BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 61, No. 3, 2013
DOI: 10.2478/bpasts-2013-0057

Design of systems with extremal dynamic properties

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Abstract. In this article the problem of determination of coefficients $a_1, a_2, \ldots, a_n$ of the characteristic equation which yield required extremal values of the solution $x(t)$ at extremal values τ of time is solved. The extremal values of $x(t)$ and τ are treated as functions of the roots $s_1, s_2, \ldots, s_n$. The analytical formulae enable to design the systems with prescribed dynamic properties. The zeros and poles can be located using the known method.

The extremal dynamic error $x(t)$ is the most important property of the behaviour of the system. This extremal value of the dynamic error has fundamental role in the chemical industry where for example overrising temperature or pressure can lead to an explosion.

A second very important property is the extremal time τ connected with the extremal value of the error. This property is essential in the electroenergetic system, which can be destroyed by the overvoltages waves.

Key words: systems, extremal dynamic properties, overvoltages waves.

1. Problem statement

We consider three sets of variables. Two of these sets are known, and knowing this we want to determine the third set.

Problem 1. There are given initial conditions $c_1, c_2, ..., c_n$ and the coefficients $a_1, a_2, ..., a_n$ of the differential equation $x^{(n)}(t) + a_1 x^{(n-1)}(t) + ... + a_n x(t) = 0$. We want to determine extremal times $\tau_1, \tau_2, ..., \tau_{n-1}$ at which the solution of $x(t)$ of the differential equation assumes extremal values $x_1, x_2, ..., x_n$.

For this problem analytic formulae of solution $x(t)$ and necessary condition $x^{(1)}(t) = 0$, and sufficient condition $x^{(2)}(t) \neq 0$ are known. Numerical solution can be obtained immediately using a computer.

Problem 2. There are given eigenvalues $s_1, s_2, ..., s_n$ of the characteristic equation of the differential equation and extremal times $\tau_1, \tau_2, ..., \tau_{n-1}$ are known. We want to determine corresponding initial conditions $c_1, c_2, ..., c_n$.

Problem 3. There are given initial conditions $c_1, c_2, ..., c_n$ and extremal times $\tau_1, \tau_2, ..., \tau_n$. We want to find corresponding coefficients $a_1, a_2, ..., a_n$ of the differential equation $\sum_{i=1}^{n} a_i x^{(i)}(t) = 0$.

2. Statement of the problem 2

Consider the n-th order linear differential equation with constant and real parameters:

$$x^{(n)}(t) + a_1 x^{(n-1)}(t) + ... + a_{n-1} x^{(1)}(t) + a_n x(t) = 0, \quad a_n \neq 0, \tag{1}$$

where

$$x^{(k)}(t) = \frac{dx^k(t)}{dt^k}$$

with initial conditions

$$x^{(i-1)}(0) = c_i, \qquad i = 1, 2, ..., n.$$

The characteristic equation for (1) is

$$s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n = 0. \tag{2}$$

We assume that the roots of (2) are simple and real: $s_j \neq s_i$ for $j \neq i$. The solution of (1) is:

$$x(t) = \sum_{k=1}^{n} A_k e^{s_k t}. \tag{3}$$

The coefficients A_k for $k = 1, 2, \ldots, ..., n$ in the explicit form are

$$A_k = \frac{a*}{\prod\limits_{v=1, v\neq k}^{n} (s_v - s_k)}, \tag{4}$$

where

$$a* = c_n - \sum_{v=1, v\neq k}^{n} s_v c_{n-1} + \sum_{v=1, v\neq k}^{n} s_v s_k c_{n-2} + ... + (-1)^{n-1} \prod_{v=1, v\neq k}^{n} s_v c_1.$$

Denote by τ the value of t for which $x(t)$ has a local extremum (if such extremum exists). Then the conditions for the extremum are:

$$x^{(1)}(\tau) = 0, \qquad x^{(2)}(\tau) \neq 0. \tag{5}$$

We consider τ and $x(\tau)$ as functions of $s_1, \ldots, s_n$ and look for solutions of the following problems:

First task. What are the conditions for $x[\tau(s_1, \ldots, s_n)]$ to have an extremum with respect to $(s_1, \ldots, s_n)$?

Second task. What are the conditions for $\tau(s_1, \ldots, s_n)$ to have an extremum with respect to $(s_1, \ldots, s_n)$?

