Jacek RYCZYŃSKI, Henryk SPUSTEK

MODELING OF LOGISTICS AND SAFETY PROCESSES USING DIFFERENTIAL EQUATIONS – SELECTED EXAMPLES

Abstract

The article describes the use of differential calculus to determine the form of differential equations family of curves. Form of differential equations obtained by eliminating the parameters of the equations describing the different family of curves. Elimination of the parameters has been performed several times by differentiation starting equations. Received appropriate form of differential equations for the case of family circles, family of curves of the second degree and the families of the logistic function.

INTRODUCTION

Uses differential equations ordinary is very much, they occur in different areas of knowledge. This is due to the presence of interdisciplinary. Most of these models is a dynamic, often non-linear (5, s. 4 - 9).

The equation is described by the ordinary defines only a derivative function. In this way, it does not designate a particular function, but some family functions, detailing the solution by adding initial conditions (3, s. 6).

Integrability equations can be obtained in several known ways, among which must be changed:

– presentation in the form of physical, mathematical;
– geometrical representation of compounds in the form of analytical;
– eliminating parameters equations by successive differential process (2, s.13).

In this article are shown examples of differential equations according to the third of the aforementioned ways.

The first two examples concern geometrical issues. Inspiration for authors fixed relationship (1) ratification of flat curves of constant curvature. Taking into account waves, which can be presented in the form of equations $y = f(x)$, it can be concluded that these curves will have fixed property in the case of curvature condition is met:

$$\frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} = const \tag{1}$$

Relationship (1) is a non-linear equation is described by second row. The third example shows family logistic function and is a reference to practical issues of econometric simulations.

1. DIFFERENTIAL EQUATION CIRCLES FAMILY IN XOY PLANE

The equation is in the form of divisions family:

$$(x-a)^2 + (y-b)^2 = R^2 \tag{2}$$

where: a, b, R are parameters and y is variable depending on x .

In order to eliminate parameters we differentiate equation three districts in relation to family. After differentiate you get:

$$2(x-a) + 2(y-b)\left(\frac{dy}{dx}\right) = 0 \tag{3}$$

that is:

$$(x-a) + (y-b)\left(\frac{dy}{dx}\right) = 0 \tag{4}$$

Second gives:

$$1+\left(\frac{dy}{dx}\right)^2 + (y-b)\left(\frac{d^2y}{dx^2}\right) = 0 \tag{5}$$

However the third:

$$2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + (y-b)\left(\frac{d^3y}{dx^3}\right) = 0 \tag{6}$$

And next after conversion:

$$3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + (y-b)\left(\frac{d^3y}{dx^3}\right) = 0 \tag{7}$$

We write equation (5) and (7) in the form as follows:

$$(y-b)\left(\frac{d^2y}{dx^2}\right) = -\left[1+\left(\frac{dy}{dx}\right)^2\right] \tag{8}$$

$$(y-b)\left(\frac{d^3y}{dx^3}\right) = -3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) \tag{9}$$

We make parties equation (8) and (9) receiving relationship:

$$\frac{\frac{d^2y}{dx^2}}{\frac{d^3y}{dx^3}} = \frac{1+\left(\frac{dy}{dx}\right)^2}{3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)} \tag{10}$$

Restated equation (9) we get equation

$$3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) - \left[1+\left(\frac{dy}{dx}\right)^2\right]\left(\frac{d^3y}{dx^3}\right) = 0 \tag{11}$$

Equation (11) is sought is described by the equation family circles in XOY plane.

2. SECOND DEGREE CURVES DIFFERENTIAL EQUATION ROSES FAMILY

A second degree curves is as follows:

$$a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + 1 = 0 \tag{12}$$

Similarly the reflection carried out at first this article, five parameters fixed to eliminate visible in the equation (12). To this end we make five differential equation (12).

