right)\right) \quad (2)$$

where T_{wh} and T_{wc} denote hot and cold water temperature, c_w and h_{wh} – specific heat capacity [kJ/kg/K] and dynamic viscosity coefficient [kg/m/h] of hot water, F_w – mass flow of water [kg/h] (forced by a difference between hot and cold water density), C_{Fr} is a geometric constant of pipes and K_r stands for a material and geometric constant of radiator.

The model (2) is applicable while $F_w > F_{wmin}$, i.e. while resident time of water in the heat exchanger is no longer than a couple of seconds. In another case a water heating dynamics formula should be employed.

The same formulae (2) may be used to describe the water heating process in a gas boiler, with T_{wh} replaced by T_{wc} and vice-versa, and an average flame temperature T_f replacing T_a. In the heating control system the gas boiler is employed as an actuating device aimed at producing the hot water of a temperature T_{wh} close to its reference value calculated by the supervisory controller optimizing the heat consumption with a presumed room temperature time profile being held.

The key issue (focused in this paper) is an appropriate modelling of heat losses streams q_i [kWh] in eq.(1) The basic formula describing the heat losses due to heat conduction across a solid wall is the 1D Fourier's law:

$$q_i(t) = -\lambda_i S \frac{\partial T_S(t,0)}{\partial x}; \qquad q_{si}(t) \stackrel{\text{def}}{=} \frac{q_i(t)}{S} = -\lambda_i \frac{\partial T_S(t,0)}{\partial x} \quad (3)$$

where $Ts(t,0)$ denotes the wall surface temperature, l_i is the thermal conductivity coefficient of the wall material, [W/(m·K)], q_{Si} is heat losses stream for 1m^2 of wall [kWh/m^2].

However, to determine the heat stream it is necessary to exploit the full 1D model of the thermal diffusion dynamics, which has the form:

$$\frac{dT_S(t,x)}{dt} = D_i \frac{\partial^2 T_S(t,x)}{\partial x^2} \qquad D_i \stackrel{\text{def}}{=} \frac{\lambda_i}{\varrho_i c_{wi}} \left[\frac{m^2}{s}\right] \quad (4)$$

The model (4) may be used to simulation the thermal conductivity process in appropriately discretised space of the wall (usually one takes a constant increment Dx of the space coordinate x), in a series of fixed time instants t_n with a constant sampling interval Dt, starting with any initial conditions $T_{si}(0)=T_s(0,i\cdot\text{D}x)$, with any boundary conditions $T_s(t, 0)$ and $T_s(t, L)$. The time and space discretization leads to the lumped parameter state-space model consisting of equations expressing the dymamic energy balace in i-th layer of the wall (i=1,… M, M=L/Dx), in which the temperature gradient is calculated with the approximating formula:

$$\left(\frac{\varrho_{i-1}c_{i-1}+\varrho_i c_i}{2}\right)\frac{\Delta T_s(n,i)}{\Delta t} = \lambda_{i-1}\frac{T_s(n,i-1)-T_s(n,i)}{\Delta x^2} - \lambda_i\frac{T_s(n,i)-T_s(n,i+1)}{\Delta x^2} \quad (5)$$

where c_i denotes a specific heat capacity of the wall material in i-th layer [J/(kg·K)] and ρ_i is its density [kg/m³].

It is noteworthy that in the formula (5) different values of λ, ρ and c_w for each layer are admitted. Let us denote:

$$D_{si} \overset{\text{def}}{=} \frac{2(\lambda_{i-1}+\lambda_i)}{\varrho_{i-1}c_{wi-1}+\varrho_i c_{wi}}\;;\quad D_{li} \overset{\text{def}}{=} \frac{2(\lambda_{i-1})}{\varrho_{i-1}c_{wi-1}+\varrho_i c_{wi}}\;;\quad D_{ri} \overset{\text{def}}{=} \frac{2(\lambda_i)}{\varrho_{i-1}c_{wi-1}+\varrho_i c_{wi}} \quad (6)$$

The equation (5) may be transformed to the form producing directly $T_s(n+1,i)$:

$$T_s(n+1,i) = T_s(n,i)\left(1-\frac{D_{si}\Delta t}{\Delta x^2}\right) + \left(\frac{D_{li}\Delta t}{\Delta x^2}T_s(n,i-1) + \frac{D_{ri}\Delta t}{\Delta x^2}T_s(n,i+1)\right) \quad (7)$$

It should be noticed that according to general rules of modeling of causal processes dynamics [5], in the formula (6) one assumes constant values of temperatures during the time interval Dt in all the layers, and the increment D$T_s(n, i)$ concerns the value averaged over a neighborhood Dx /2 of the point $x_i = i$·Dx. It points that the heat stream q_i in eq.(3), playing the key role in the simulation model in eq.(1), may be highly sensitive to values Dx and Dt taken arbitrarily. This sensitivity may be reduced a bit by using the following (more accurate) formulae:

$$T_s(n+1,i) = T_s(n,i)\propto_i + (1-\propto_i)\left(\frac{D_{li}}{D_{si}}T_s(n,i-1) + \frac{D_{ri}}{D_{si}}T_s(n,i+1)\right) \quad (8)$$

where: $\propto_i \overset{\text{def}}{=} \exp\left(-\frac{D_{si}\Delta t}{\Delta x^2}\right)$

A general rule of thumb for assigning the value for Dt for simulations is [4], that the following relations should be held:

$$\max_i \frac{D_{si}\Delta t}{\Delta x^2} < 0.1 \qquad \text{i.e.} \qquad \frac{D_{si}\Delta t}{\Delta x^2} \cong 1-\propto_{si} \quad (9)$$

In order to evaluate a precision of calculations based on eq.(8) simulations of the heat transfer across a typical building partion were performed. First, a „homogeneous" brick wall was taken under considerations of 0.45 m thickness, then a more realistic partition consisting of two layers: 0.3 m of brick and 0.15 m of typical polystyrene insulating layer. Thermal parameters of such a wall (taken from [6]) are presented in Table 2.

Selected sequence of temperature profiles across the partition after step-wise change of left border temperature from 15°C to 20°C are shown in Fig.1, assuming the homogeneous brick wall

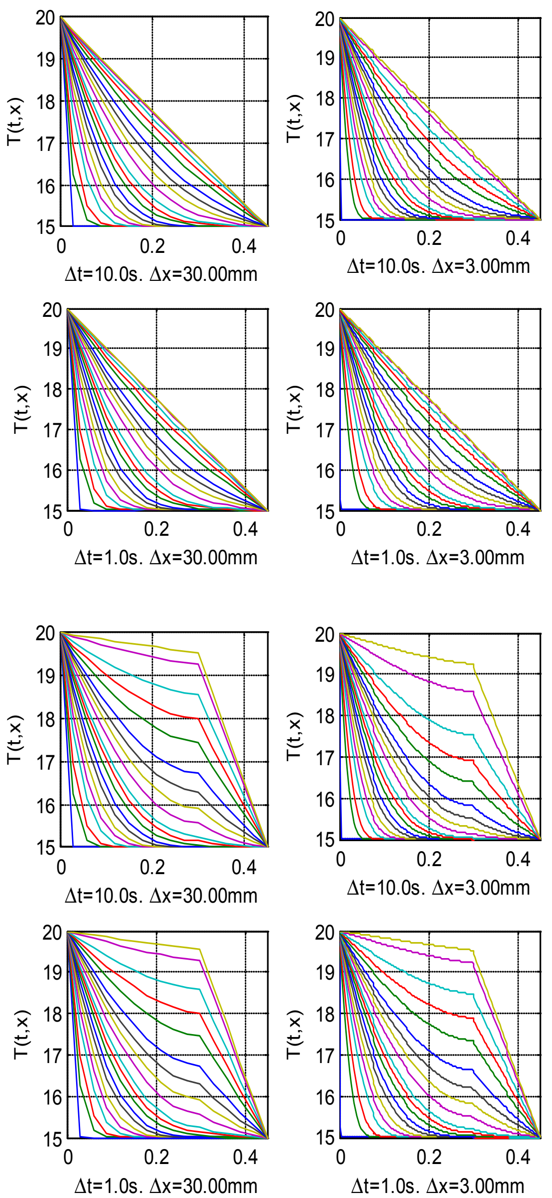

Figure 1. Selected temperature profiles across the homogeneous brick wall (left subfigure) and the two-layer wall: brick 0.3m and polystyrene 0.15 m (right subfigure) calculated with eq.(8) for different Dt and Dx.