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Third task is a conjunction of the first and second task. The equations

$$\frac{\partial x}{\partial s_i} = 0, \qquad i = 1, ..., n, \tag{6}$$

$$\frac{\partial \tau}{\partial s_i} = 0, \qquad i = 1, ..., n, \tag{7}$$

represent necessary conditions for existence of the local extremum of these functions.

3. Solution of the tasks

3.1. Task 1. In the paper [1] it is proved that the necessary condition for $x[\tau(s_1, \ldots, s_n)]$ to have an extremum with respect to $(s_1, \ldots, s_n)$ is

$$(-1)^n \tau^n \prod_{k=1}^{n} A_k = 0. \tag{8}$$

From this we conclude that either

$$\tau = 0 \tag{9}$$

which means that $c_2 = 0$ (10)
or $A_k = 0$, for some value of k, k from $[1, \ldots, n]$ (11)
which gives some relations between the roots and initial conditions.

3.2. Task 2. In the paper [1] it is proved that the necessary conditions for $\tau(s_1, \ldots, s_n)$ to have an extremum with respect to $(s_1, \ldots, s_n)$ are

$$(-1)^n \prod_{k=1}^{n} A_k \prod_{k=1}^{n} s_k \tau^{n-1} \left[\tau + \sum_{k=1}^{n} \frac{1}{s_k}\right] = 0. \tag{12}$$

From (12) results that:

$$\tau = 0 \tag{13}$$

(similarly as in task1), or

$$A_k = 0, \text{ for some value of } k, \; k \text{ from } [1, \ldots, n] \tag{14}$$

(similarly as in task 1), or

$$\tau = -\sum_{k=1}^{n} \frac{1}{s_k} = \frac{a_{n-1}}{a_n}. \tag{15}$$

In the paper [2] another necessary condition has been found, i.e.:

$$D_n(\tau) = \begin{vmatrix} c_1 & c_2 & c_3 & c_4 & \ldots & c_{n-1} & c_n \\ -\frac{a_{n-2}}{a_n} & \tau & -1 & 0 & \ldots & 0 & 0 \\ -\frac{a_{n-3}}{a_n} & 0 & \tau & -2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\frac{a_1}{a_n} & 0 & 0 & 0 & \ldots & \tau & 2-n \\ -\frac{1}{a_n} & 0 & 0 & 0 & \ldots & 0 & \tau \end{vmatrix} = 0. \tag{16}$$

Substituting $\tau = \frac{a_{n-1}}{a_n}$ into (16) gives another necessary condition

$$D_n = \begin{vmatrix} c_1 & c_2 & c_3 & c_4 & \ldots & c_{n-1} & c_n \\ a_{n-2} & -a_{n-1} & a_n & 0 & \ldots & 0 & 0 \\ a_{n-3} & 0 & -a_{n-1} & 2a_n & \ldots & 0 & 0 \\ a_{n-4} & 0 & 0 & -a_{n-1} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1 & 0 & 0 & 0 & \ldots & -a_{n-1} & (n-2)a_n \\ 1 & 0 & 0 & 0 & \ldots & 0 & -a_{n-1} \end{vmatrix} = 0. \tag{17}$$

In the special case of one multiple root of Eq. (2) the necessary conditions for the solution of the task 1 involving (8), (9), (10) are true and for the solution of the task 2 we have similarly that [5]:

$$\tau = \frac{-n}{s}, \tag{18}$$

$$x(\tau) = e^{-n} \left[A_n \left(\frac{a_{n-1}}{a_n}\right)^{n-1} + \ldots + A_2 \frac{a_{n-1}}{a_n} + A_1\right], \tag{19}$$

where

$$A_k = \sum_{i=0}^{k} \frac{x^{(k-i)}(0) \, (-1)^i \, s^i}{i! \, (k-i)!}, \qquad k = 1, 2, ..., n \tag{20}$$

and

$$\frac{d\tau}{ds}\Big|_{\tau=-\frac{n}{s}} = \sum_{i=0}^{n-1} \frac{x^{(i)}(0) \, (-1)^i \, s^{n-1-i}}{i!} \cdot \sum_{j=0}^{n-i-1} \frac{n^{j-1} \, (n-j-i)}{j!} = 0, \tag{21}$$

$$\frac{d^2\tau}{ds}\Big|_{\tau=-\frac{n}{s}} = \frac{A_{n-1} n^n \, (-1)^n \, s^{-1}}{\sum\limits_{j=1}^{n-1} A_j \, (-1)^j \cdot j \cdot n^{j-2} \cdot s^{n-j+1} \, (n-j+1))}. \tag{22}$$