As a result of subsequent steps abstract we have: after the first

$$2a_1x + 2a_2y\frac{dy}{dx} + a_3y + a_3x\frac{dy}{dx} + a_4 + a_5\frac{dy}{dx} = 0 \tag{13}$$

after the second step:

$$\begin{aligned} &2a_1 + 2a_2y\frac{d^2y}{dx^2} + 2a_2\left(\frac{dy}{dx}\right)^2 + 2a_3\left(\frac{dy}{dx}\right) \\ &+ a_3\left(\frac{d^2y}{dx^2}\right) + a_5\left(\frac{d^2y}{dx^2}\right) = 0 \end{aligned} \tag{14}$$

after the third step:

$$\begin{aligned} &2a_2y\left(\frac{d^3y}{dx^3}\right) + 2a_2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 4a_2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) \\ &+ 2a_3\left(\frac{d^2y}{dx^2}\right) + a_3\left(\frac{d^3y}{dx^3}\right) + a_5a_3\left(\frac{d^3y}{dx^3}\right) = 0 \\ &2a_2y\left(\frac{d^3y}{dx^3}\right) + 6a_2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 2a_3\left(\frac{d^2y}{dx^2}\right) \\ &+ (a_3 + a_5)\left(\frac{d^3y}{dx^3}\right) = 0 \end{aligned} \tag{15}$$

after the fourth step:

$$\begin{aligned} &8a_2\left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right) + 2a_2y\left(\frac{d^4y}{dx^4}\right) + 6a_2\left(\frac{d^2y}{dx^2}\right)^2 \\ &+ 2a_3\left(\frac{d^3y}{dx^3}\right) + (a_3 + a_5)\left(\frac{d^4y}{dx^4}\right) = 0 \end{aligned} \tag{16}$$

after the fifth step:

$$\begin{aligned} &20a_2\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^3y}{dx^3}\right) + 10a_2\left(\frac{dy}{dx}\right)\left(\frac{d^4y}{dx^4}\right) + 2a_2y\left(\frac{d^5y}{dx^5}\right) \\ &+ 2a_3\left(\frac{d^4y}{dx^4}\right) + (a_3 + a_5)\left(\frac{d^5y}{dx^5}\right) = 0. \end{aligned} \tag{17}$$

Eliminating parameters: a_1, a_2 and a_3 from the equations (15 - 17). We Convert equation as follows:

$$\begin{aligned} &2a_2y\left(\frac{d^3y}{dx^3}\right) + 6a_2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 2a_3\left(\frac{d^2y}{dx^2}\right) = \\ &-(a_3 + a_5)\left(\frac{d^3y}{dx^3}\right) \end{aligned} \tag{18}$$

$$\begin{aligned} &8a_2\left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right) + 2a_2y\left(\frac{d^4y}{dx^4}\right) + 6a_2\left(\frac{d^2y}{dx^2}\right)^2 \\ &+ 2a_3\left(\frac{d^3y}{dx^3}\right) = -(a_3 + a_5)\left(\frac{d^4y}{dx^4}\right) \end{aligned} \tag{19}$$

$$\begin{aligned} &20a_2\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^3y}{dx^3}\right) + 10a_2\left(\frac{dy}{dx}\right)\left(\frac{d^4y}{dx^4}\right) + 2a_2y\left(\frac{d^5y}{dx^5}\right) \\ &+ 2a_3\left(\frac{d^4y}{dx^4}\right) = -(a_3 + a_5)\left(\frac{d^5y}{dx^5}\right) \end{aligned} \tag{20}$$

We divide equation (18) by (19) and (19) by (20) and the equations:

$$\begin{cases} \dfrac{2a_2y\left(\frac{d^3y}{dx^3}\right) + 6a_2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 2a_3\left(\frac{d^2y}{dx^2}\right)}{8a_2\left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right) + 2a_2y\left(\frac{d^4y}{dx^4}\right) + 6a_2\left(\frac{d^2y}{dx^2}\right)^2 + 2a_3\left(\frac{d^3y}{dx^3}\right)} = \dfrac{\frac{d^3y}{dx^3}}{\frac{d^4y}{dx^4}} \\[3ex] \dfrac{8a_2\left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right) + 2a_2y\left(\frac{d^4y}{dx^4}\right) + 6a_2\left(\frac{d^2y}{dx^2}\right)^2 + 2a_3\left(\frac{d^3y}{dx^3}\right)}{20a_2\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^3y}{dx^3}\right) + 10a_2\left(\frac{dy}{dx}\right)\left(\frac{d^4y}{dx^4}\right) + 2a_2y\left(\frac{d^5y}{dx^5}\right) + 2a_3\left(\frac{d^4y}{dx^4}\right)} = \dfrac{\frac{d^4y}{dx^4}}{\frac{d^5y}{dx^5}} \end{cases} \tag{21}$$