Table 2. Thermal parameters of typical building partitions and resultant thermal diffusion coefficients (source: [6], author's own recalculations)

Material of partition	Thermal conductivity [kWh/(m K) l_i	Density [kg/m³] r_m	Specific heat capacity [kJ/(kg K)] c_m	Thermal diffusion coefficient [m²/h] D_m	Partition size [m] L_m
Brick layer	2.88	1800	0.88	1.8182*10⁻³	0.45 0.30
Isolation layer (polystyrene)	0.126	40	1.46	2.1575*10⁻³	0.15

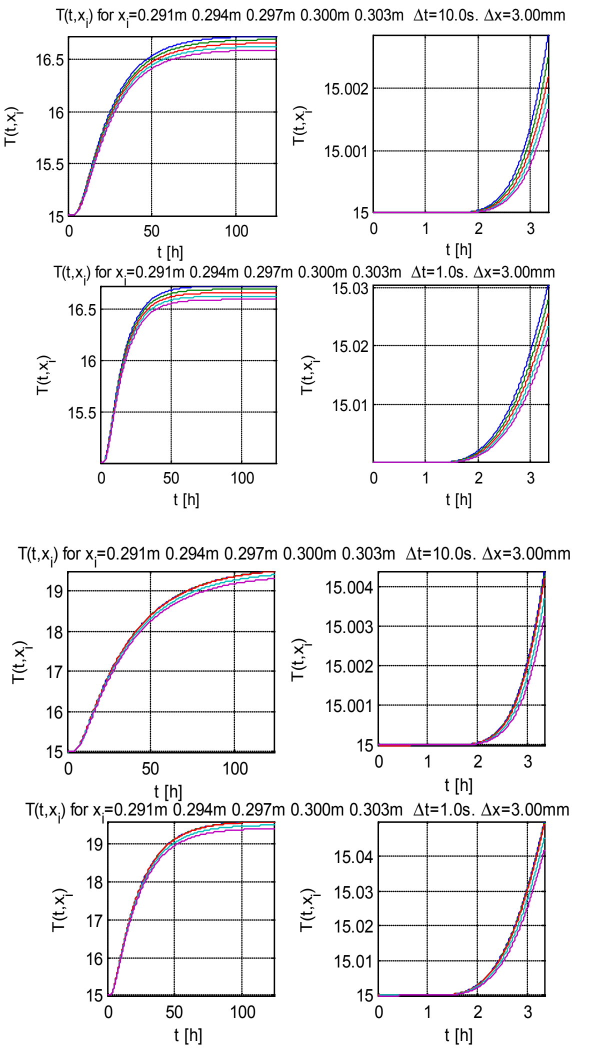

Figure 2. Time profiles of temperature at selected points calculated with eq.(8): left subfigures – homogeneous brick wall; right subfigures – the two-layer wall: brick 0.3m and polystyrene 0.15m.

(left subfigure) and the two layer wall: brick 0.3m and polystyrene 0.15m (right subfigure). Calculation time was very short – ranging from 0.14s. for 43200 time samples and 15 layers to 4s. for 432000 time points and 150 layers. It may be seen that there are no significant differences depending on D*t* and D*x*. Time profiles of the temperature at selected points x_i (close to x=0.3m) are shown in Fig.2 (calculated with D*x*=0.003m, D*t*=10s and 1s.). Initial sections of the transient responses are exposed.

Very long transient response (ca. 100h) is specific for this process. Differences in transient time and initial lags (visible in Fig.2) for D*t*=10s (upper subfigures) and D*t*=1s (lower subfigures) point that proper selection of D*t* is of importance, and D*t*=10s is too large. However, the role of proper selection of D*t* and D*x* becomes evident when consider the heat losses stream q_i, which is crucial in the model (1). Let $q_T(t,0)=q_i/S$ denotes the heat stream [kWh/m^2] calculated numerically by using eq.(8):

$$q_T(t,0) = \frac{\lambda_m}{\Delta x}\left(T_s(t,0) - T_s(t,\Delta x)\right) \qquad (10)$$

Results of calculations of q_T with different D*t* and D*x* are presented in Fig. 3.

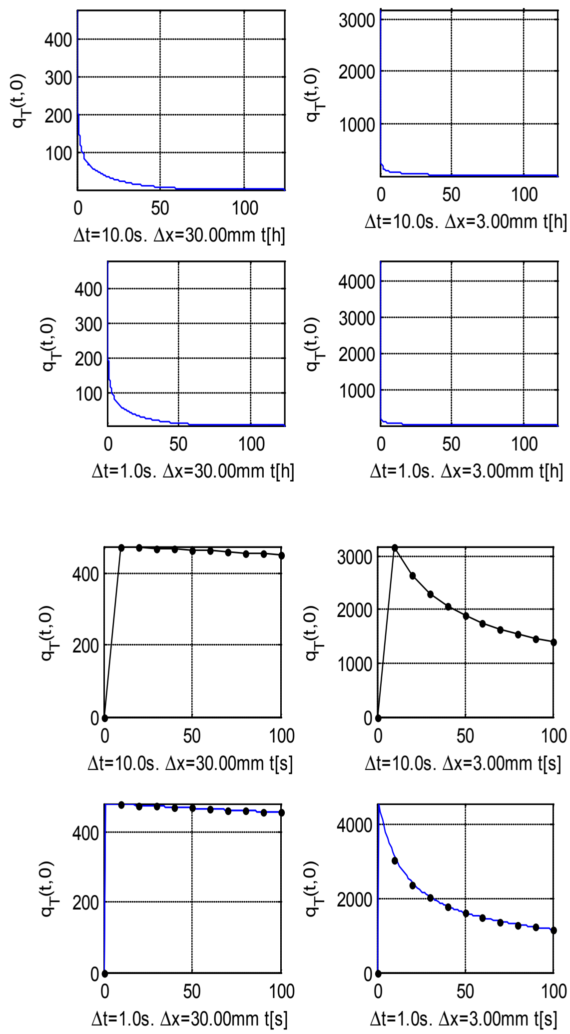

Figure 3. Time profiles of heat losses stream (eq.10) calculated for the homogeneous wall with different Dt and Dx, presented in full time interval in hours (left subfigures) and at initial time section of transient response in seconds (right subfigures, points show 10s intervals)

Huge differences between values found with D*t*=10s and 1s, as well as with D*x*=30mm and 3mm are clearly seen in large time interval (up to 50h), but especially at very initial section of transient response (see right subfigure). It should be noticed that the formulae (8) yield values of q_T averaged over D*t*, hence $q_T(0,0)\equiv 0$ is assumed, and differences visible for the same D*x* might be acceptable in more approximate calculations. Nevertheless, the effect of D*x* is due to the temperature gradient averaging, so it should be eliminated as a typical artefact, because in eq.(1) estimation of the outgoing heat stream is needed. The question what value for D*x* should be taken to reach demanded accuracy of calculations may be answered by analysis of analytical solutions of eqs. (3, 4). It is taken under consideration in the next section.

Formal and numerical properties of analytical solutions of the thermal conductivity process

For the diffusion process of constant diffusion coefficient D, with some special initial and boundary conditions, there are analytical solutions of equations (4), (5). In particular, when at t=0 (initial condition) the temperature across the wall is zero $T(0,x)$=0, x (L denotes the wall thickness), and temperature T_L at the left bound of the wall $T_L=T(t, 0)=1$, t≥0, while at the right bound $T_R=T(t, L)=0$, t≥0 (boundary conditions for thermal conductivity) [6, 7], the analytical solution of eqs (4), (5) for the unit step change of T_L , i.e. the step response denoted as $h_{TL}(x, t)$ is:

$$h_{TL}(t,x) = \left(1 - \frac{x}{L}\right) - \frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sin\left(\frac{i\pi x}{L}\right) exp\left(-\frac{i^2\pi^2 Dt}{L^2}\right) \qquad (11)$$

By virtue of eq.(2) linearity, the formula (11) may be viewed as the universal model of the process. It results from a mathematical trick. The first term at the right hand side writes the final steady state profile, the first term in the series is a special function expressed in the form of Fourier series [7]:

$$\frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sin\left(\frac{i\pi x}{L}\right) = \begin{cases} 0 & for\ x = 0 \\ \left(1-\frac{x}{L}\right) & for\ x\epsilon(0,L)' \end{cases} \qquad (12)$$

which is used to compensate the final profile at t=0 except for the boundary value. Then, it is dumped to zero with $t \to \infty$ by the exponential term to produce the solution. As the matter of fact it is composed of two periodical functions of $2L$ period [9]:

$$f_1(x) = \frac{2}{\pi}\sum_{i=1}^{\infty}\left(1-(-1)^i\right)\frac{1}{i}\sin\left(\frac{i\pi x}{L}\right) = \begin{cases} -1 & for\ x\epsilon(-L,0) \\ 0 & for\ x = \{-L,0,L\} \\ 1 & for\ x\ \epsilon(0,L) \end{cases} \qquad (13)$$

$$f_2(x) = \frac{2}{\pi}\sum_{i=1}^{\infty}(-1)^i\frac{1}{i}\sin\left(\frac{i\pi x}{L}\right) = \begin{cases} -x/L & for\ \ x\epsilon(-L,L)\backslash\{0\} \\ 0 & for\ x = \{-L,0,L\} \end{cases} \qquad (14)$$

Plots of the above functions vs. x reached with a finite iteration number ($i<n_{max}$) are shown in Fig.4.

As it is seen in Fig.4 the sums in eqs.(11–13) are rather quickly converging, even at t=0 and x=0. In turn, properties of the functions $h_1(t, x)$, $h_2(t, x)$ based on the series f_1 and f_2 completed with the exponential time factors (like in eq.11), as well as of the sum h_1+h_2, viewed versus x coordinate in the range $-L<x<L$, (L=0.45) are shown in Fig.5 for selected time instants the same as in Fig.1. (calculation were carried-out for the brick wall of L=0.45 – see Table 1 for parameters).

It may be seen in the lower-right subfigure that the function $h_2(t, x)$ describes the physical heat conduction process but only for (outside this interval it is of no physical meaning, although it remains a formal solution of eq.(4) for any x).