Transforming the first equation the equations (21) we have:

$$\begin{aligned} &\left[2a_2y\left(\frac{d^3y}{dx^3}\right) + 6a_2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 2a_3\left(\frac{d^2y}{dx^2}\right)\right]\left(\frac{d^4y}{dx^4}\right) = \\ &= \left[\begin{aligned} &8a_2\left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right) + 2a_2y\left(\frac{d^4y}{dx^4}\right) + 6a_2\left(\frac{d^2y}{dx^2}\right)^2 \\ &+ 2a_3\left(\frac{d^3y}{dx^3}\right) \end{aligned}\right]\left(\frac{d^3y}{dx^3}\right) \end{aligned}$$

$$\begin{aligned} &\left[4a_2\left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right) + a_2y\left(\frac{d^4y}{dx^4}\right) + 3a_2\left(\frac{d^2y}{dx^2}\right)^2\right]\left(\frac{d^5y}{dx^5}\right) - \\ &- \left[\begin{aligned} &10a_2\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^3y}{dx^3}\right) + 5a_2\left(\frac{dy}{dx}\right)\left(\frac{d^4y}{dx^4}\right) \\ &+ a_2y\left(\frac{d^5y}{dx^5}\right) \end{aligned}\right]\left(\frac{d^4y}{dx^4}\right) = \\ &a_3\left[\left(\frac{d^4y}{dx^4}\right)^2 - \left(\frac{d^3y}{dx^3}\right)\left(\frac{d^5y}{dx^5}\right)\right]. \end{aligned} \tag{22}$$

By converting the second equation equations (21) we have:

$$\left[\begin{array}{l} 8a_2\left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right)+2a_2y\left(\frac{d^4y}{dx^4}\right) \\ +6a_2\left(\frac{d^2y}{dx^2}\right)^2+2a_3\left(\frac{d^3y}{dx^3}\right) \end{array}\right]\left(\frac{d^5y}{dx^5}\right)=$$

$$=\left[\begin{array}{l} 20a_2\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^3y}{dx^3}\right)+10a_2\left(\frac{dy}{dx}\right)\left(\frac{d^4y}{dx^4}\right) \\ +2a_2y\left(\frac{d^5y}{dx^5}\right)+2a_3\left(\frac{d^4y}{dx^4}\right) \end{array}\right]\left(\frac{d^4y}{dx^4}\right)$$

$$\left[4a_2\left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right)+a_2y\left(\frac{d^4y}{dx^4}\right)+3a_2\left(\frac{d^2y}{dx^2}\right)^2\right]\left(\frac{d^5y}{dx^5}\right)- \tag{23}$$

$$-\left[\begin{array}{l} 10a_2\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^3y}{dx^3}\right) \\ +5a_2\left(\frac{dy}{dx}\right)\left(\frac{d^4y}{dx^4}\right) \\ +a_2y\left(\frac{d^5y}{dx^5}\right) \end{array}\right]\left(\frac{d^4y}{dx^4}\right)=a_3\left[\begin{array}{l} \left(\frac{d^4y}{dx^4}\right)^2- \\ \left(\frac{d^3y}{dx^3}\right) \\ \left(\frac{d^5y}{dx^5}\right) \end{array}\right]$$

Then dividing parties equation (22) and (23) we set the factor a_2 and by inserting the expression for equation (22) we set the factor a_3 . In the end, after you insert both according to one of the equations (20) is sought is described by the equation:

$$9\left(\frac{dy}{dx}\right)^2\left(\frac{d^5y}{dx^5}\right)-45\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^3y}{dx^3}\right)\left(\frac{d^4y}{dx^4}\right)+40\left(\frac{d^3y}{dx^3}\right)^3=0 \tag{24}$$