By using the same trick related to the unit step change of the right hand side temperature T_R one arrives at the step response formula $h_{TR}(t, x)$ for the second input T_R of the process:

$$h_{TR}(t,x) = \left(\frac{x}{L}\right) + \frac{2}{\pi}\sum_{i=1}^{\infty}\frac{(-1)^i}{i}\sin\left(\frac{i\pi x}{L}\right) exp\left(-\frac{i^2\pi^2 Dt}{L^2}\right) \qquad (15)$$

A simultaneous unit step-wise change of both T_L and T_R gives the typical thermal diffusion process, which may be described by using only $h_1(t, x)$. It leads to the following diffusion step-response formula $h_{TH}(t, x)$ (not exploited in this paper):

$$h_{TH}(t,x) = 1 - \frac{1}{\pi}\sum_{i=1}^{\infty}\left(1-(-1)^i\right)\frac{1}{i}\sin\left(\frac{i\pi x}{L}\right) exp\left(-\frac{i^2\pi^2 Dt}{L^2}\right) \qquad (16)$$

The values of the functions $h_{TL}(t, x)$, $h_{TR}(t, x)$ and $h_{TH}(t, x)$ may be calculated individually for any x and t irrespective their neighbourhood, i.e with no effect of Dx and Dt. The series in eqs.(11, 15, 16) converges rather quickly (it is enough to take i<300), so calculation of one time profile of e.g. h_{TL} at a given x with Dt=1s up to t=120h takes a couple of seconds (a bit more then solving of the full set of eqs.(8)). Space profiles of $h_{TL}(t, x)$,

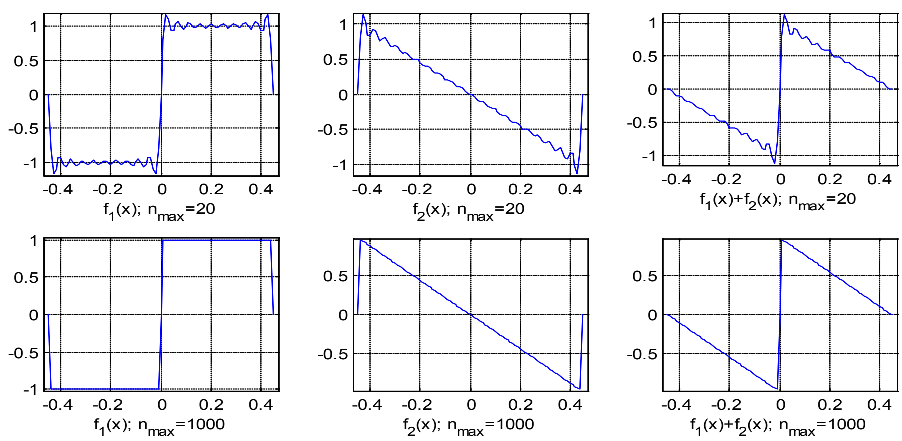

Figure 4. Analytical functions $f_1(x)$ – eq.(12), $f_2(x)$ – eq.(13) and $f_1(x)+f_2(x)$ – eq.(11) calculated with finite number of iterations $i<n_{nax}$ (Dx=3mm)

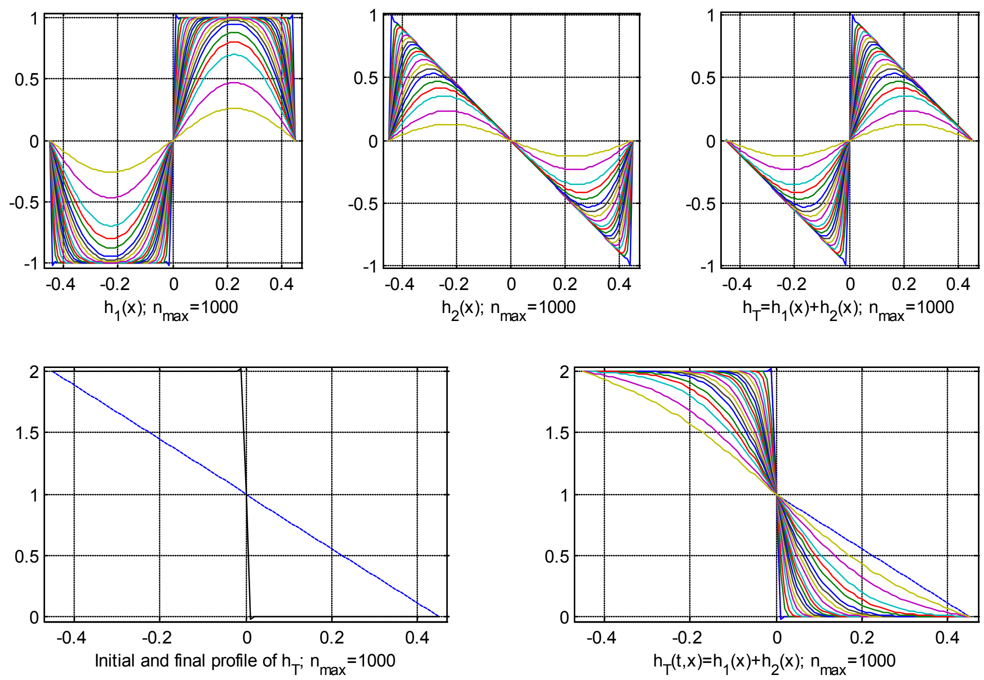

Figure 5. Space coordinate x profiles of analytical functions $h_1(t, x)$, $h_2(t, x)$ and $h_{TL}(t, x)$ calculated at selected time instants (the same as in Fig.1) with finite number of iterations i<1000 (Dx=3mm, brick wall)

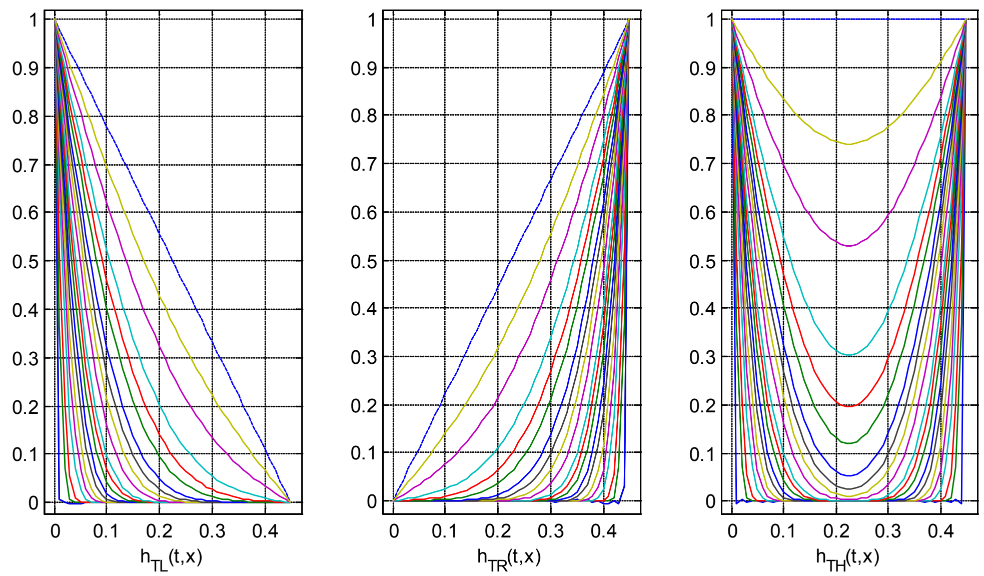

Figure 6. Space coordinate x profiles of analytical functions $h_{TL}(t, x)$, $h_{TR}(t, x)$ and $h_{TH}(t, x)$ calculated at selected time instants (the same as in Fig.1) with finite number of iterations i<1000 (Dx=3mm, brick wall)

$h_{TR}(t, x)$ and $h_{TH}(t, x)$ for the selected time instants are presented in Figure 6.

Due to the linearity of eqs (4) and (5) one may exploit the formulae (11, 15) as the universal step response model of the heat transfer process at any point x, excited by changes of the left $T_L(t)$ and right $T_R(t)$ temperature. Let us assume an initial steady-state conditions (for t=0).

$$T(0\_, 0) = T_{L0} = const;\ T(0\_, L) = T_{R0} = const;$$
$$T_0(0, x) = (T_{L0} - T_{R0})\left(1 - \frac{x}{L}\right) + T_{R0} \qquad (17)$$

The step response to a simultaneous step-wise change in the boundary conditions for $t \geq 0$:

$$T(t,0) = T_L = const, \qquad T(t,L) = T_R = const; \tag{18}$$

can be expressed in the form following (exploiting additivity of excitation effects in linear transforms):

$$T(t,x) = (T_{L0} - T_{R0})\left(1 - \frac{x}{L}\right) + T_{R0} + (T_L - T_{L0})h_{TL}(t,x) + (T_R - T_{R0})h_{TR}(t,x) \tag{19}$$

The model (19) may be employed in any simulation procedure by calculating its convolution with any time profile of $T_L(0, t)$. For causal systems it may be expressed by the general convolution formula:

$$y(t) = \sum_k \int_{\tau=-\infty}^{\infty} u_k(t-\tau) g_k(\tau) d\tau \, ; \qquad y(t) = \sum_k \int_{\tau=0}^{\infty} u_k(t-\tau) g_k(\tau) d\tau \tag{20}$$

where $u_k()$ stands for a k-th input time profile, $g_k(t)$ denotes the impulse-response of the considered process attributed to the k-th input (for causal systems $g(\tau \le 0) \equiv 0$. It may be also written in the alternative form, involving the step-response $h(\tau)$ ($h(\tau \le 0) \equiv 0$), which for $u_k(t \le 0) \equiv 0$ is:

$$y(t) = \sum_k \int_{\tau=0}^{t} \frac{du_k}{dt}(t-\tau) h_k(\tau) d\tau; \quad h(t) \stackrel{\text{def}}{=} \int_0^t g(\tau) d\tau; \tag{21}$$