3. EQUATION GROWTH OF ROBERTSON

As an example, the last selected family curves applicable practical issues in economic forecasting. Family logistic curve is uniquely privileged place. Logistic curve interest dating back to the end of XIX century. He was a pioneer. F. Verhulst, and a major proponent of first use. F. Pearl. He gave it a final form to be used with great success for today. demographers had been (e.g. , A. J. Lottery) and biologists as the first attempt to use a logistic curve to the right development of the population. Logistic curve can be well adapted to a temporary increase in quantitative imaging living organisms in conditions to be determined (referred to reserve a specific food or food supply). This curve may well describe a quantitative increase in living organisms, where there is no human intervention even in their lives, for example, in major science does not accessible areas (3, s. 20). The increase in human population does not develop exactly according to this curve. It affects the many socio-economic conditions affecting the food resources. Did not contest the possibility of using logistic curve as universal rights development of the population. It has been shown that the development of population Sweden has averaged in accordance with logistic curve. This development in the past two centuries were made almost in isolation (the largely external factors) .logistic function well describes demand for durable goods, such as cars in highly developed countries, on TV, on motorcycles and mopeds, motorcycles and scooters, on the radios (1, s.29). It reflects the increase in the number of telephone subscribers in Poland. In Poland the logistic function to test demand for durable goods. On the basis of the data concerning the state of motorcycles and bicycles in rural households in Poland in the years 1951-1962 was extrapolated resulting function for the years 1963-1966. It was emphasized, that logistic function may give good forecasts short-term and medium-term, and long-term forecasts are risky. Using the logistic performance we can describe the relationship of length of service in a fixed position in the case of workers who are aged 50 years (beyond this age limit performance decreases). It should be noted, however, that the impact of technological progress we skip the above approximations. Using the logistic change of the speed you can describe certain phenomena, such as chemical dissolution rate is salt in the water. Similarly, the dependency many reactions from their duration, especially fusion reactions, e.g. reaction water with carbide, as a result of which we receive acetylene. W. Winkler suggested the use logistic curve to describe density change network of railways within a specified area. Logistic function was used in medicine (e.g. development of certain diseases in the body) and pharmacology (dose regimen certain drugs). There have been attempts to use logistic curve in rehabilitation and anthropology.

A logistic function (figure 1) can be obtained starting from physical considerations. Logistic curve equation we can describe all phenomena, which is directly proportional to speed development of growth factor called momentum and the distance of the increase of saturation level (a-y), which we can call a factor in braking, because the time the value of this factor decreases.

Considerations of physics we have is described by the following differential equation:

$$\frac{dy}{dx}=ky(a-y) \tag{25}$$

Where: k>0 is the factor proportionality, a saturation level (ordinate, is used asymptotic logistic curve), bearing the **name of Robertson growth**, (6, s. 365 - 373).

We split variables in equation (24):

$$dy=ky(a-y)dx$$

$$\frac{dy}{y(a-y)}=kdx$$

We advocate a solution of the form: $y=A\exp(\alpha x)$ (26) .

After you get the following equivalent forms transformation analytical logistics functions:

$$y=\frac{a}{1+b\exp(-cx)} \tag{27}$$

$$y=\frac{a}{a^*+b^*\exp(-cx)} \tag{28}$$

$$y=\frac{a}{1+\exp(b_1-cx)} \tag{29}$$

$$y=\frac{1}{a^*+\exp-c(x-T)} \tag{30}$$

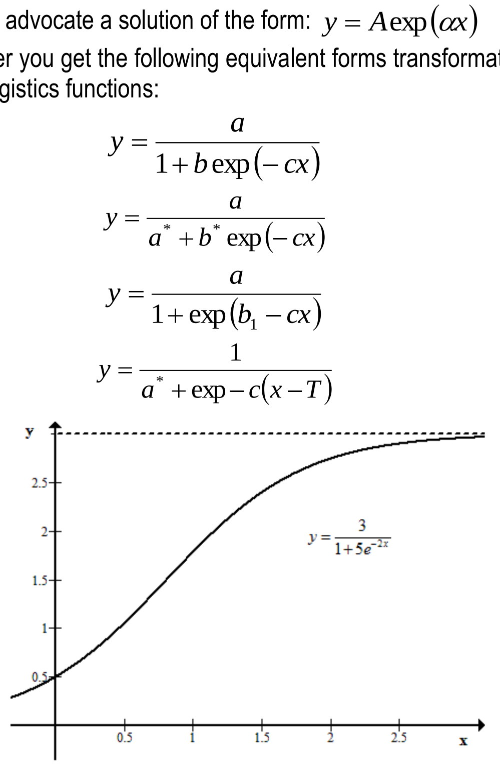

***Fig. 1.** Example logistic function*

In practice there is usually a string of empirical and based on them we are seeking to determine the function of one of these four characters. The logistic function in predicting economic processes gave rise among the once econometricians many controversies, difficulties in estimating parameters, that these features were rarely used in practice. Only when the 1927H. Hotelling has developed a simple and easy to use method of estimating parameters logistic function form:

$$Y_t = \frac{\alpha}{1+\beta e^{-\gamma t}} \qquad (t = 1,\dots,n) \tag{31}$$

where: $\alpha, \beta, \gamma > 0$ (compare with dependence (27).

α - the process is called saturation level growth ,

$\beta > 1$ - ensures that this function has a point of inflection.