Having the formulae for $h_T(t, x)$ we may easily derive the formulae for cheat stream $q_s(t, x)$ according to the Fourier's law (3). The formula for the step response of heat flow $h_{qTL}(t, x)$ at any x-point and time t, to step change in temperature DT_L=1 has the following form:

$$h_{qTL}(t,x) = \frac{\lambda}{L} + \frac{2\lambda}{L} \sum_{i=1}^{\infty} \cos\left(\frac{i\pi x}{L}\right) exp\left(-\frac{i^2\pi^2 Dt}{L^2}\right) \tag{22}$$

In the same way one may express the step response $h_{qTR}(t_n, x)$ to unit step change DT_R=1:

$$h_{qTR}(t,x) = -\frac{\lambda}{L} - \frac{2\lambda}{L} \sum_{i=1}^{\infty} \cos\left(i\pi\left(1 - \frac{x}{L}\right)\right) exp\left(-\frac{i^2\pi^2 Dt}{L^2}\right) \tag{23}$$

In household heating calculations the most important issue is appropriately accurate modeling of the heat stream $q(t, 0)$ outgoing a heated room (see eq.1). To this aim the one may employ directly the eq.(22). However the series in eq.(22) converges very slowly for x>0, and it does not converge at x=0 and t=0. In particular, for t=0 the formula (22) represents the Dirac delta function δ(0) in its Fourier series form [8, 9]:

$$\frac{2}{L} \sum_{i=1}^{\infty} \cos\left(\frac{i\pi x}{L}\right) = \begin{cases} 1/\partial x & for\ x = \{0, \pm 2L\} \\ 0 & for\ x \ne \{0, \pm 2L\} \end{cases} \tag{24}$$

Hence one may consider calculation of $q(t, 0)$ by using the approximating formula, like in eq. (10):

$$q(t,0) = \frac{\lambda}{\Delta x}\left(h_{TL}(t,0) - h_{TL}(t,\Delta x)\right) \tag{25}$$

which converges much better (see Fig. 6).

In order to help resolving of the above dilemma one may examine effects of using of eq. (22) and eq. (25) with different number if iterations and different Dx. Results of such calculations are compared in Fig. 7. When viewing this figure one can see very strong effect of Dx to the formula (24) and its noticeable influence on eq. (22). Notice that initial transients response calculated with Dx=4.5mm is quite different when using eq. (22) and eq. (25). It remains significant for Dx=0.045mm (compare upper and lower subfigures), hence its value closer to 0.00045 seems to be adequate for eq. (25). The approximating formula (25) is also noticeably sensitive to the number of iteration, but tis effect is much stronger for the formula (22). Notice that calculations of h_{qTL}(0,Dx) do not fit the formula (24) until very large I_{max} is taken (I_{max}=3 600 000 was applied). The initial value in the central lower subfigure (I_{max}=3 600, Dx=0.0045mm) is ca 3.6810[4] instead of its actual value i.e. 0 (see eq.24), which is reachable with much larger I_{max} (see bold point at x=0.0045mm). It should be noticed that, in fact, the values for x=0 counts simply the iterations (the formula (22) does not converge), i.e. $h_{qTL}(0,0) = \frac{\lambda}{L}(1 + 2I_{max})$. It yields ca. 4.6 10^4 and 4.6 10^7, respectively (see values in lower left-right and right-right subfigures), that (according to eq. 24) may be viewed as averaged values over δx=6.25 10^{-2} mm and δx=6.25 10^{-5} mm.

Nevertheless as the matter of fact, in practical simulations we need formulae for the heat streams averaged over a time interval Dt. Such a formula may be derived by integration of eq. (22), i.e.:

$$h_{qTL}(t_n,x) = \frac{\lambda}{L} - \frac{2\lambda L}{\Delta t \pi^2 D} \sum_{i=1}^{\infty} \frac{1}{i^2} \cos\left(\frac{i\pi x}{L}\right)\left(exp\left(-\frac{i^2\pi^2 D t_n}{L^2}\right) - exp\left(-\frac{i^2\pi^2 D(t_n - \Delta t)}{L^2}\right)\right) \tag{26}$$

It may be used directly to calculate $q(t, 0)$ with a lower I_{max}, as the series in eq. (26) converges much better than in eq.(22) [9]). One may also employ the approximating formula (25) with $h_{TL}()$ averaged in the same way. Results of such calculations are compared in Figure 8.

It may be seen that this time the sensitivity of the both formulae to Dx is much weaker, providing that Dx<1mm. Results obtained with approximating formula and eq. (26) are similar (compare corresponding subfigures in Fig. 7 and 8). It should be noticed that calculation with eq. (26) are less time consuming than approximating one.

In Figure 9. the step response of the heat losses stream calculated in the discretized state-space – eqs. (8) (recursive model) is confronted with that produced by the analytical formula (26). A relatively small value Imax=2000 was taken and Dt=1s

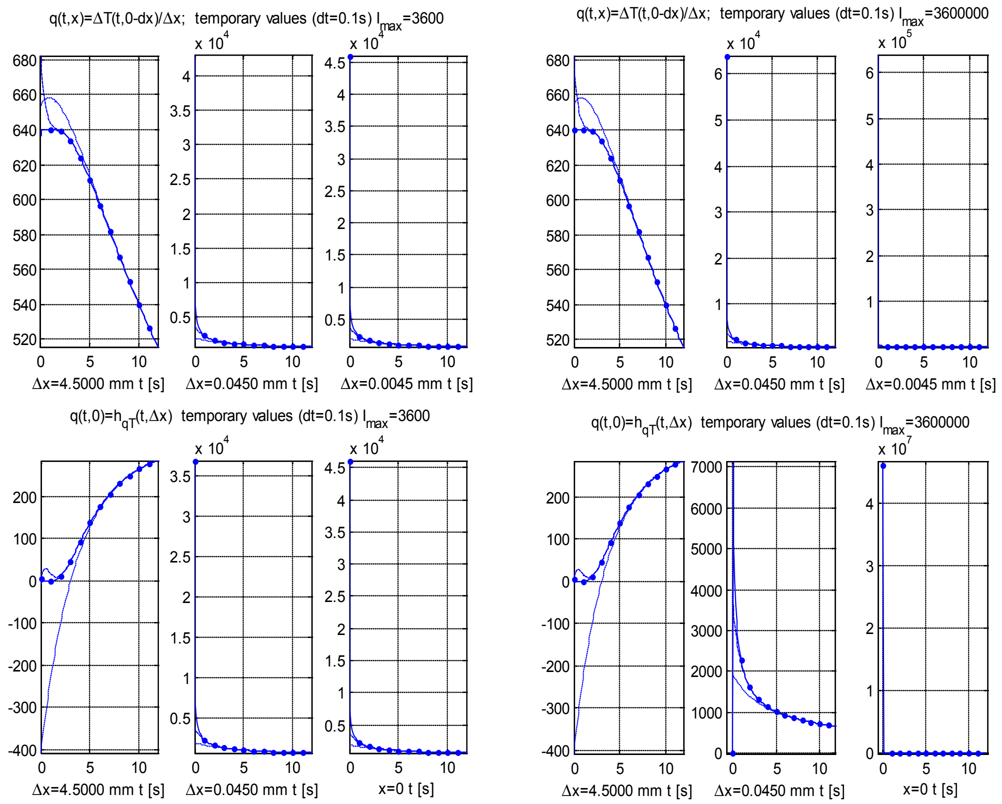

Figure 7. Step response of heat losses stream $q_q(t, 0)$, at very initial time section, calculated for the brick wall with approximating equation (27) (upper subfigures) and with the analytical formula (22) $h_{qTL}(t, x=Dx)$ (lower subfigures), reached with the number of iterations I_{max}=3600 (left subfigures) and I_{max}=3 600 000 (right subfigures). Results obtained for i=150 are also shown as point-lines, and for i=300 as dotted lines.

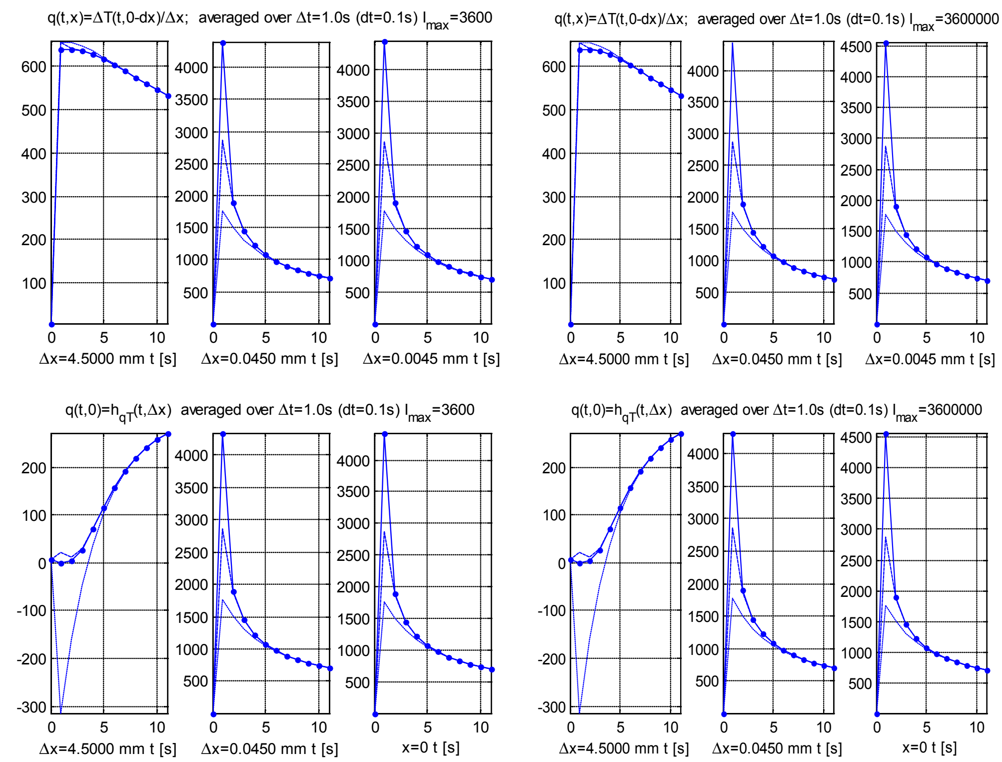

Figure 8. Step response of heat losses stream $q_q(t, 0)$, at very initial time section, averaged over the time sampling interval Dt=1s, calculated for the brick wall with approximating equation (25) (upper subfigures) and the analytical formula (26) (lower subfigures) reached with the number of iterations I_{max}=3600 (left subfigures) and I_{max}=3 600 000 (right subfigures). Results obtained for i=150 are also shown as point-lines, and for i=300 – dotted lines

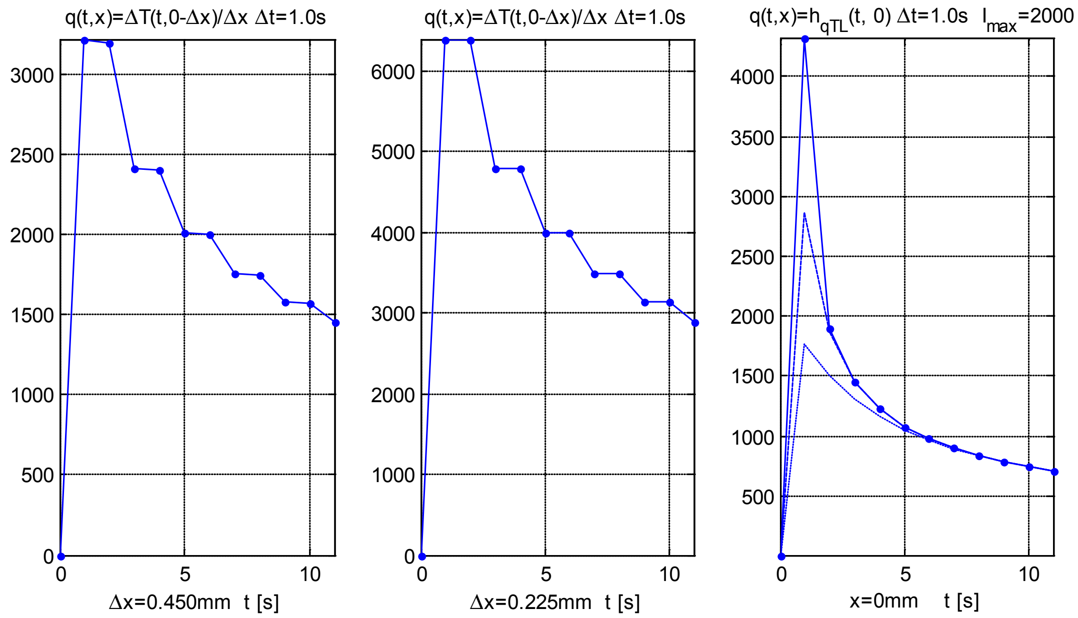

Figure 9. Comparison of step response of heat losses stream $q_q(t, 0)$, at very initial time section, calculated in the discretized state-space model - eqs.(8) with two different Dx (left and central subfigures), and by employing the analytical formula (26) for x=0 (right subfigure), calculated for the brick wall. In the right subfigure results obtained for i=150 are shown as point-lines, and for i=300 – dotted lines

the same as in Fig.8. The recursive model was applied with Dx=0.45mm and Dx=0.225mm. The plots shown in left and central subfigures reveal significant modelling errors being an effect of too large Dt value (Dt/Dx ratio is too large, so smaller values of Dx are unacceptable when Dt=1s – see also right subfigures in Fig. 3).

The above picture gives evidence for advantages of the analytical formula (26), when compared to the recursive model. Thus, for homogeneous partitions it may be recommended as the appropriate formula enabling for calculation the heat losses step response, to be stored and used as the basis of convolution model (21) in simulation procedures.

So far as the diffusion coefficient D and the thermal conductivity coefficient λ are constant across the wall, the equations (11, 13, 26) enable us to calculate the temperature T and cheat flow q at any point x and at any time t_n with no information on their values at neighboring sites $x- \partial x$, $x+ \partial x$.

Let as recall the realistic case, when the wall consists of two layers: the left layer is of L_L thickness, D_L diffusion coefficient and λ_L thermal conductivity coefficient, while these parameters for the right layer are L_R, D_R and l_R (see Table 2). In this case the cheat transfer process formulae cannot be derived in so simple way, as eq.(11) and further. It is possible indeed to derive a function $f_{Th}(x)$ in form of the Fourier series, compensating the steady-state x-profile of temperature, like eq. (12), i.e.:

$$f_{Th}(x) = \frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sin\left(\frac{i\pi x}{L}\right)\left(1+\frac{L}{\pi}\frac{(b_L-b_R)}{i}\sin\left(\frac{i\pi L_L}{L}\right)\right) \quad (27)$$

where b_L and b_R denote slope of the steady-state profile in the left and right layer (see Fig. 1):

$$b_L \overset{\text{def}}{=} -\frac{\lambda_R}{\lambda_L L_R+\lambda_R L_L}, \qquad b_R \overset{\text{def}}{=} -\frac{\lambda_L}{\lambda_L L_R+\lambda_R L_L} \quad (28)$$

However, original attempts to derive simple formulae for exponential dumping factors, like in eq. (11), fulfilling the eq. (4) at the interlayer border point ($x=x_b\overset{\text{def}}{=}L_L$) did not succeed. Hence, in the sequel a semi-analytical approach is proposed, exploiting the superposition law for effects of $T(t, x_b)$ on $T(t, x\neq x_b)$ and $q(t, x\neq x_b)$.

Let us take hypothetically that $T_b T(0, x_b)$ may be changed by 1°C. The response of $T(t, x<x_b)$ and $T(t, x>x_b)$ to such excitation, i.e. the step response model h_{TbL} and h_{TbR}) may be derived by using the same trick as in eq. (11), but related separately to the left and right hand side of x_b (or more generally – to left (vs. x_b) and right homogeneous sections of the wall):

$$h_{TbR}(t,x) = 1-\left(\frac{x-x_b}{L_R}\right)-\frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sin\left(i\pi\frac{(x-x_b)}{L_R}\right)exp\left(-\frac{i^2\pi^2 D_R t}{L_R^2}\right), \quad x \geq x_b \quad (29)$$

$$h_{TbL}(t,x) = \frac{x}{L_L}-\frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sin\left(i\pi\left(1-\frac{x}{L_L}\right)\right)exp\left(-\frac{i^2\pi^2 D_L t}{L_L^2}\right), \quad x < x_b \quad (30)$$

Similarly, the step response of the heat flow may be written by proper modifications of eq. (26):

$$h_{qTbR}(t_n, x \geq x_b) = \frac{\lambda_R}{\Delta t}\left[\frac{\Delta t}{L_R} - \frac{2L_R}{\pi^2 D_R}\sum_{i=1}^{\infty}\frac{1}{i^2}\cos\left(i\pi\frac{(x-x_b)}{L_R}\right)\left(exp\left(-\frac{i^2\pi^2 D_R t_n}{L_R^2}\right) - exp\left(-\frac{i^2\pi^2 D_R t_{n-1}}{L_R^2}\right)\right)\right] \quad (31)$$

$$h_{qTbL}(t_n, x < x_b) = -\frac{\lambda_L}{\Delta t}\left[\frac{\Delta t}{L_L} - \frac{2L_L}{\pi^2 D_L}\sum_{i=1}^{\infty}\frac{-1^i}{i^2}\cos\left(\frac{i\pi x}{L_L}\right)\left(exp\left(-\frac{i^2\pi^2 D_L t_n}{L_L^2}\right) - exp\left(-\frac{i^2\pi^2 D_L t_{n-1}}{L_L^2}\right)\right)\right] \quad (32)$$

The effect of continuous changes of $T_b(t)$ on $T(t, x \neq x_o)$ may be calculated with the convolution model (21) for a fixed x (irrespective its effect at another x), e.g. for $x > x_g$ we have:

$$T(t, x > x_b) = \int_{\tau=0}^{\infty} h_{TbR}(\tau, x)\frac{dT_b(t-\tau)}{dt}d\tau \quad (33)$$

In a typical real heat conduction process $T_b(t)$ is produced by changes in a border temperature, say T_L(t). Thus, if $T_b(t)$ is the response to the unit step change in T_L the following equality is held:

$$h_{TL}(t, x > x_b) = \int_{\tau=0}^{\infty} h_{TbR}(\tau, x)\frac{dT_b(t-\tau)}{dt}d\tau = 1 - \frac{x}{L} - \frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sin\left(\frac{i\pi x}{L}\right)exp\left(-\frac{i^2\pi^2 Dt}{L^2}\right) \quad (34)$$