The above well-suited to bringing empirical data relating to cases under investigation monotonic growth phenomena captured in a long period of time, the growth rate decreasing to zero. This function is non-linear with its three parameters

Verhust proposed following dynamic model, which is a variety according (25):

$$Y_t = \frac{\alpha}{1+\beta e^{-\gamma t}} \tag{32}$$

In the above depending on the parameter is interpreted as the maximum possible size of the population. Randomization model (32) leads to a family of logistic curves (Figure 2).

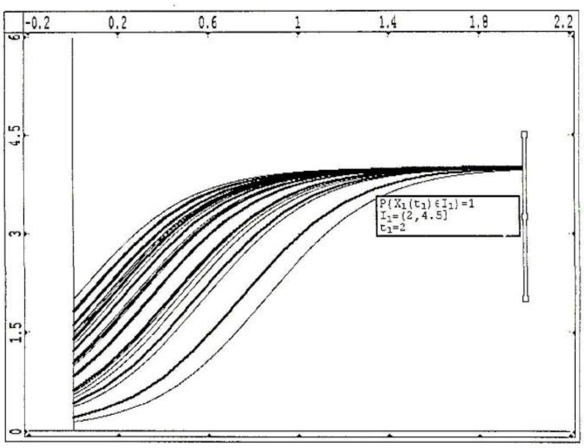

Fig. 2. *Family logistics functions being the solution to the deterministic equations Verhulst, with a random initial conditions [4, p. 136]*

SUMMARY

Examples are presented in the article usefulness account derivative in resolving issues that describe family selected waves. Due to the limited volume article mentioned only three examples. Content contained here can be extended to other examples, both geometry and other disciplines, as well as of computer simulations presented examples

BIBLIOGRAFIA

1. Goryl A., Jędrzejczyk Z., Kukuła K., Osiewalski J., Walkosz A., *Wprowadzenie do ekonometrii w przykładach i zadaniach*, Wyd. PWN, Warszawa 1996.
2. Gutowski R., *Równania różniczkowe zwyczajne*, Wyd. WNT, Warszawa 1971.
3. Gutenbaum J., *Podstawy modelowania matematycznego. Cz. B. Modele dynamiczne*, Wyd. WSISiZ, Warszawa 2001.
4. Janicki A., Izydorczyk A., *Komputerowe metody w modelowaniu stochastycznym*, Wyd. WNT, Warszawa 2001.
5. Przemieniecki J. S., *Mathematical Methods in Defense Analyses*, Air Force Institute of Technology, Washington 1994.
6. Ryczyński J., *Modelowanie systemów bezpieczeństwa - podstawy praktyki inżynierskiej*, WSOWL, Wrocław 2014.
7. Smolik S., *Wyznaczanie parametrów krzywej logistycznej*, Przegląd Statystyczny R. XXXII - zeszyt 4 – 1984.

MODELOWANIE SYSTEMÓW LOGISTYCZNYCH I SYSTEMÓW BEZPIECZEŃSTWA Z WYKORZYSTANIEM RÓWNAŃ RÓŻNICZKOWYCH - WYBRANE PRZYKŁADY

Streszczenie

W artykule przedstawiono zastosowanie rachunku różniczkowego do wyznaczania postaci równań różniczkowych rodziny krzywych. Postać równań różniczkowych otrzymano w wyniku eliminacji parametrów z równań opisujących rodzinę poszczególnych krzywych. Eliminacja parametrów została wykonana w wyniku kilkukrotnego różniczkowania równań wyjściowych. Otrzymano stosowne postaci równań różniczkowych dla przypadku rodziny okręgów, rodziny krzywych drugiego stopnia oraz rodziny funkcji logistycznej.

Autorzy:

dr inż. **Jacek RYCZYŃSKI** – adiunkt, Zakład Inżynierii Bezpieczeństwa, Wydział Nauk o Bezpieczeństwie, Wyższa Szkoła Oficerska Wojska Lądowych, 51-150 Wrocław, ul. Czajkowskiego 109, Tel. +48 71 261 658 478, jacek.ryczynski@gmail.com.

dr hab. **Henryk SPUSTEK** – profesor, Kierownik Zakładu Nauk o Bezpieczeństwie, Wydział Prawa i Administracji, Uniwersytet Opolski w Opolu; 45-060 Opole; ul. Katowicka 87a. Tel.: +48 77 452 75 00, wpia@uni.opole.pl