In a similar way one may express the step response $h_{TL}(t, x < x_g)$. However, in this case a dual impact of DT_L on h_{TL}(t, x) must be taken into account. First it is the direct effect of the left side disturbance DT_L expressed by the step-response formula denoted as $h_{TLT}(t, x)$:

$$h_{TLT}(t, x < x_g) = 1 - \frac{x}{L_L} - \frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sin\left(i\pi\frac{x}{L_L}\right)exp\left(-\frac{i^2\pi^2 D_L t}{L_L^2}\right), \quad x < x_b \quad (35)$$

and then a back effect of $dT_g(t)/dt$ expressed by $h_{TbL}(t, x)$ and used in the convolution formula like in eq. (33). The superposition of both these effect leads to the formula:

$$h_{TL}(t, x < x_b) = h_{TLT}(t, x < x_g) + \int_{\tau=0}^{\infty} h_{TbL}(\tau, x)\frac{dT_b(t-\tau)}{dt}d\tau \quad (36)$$

To check the formulae (34, 36) accuracy we made numerical calculations for eq. (34), assuming the homogeneous wall ($D_R = D_L = D$), by employing the analytical formula for $T_b(t) = h_{TL}(t, x_b)$, which is:

$$T_b(t) = h_{TL}(t, L_L) = \frac{L_R}{L} - \frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sin\left(\frac{i\pi L_L}{L}\right)exp\left(-\frac{i^2\pi^2 Dt}{L^2}\right) \quad (37)$$

and: $$\frac{dT_b(t)}{dt} = \frac{2\pi D}{L^2}\sum_{i=1}^{\infty} i * \sin\left(\frac{i\pi L_L}{L}\right)exp\left(-\frac{i^2\pi^2 Dt}{L^2}\right) \quad (38)$$

One may proof that:

$$\int_{\tau=0}^{\infty} h_{TbR}(\tau, x)\frac{dT_b(t-\tau)}{dt}d\tau = \frac{L-x}{L_R}T_b - I_2 \quad (39)$$

Here I_2 stands for the following integral:

$$I_2 \stackrel{\text{def}}{=} \frac{2}{\pi}\frac{L_R}{L}\int_{\tau=0}^{\infty}\sum_{j=1}^{\infty}\frac{1}{j}\sin\left(j\pi\frac{(x-L_L)}{L_R}\right)exp\left(-\frac{j^2\pi^2 D_R\tau}{L_R^2}\right)\frac{dT_b(t-\tau)}{dt}d\tau \quad (40)$$

$$I_2 = \frac{4}{\pi^2}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{ij}\frac{1}{v}\sin\left(j\pi\frac{(x-L_L)}{L_R}\right)\sin\left(\frac{i\pi L_L}{L}\right)\left(exp\left(-(1-v)\frac{i^2\pi^2 Dt}{L^2}\right) - exp\left(-\frac{i^2\pi^2 Dt}{L^2}\right)\right) \quad (41)$$

where $$v \stackrel{\text{def}}{=} \left(1 - \frac{j^2 L^2 D_R}{i^2 L_R^2 D_L}\right) \quad (42)$$

The above calculations faced numerical problems, as in many components of the series (41) v is very close to zero, although there exists a finite limit. It was found that in the case where substitution solves the problem.

In a similar way the heat-stream step response $h_{qTbqL}(t, x < x_b)$ and $h_{qTbqR}(t, x > x_b)$ may be derived involving $h_{qTbL}(t, x)$ and $h_{qTbL}(t, x)$, respectively, in the convolution like in eqs. (33, 36) and completed with $h_{qTLqL}(t, x)$ representing the direct effect of DT_L to $h_{qTbqL}(t, x < x_b)$ – see eq. (32):

$$h_{qTLqL}(t, x) \stackrel{\text{def}}{=} \frac{\lambda_L}{\Delta t}\left[\frac{\Delta t}{L_L} - \frac{2L_L}{\pi^2 D_L}\sum_{i=1}^{\infty}\frac{1}{i^2}\cos\left(\frac{i\pi x}{L_L}\right)\left(exp\left(-\frac{i^2\pi^2 D_L t_n}{L_L^2}\right) - exp\left(-\frac{i^2\pi^2 D_L t_{n-1}}{L_L^2}\right)\right)\right] \quad (43)$$

Through numerous calculations, it has been shown that the formulas (34, 36) give sufficiently good results with errors of the order of 10^{-6}, when confronted with effects of analytical formula (11). Interestingly, it was stated that the contribution of the integral I_2 to the value of $h_{T()}$ is in the order of 10^{-3} , that is shown in Fig.10 (see upper-right plot). Thus, in rough estimates, it can be taken that

$$T(t, x > x_g) \approx \frac{L-x}{L_R}T_b(t) \text{ and } T(t, x < x_g) \approx \frac{L-x}{L_L}T_b(t) \quad (44)$$

Now, let us recall the heterogeneous wall, for which the eq. (37) is not applicable so that a new formula for $T_b(t)$ must be found satisfying eq. (4). In the border layer of the following energy balance equation is valid:

$$\frac{\partial x}{2}(\rho_L c_L + \rho_R c_R)\frac{dT_b(t)}{dt} = \lambda_L\frac{\partial T\left(t,\ L_L - \frac{\partial x}{2}\right)}{\partial x} - \lambda_R\frac{\partial T\left(t,\ L_L + \frac{\partial x}{2}\right)}{\partial x} \quad (45)$$

Thus we have to express the derivatives $\lambda_R\frac{\partial T}{\partial x}, \lambda_L\frac{\partial T}{\partial x}$ (i.e. the input and output heat streams) as affected separately: first directly by change DT_L =1°C and back – by continuous changes in T_b due to D$T_{L,}$ with the convolution formula (21), like in eqs. (33, 36) for temperature calculations.

No analytical solver was found for $T_b(t)$ based on eq.(45), hence the step-response $h_{TTb}(t)$ to DT_L =1°C must be solved in a numerical way for the sequence of discrete time values t_n with a fixed Dt and for an appropriately small Dx. Having in mind the pictures shown in Figs. 7 and 8, it should be done by employing the analytical formulae for the heat streams averaged over Dt interval: $h_{qTLqL}(t, L_L$-D$x/2)$ – see eq. (43), $h_{qTbR}(t$, D$x/2)$ – eq. (31) and, $h_{qTbL}(t, L_L$-D$x/2)$ – eq. (32). The equation (45) discretized in space and time produces the step response model (notation will be used in sequel for better readability). The initial conditions are as follows:

$$\Delta T_b(0) = 0; \quad h_{qTLqL}\left(0, L_L - \frac{\partial x}{2}\right) = 0; \quad h_{qTbL}\left(0, L_L - \frac{\partial x}{2}\right) = 0; \quad h_{qTbR}\left(0, L_L + \frac{\partial x}{2}\right) = 0 \quad (46)$$

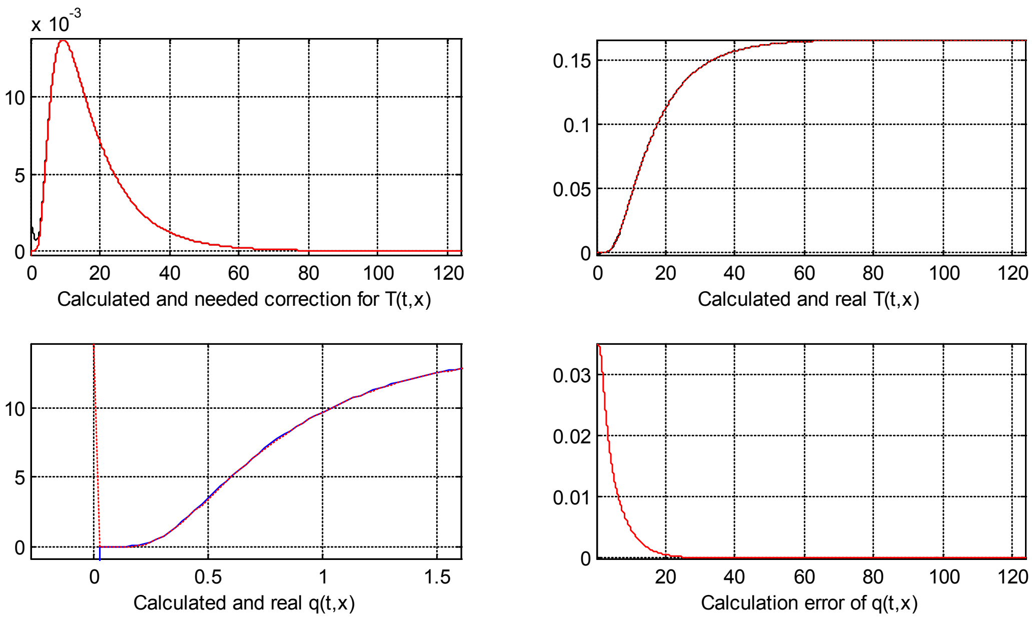

Figure 10. Temperature and heat stream time profiles calculated for x=0.4 using eqs.(37-43), confronted with results obtained using analytical formulae (11) and (22). Correction shown in the upper-left subfigure is produced by I_2 in eq.(39). Large error for $q(0, x)$ in the lower-right plot – point line - is produced by eq.(22)

The formula for $h_{TdTb}(t_n) \stackrel{\text{def}}{=} \Delta T_b(t_n)$ derived from eq. (45) may be written shortly in the following form:

$$\Delta T_b(t_n) = \frac{2\Delta t}{(\rho_L c_L+\rho_R c_R)\Delta x}\left(h_{qTLqL}\left(t, L_L-\frac{\Delta x}{2}\right)-\sum_{m=1}^{n} h_{qTbTb}\left(t_m, L_L-\frac{\Delta x}{2}\right)\Delta T_b(t_{n-m})\right) \quad (47)$$

where $h_{qTbTb}(\square)$ denotes the cumulated feed-back effect of $T_b()$ to itself via heat transfer:

$$h_{qTbTb}\left(\tau_m, L_L-\frac{\partial x}{2}\right) = h_{qTbR}\left(t_n, L_L+\frac{\Delta x}{2}\right) - h_{qTbL}\left(t_n, L_L-\frac{\Delta x}{2}\right) \quad (48)$$

Let D_{LR} and D_{RL} denote thermal diffusion coefficients for the left and right halves of the layer:

$$D_{LR} \stackrel{\text{def}}{=} \frac{2\lambda_L}{(\rho_L c_L+\rho_R c_R)} \quad D_{RL} \stackrel{\text{def}}{=} \frac{2\lambda_R}{(\rho_L c_L+\rho_R c_R)} \quad D_b \stackrel{\text{def}}{=} \frac{D_{LR}}{L_L}+\frac{D_{RL}}{L_R} = \frac{2}{L_R L_L}\frac{\lambda_L L_R+\lambda_R L_L}{(\rho_L c_L+\rho_R c_R)} \quad (49)$$

and $S_{qTLqL}(t, L_L$-Dx/2), $S_{qTbL}(t, L_L$-Dx/2) $S_{qTbR}(t, L_L$+Dx/2) denote the series in $h_{qTLqL}(t, L_L$–Dx/2), $h_{qTbL}(t, L_L$-Dx/2) and $h_{qTbR}(t, L_L$+Dx/2) in eqs.(31, 32, 43), respectively:

$$S_{qTbL}\left(t_n, L_L-\frac{\Delta x}{2}\right) = -\frac{2L_L}{\pi^2 D_L}\sum_{i=1}^{\infty}\frac{1}{i^2}\cos\left(i\pi\frac{\Delta x}{2L_L}\right)\left(exp\left(-\frac{i^2\pi^2 D_L t_n}{L_L^2}\right)-exp\left(-\frac{i^2\pi^2 D_L t_{n-1}}{L_L^2}\right)\right) \quad (50)$$

$$S_{qTbR}\left(t_n, L_R+\frac{\Delta x}{2}\right) = -\frac{2L_R}{\pi^2 D_R}\sum_{i=1}^{\infty}\frac{1}{i^2}\cos\left(i\pi\frac{\Delta x}{2L_R}\right)\left(exp\ -\frac{i^2\pi^2 D_R t_n}{L_R^2}\right)-exp\left(-\frac{i^2\pi^2 D_R t_{n-1}}{L_R^2}\right)\Bigg) \quad (51)$$

$$S_{qTLqL}\left(t_n, L_L-\frac{\Delta x}{2}\right) = -\frac{2L_L}{\pi^2 D_L}\sum_{i=1}^{\infty}\frac{-1^i}{i^2}\cos\left(i\pi\frac{\Delta x}{2L_L}\right)\left(exp\left(-\frac{i^2\pi^2 D_L t_n}{L_L^2}\right)-exp\left(-\frac{i^2\pi^2 D_L t_{n-1}}{L_L^2}\right)\right) \quad (52)$$

Substitution of eqs. (50–52) to eq.(48, 47) and a simple transformation leads to the required formula for the step –response $h_{TTb}(t_n)$, for n=1, 2, ….:

$$\Delta T_b(t_n) = \frac{D_{LR}}{\Delta x}\left(\frac{\Delta t}{L_L}-S_{qTbTb}\left(t_n, L_L-\frac{\Delta x}{2}\right)\right)-\frac{D_b}{\Delta x}h_{TTb}(t_{n-1}) + \\ -\frac{1}{\Delta x}\sum_{m=1}^{n}\left(D_{LR}S_{qTbL}\left(t_n, L_L-\frac{\Delta x}{2}\right)+D_{RL}S_{qTbR}\left(t_m, L_L+\frac{\Delta x}{2}\right)\right)\Delta T_b(t_{n-m}) \quad (53)$$

and finally:

$$h_{TTb}(t_0=0)=0; \qquad h_{TTb}(t_n) = h_{TTb}(t_{n-1})+\Delta T_b(t_n) \quad (54)$$

Notice that all of the expressions (50-53) must be calculated for x, where their convergence is weak, thus numerical problems may be expected.

In order to evaluate the formula (53) accuracy we have applied it to the homogeneous brick-wall (see Table 2), and the series (denotes as T_bn(t_n) – numerical) obtained in this way was confronted with that calculated using the analytical formula (37) (denotes as T_ba(t_n) – analytical). The results are shown in Fig.11. One can see in this figure that the accuracy of the model (53) is very good (maximum relative error of T_bn(t_n) is less than 0.03%). However, it should be emphasized that the achievement of such precision requires the use of very low values for Dx=5*10^{-6}L [m], rather short sampling (and averaging) interval Dt=1s, and rather large number of components in the series (50−52) ($I_{\max}$=4000). Especially, effect of Dx is very significant. It has been found that Dx=1*10^{-4}L already gives errors of the order of 1%, while Dx-=1*10^{-6}L leads to numerical instability of eq. (53,53). Large I_{max} makes the calculation time consuming, but we need to calculate the series T_bn(t_n) only once, and it may be then used in multiple simulations as the convolution model. The left subfigure in Fig.11 illustrates contribution of main components of eq.(53) to the DT_b, i.e. DT_{bL} – effect of the left (original) excitation T_L=1°C and DT_{bL} – feed-back effect of $T_b(t)$.

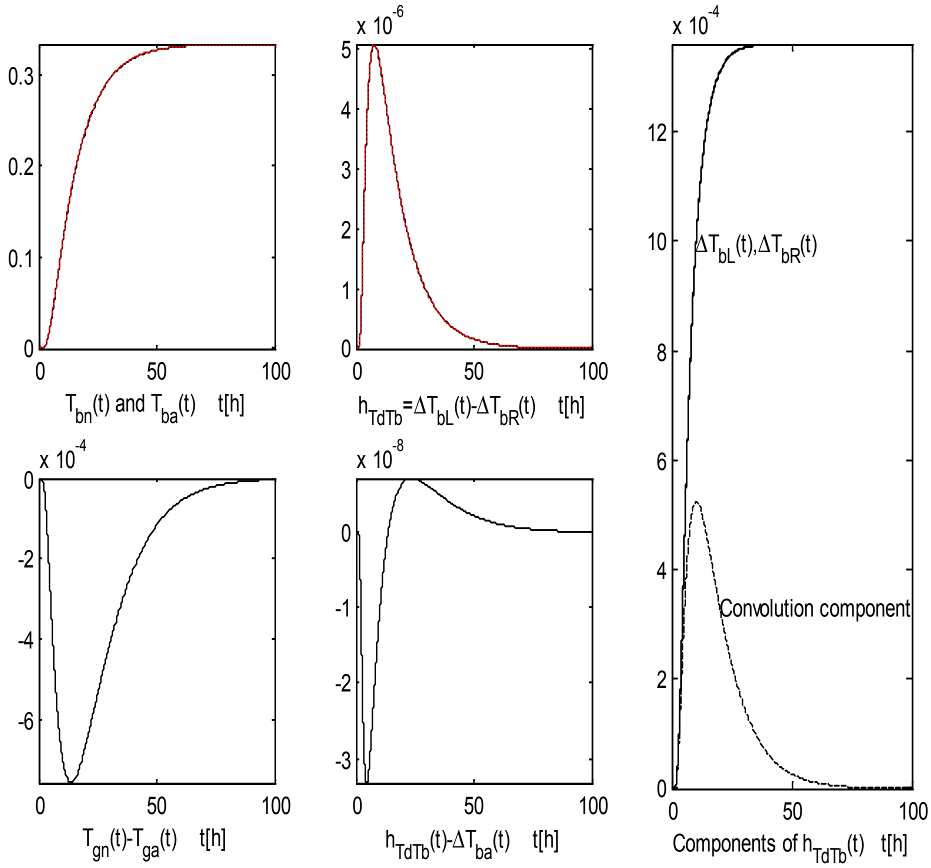

Figure 11. Effects of applying the semi-analytical formula (53) to the homogeneous wall, compared with the temperature $T_b(t)$ profile obtained by using the analytical formula (37). Heat streams (50-52) calculated with Dx=5*10^{-6}L [m], averaged over Dt=1s. The modeling errors shown in left subfigures. Contribution of components of eq.(53) is shown in the right subfigure (dotted line – effect of the convolution term in eq. (53)

Figure 12 illustrate differences in dynamics of heat streams involved in the energy balance equation (45).

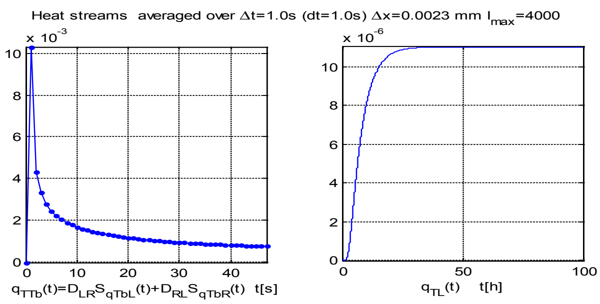

Figure 12. Differences in dynamics of heat streams $q_{TTb}(t)=h_{qTbTb}$ – see eq.(48), and $q_{TL}(t)=h_{qTLqL}$ – see eq.(43), going to the layer of $x=L_L$ due to the step response of $T_b(t)$ – (left subfigure) and from the original excitation DT_L =1°C (right subfigure). Calculations made for the homogeneous wall

Now, there are no more obstacles to use the formulae (53, 54) to calculate the step response of T_b, to the unit step change of T_L, i.e. $h_{TTb}(t, L_L)=T_b(t)$, for a heterogeneous two-layer partition, as it needs only proper values for D_L, D_R, λ_L and l_R to be taken.

Results of such calculations, made for the wall characterized in Table 2, and treated before with the state-space model (7), are presented in Fig. 13 and Fig. 14.

Finally, we may derive the most demanded semi-analytical formula for the step response of heat losses stream to the unit step change DT_b=1°C, using the approach as in eq. (46), and employing the step response $h_{TdTb}(t)$=DT_b(t) calculated once with eq. (53) and stored for the considered wall. As in the eqs. (42, 47), by virtue of superposition law the following equation may be written:

$$h_{qTL0}(t_n) = h_{qTLqL}(t_n,0) - \sum_{m=1}^{n} h_{qTbL}(t_m,0)\Delta T_b(t_{n-m}) \qquad (55)$$

The step responses $h_{qTLqL}(t_n,0)$ and $h_{qTbL}(t_n,0)$ may be calculated once for n=1,... n_{max}, (due to causality $h_{qTLqL}(0,0) \equiv 0$ and $h_{qTbL}(0,0) \equiv 0$), by using the formulae (43) and (32) with x=0:

$$h_{qTLqL}(t,0) = \frac{\lambda_L}{\Delta t}\left[\frac{\Delta t}{L_L} - \frac{2L_L}{\pi^2 D_L}\sum_{i=1}^{\infty}\frac{1}{i^2}\left(exp\left(-\frac{i^2\pi^2 D_L t_n}{L_L^2}\right) - exp\left(-\frac{i^2\pi^2 D_L t_{n-1}}{L_L^2}\right)\right)\right] \qquad (56)$$

$$h_{qTbL}(t_n,0) = -\frac{\lambda_L}{\Delta t}\left[\frac{\Delta t}{L_L} - \frac{2L_L}{\pi^2 D_L}\sum_{i=1}^{\infty}\frac{-1^i}{i^2}\left(exp\left(-\frac{i^2\pi^2 D_L t_n}{L_L^2}\right) - exp\left(-\frac{i^2\pi^2 D_L t_{n-1}}{L_L^2}\right)\right)\right] \qquad (57)$$

In the similar way one can derive the step response of DT_b to the unit step change of the right side temperature DT_R=1°C (denotes as – see eqs. (46-53)), and then derive the formula like

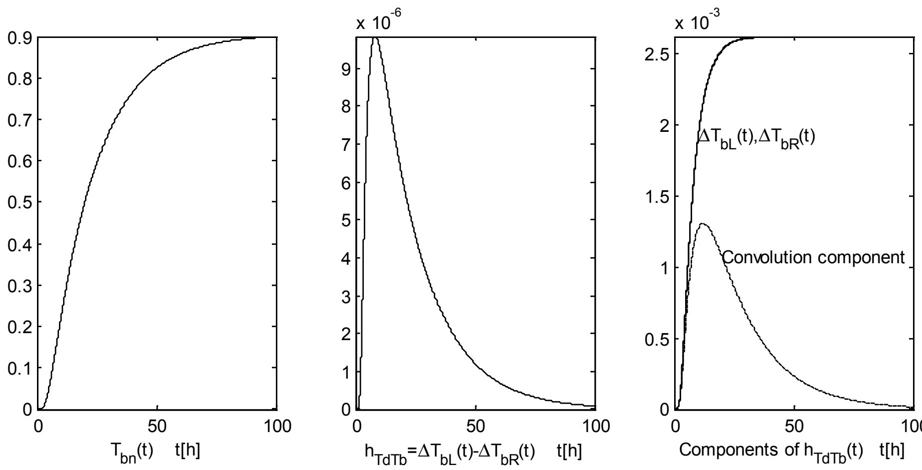

Figure 13. Temperature $T_b(t)$ profile obtained by using the semi-analytical formula (53), calculated for the two-layer wall (brick and isolation) with the interlayer border at $x=L_L=0.3$m (see Table 2 for further parameters). Heat streams (50-52) calculated with Dx=5*10^{-6}L [m], averaged over Dt=1s. Contribution of components of eq.(53) is shown in the right subfigure (dotted line – effect of the convolution term in eq.53)

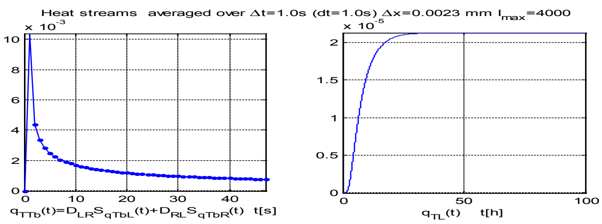

Figure 14. Differences in dynamics of heat streams $q_{TTb}(t)=h_{qTbTb}$ – see eq.(48), and $q_{TL}(t)=h_{qTLqL}$ – see eq.(43), calculated for the two layer wall (see Figs.12, 13 for more explanations)

(55) for step response of heat losses at x=0 to DT_R (in this case it depends only on DT_b):

$$h_{qTR0}(t_n) = -\sum_{m=1}^{n} h_{qTbL}(t_m, 0)\ h_{TRdTb}(t_{n-m}) \qquad (58)$$

The series and calculated ones for a considered wall constitute the complete model for heat losses $q_i(t_n)$, which may be exploited in any simulations by using the convolution formula:

$$q_i(t_n) = \sum_{m=1}^{n} \left(h_{qTL0}(t_m)\Delta T_L(t_{n-m}) + h_{qTR0}(t_m)\Delta T_R(t_{n-m}) \right) \qquad (59)$$

Effects of calculation of the step response with eq. (55) (index a) compared to the step response of heat losses obtained by applying the state-space model (8) (index n) with Dx={0.003,10^{-3}L, 5*10^{-4}L} and Dt=0.1s are shown in Fig. 15 (effects of eq.8 were averaged over Dt=1s, like in eq. 55). It may be seen that er-

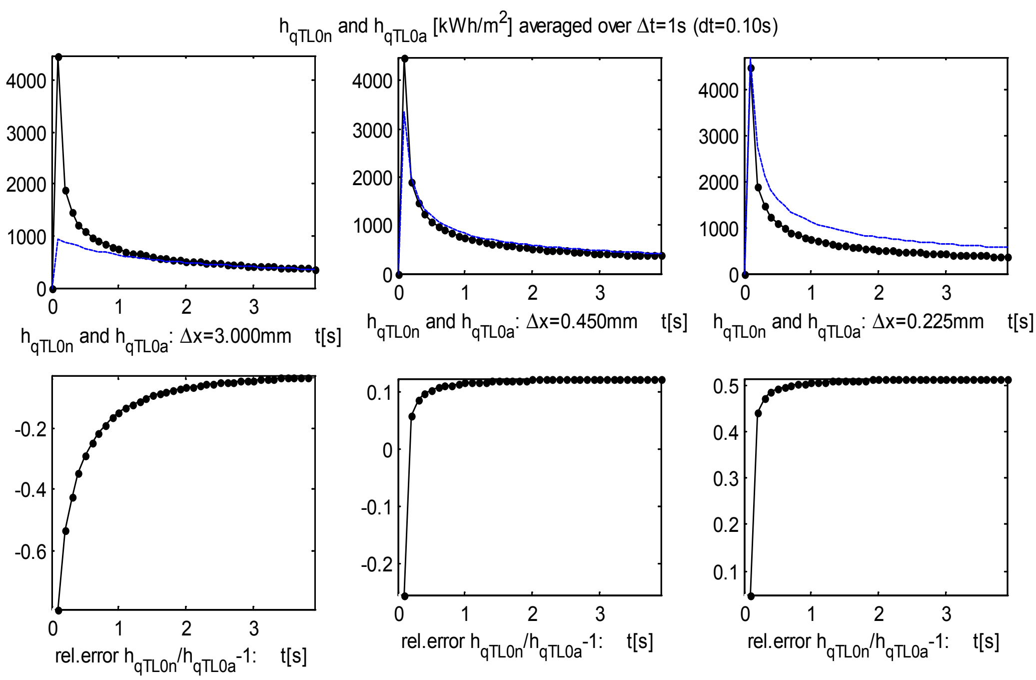

Figure 15. Comparison of heat losses step response at an initial time section, calculated with the model (55) – point-lines and by employing the discretized state-space model (8). – dotted lines

rors of eq. (8) are significant, and they a minimal for Dx=10^{-3}/ L=0.45mm. Larger Dt and Dx lead to much higher errors, but for the lower Dx the relation Dt/Dx is too large, that produces numerical errors. The small Dt makes the model (8) c.a. twice more time consuming then the formula (58).

Conclusions

Analytical formulae for heat transfer available in literature [7, 8] make possible calculation of step response of temperature and heat losses for any homogenous building partition, that may be then directly applied in simulation instead of state space (recursive) model. In the paper semi-analytical formulae was derived for two-layer walls (typical building partitions), which was found as more accurate and less time-consuming then the state-space model. Hence, it may be recommended for real-time simulations demanded in modern heating control systems. Moreover, the proposed model may be used to check accuracy of simplified calculations with the state-space formulae, and to adjust discretization parameters Dt and Dx.

Acknowledgments

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