3.3. Particular cases. Let us consider 3-rd order differential equation

$$x^{(3)}(t) + a_1 x^{(2)}(t) + a_2 x^{(1)}(t) + a_3 x(t) = 0 \tag{23}$$

with the initial conditions

$$\left.\begin{array}{l} x(0) = c_1 \\ x^{(1)}(0) = c_2 \\ x^{(2)}(0) = c_3 \end{array}\right\}. \tag{24}$$

Solution of Eq. (23) is

$$\begin{aligned} x(t) &= \frac{c_3 - c_2(s_2 + s_3) + c_1 s_2 s_3}{(s_1 - s_2)(s_1 - s_3)} e^{s_1 t} \\ &+ \frac{c_3 - c_2(s_3 + s_1) + c_1 s_3 s_1}{(s_2 - s_3)(s_2 - s_1)} e^{s_2 t} \\ &+ \frac{c_3 - c_2(s_1 + s_2) + c_1 s_1 s_2}{(s_3 - s_1)(s_3 - s_2)} e^{s_3 t}, \end{aligned} \tag{25}$$

where s_1, s_2, s_3 are the roots of the equation

$$s^3 + a_1 s^2 + a_2 s + a_3 = 0. \tag{26}$$

The derivative of $x(t)$ is equal

$$x^{(1)}(t) = \frac{s_1[c_3 - c_2(s_2 + s_3) + c_1 s_2 s_3]}{(s_1 - s_2)(s_1 - s_3)} e^{s_1 t} + \frac{s_2[c_3 - c_2(s_3 + s_1) + c_1 s_3 s_1]}{(s_2 - s_3)(s_2 - s_1)} e^{s_2 t} + \frac{s_3[c_3 - c_2(s_1 + s_2) + c_1 s_1 s_2]}{(s_3 - s_1)(s_3 - s_2)} e^{s_3 t}. \tag{27}$$

The necessary condition for the extremum $x(t)$ is

$$x^{(1)}(t)\,|_{t=\tau} = 0. \tag{28}$$

The necessary condition (15) takes the form

$$\tau_1 = \frac{a_2}{a_3} \tag{29}$$

and the second necessary condition (16) becomes

$$D_3(\tau) = \begin{vmatrix} c_1 & c_2 & c_3 \\ -\dfrac{a_1}{a_3} & \tau & -1 \\ -\dfrac{1}{a_3} & 0 & \tau \end{vmatrix} = 0 \tag{30}$$

or, in explicit form

$$a_3 c_1 \tau^2 + (c_3 + a_1 c_2)\tau + c_2 = 0. \tag{31}$$

Dividing Eq. (31) by $\tau - \dfrac{a_2}{a_3}$ gives

$$\tau a_3 c_1 + c_3 + a_1 c_2 + a_2 c_1 = 0 \tag{32}$$

and the rest of division, which must be equal to zero, as another necessary condition

$$a_2^2 c_1 + (c_3 + a_1 c_2) a_2 + a_3 c_2 = 0. \tag{33}$$

Elimination of c_3 from Eq. (32) using Eq. (33) gives finally

$$\tau_2 = \frac{c_2}{a_2 c_1}. \tag{34}$$

Elimination of c_2 gives

$$\tau_2 = -\frac{c_3 + c_1 a_2}{(a_3 + a_2 a_1) c_1} \tag{35}$$

and elimination of c_1 results in

$$\tau_2 = -\frac{c_2 a_2}{c_2 a_3 + a_1 a_2 c_2 + a_2 c_3}. \tag{36}$$

Equation (28) can be written in more convenient form:

$$[s_1(s_2-s_3)e^{s_1\tau} + s_2(s_3-s_1)e^{s_2\tau} + s_3(s_1-s_2)e^{s_3\tau}]\, c_3 + \left[s_1(s_3^2-s_2^2)e^{s_1\tau} + s_2(s_1^2-s_3^2)e^{s_2\tau} + s_3(s_2^2-s_1^2)e^{s_3\tau}\right] c_2 + s_1 s_2 s_3 \left[(s_2-s_3)e^{s_1\tau} + (s_3-s_1)e^{s_2\tau} + (s_1-s_2)e^{s_3\tau}\right] c_1 = 0. \tag{37}$$

Assuming c_1, c_2, c_3 as different from zero and using Eqs. (33), (37) we can calculate the unknowns $\dfrac{c_2}{c_1}$ and $\dfrac{c_3}{c_1}$.

Denote

$$B_3 = [s_1(s_2-s_3)e^{s_1\tau} + s_2(s_3-s_1)e^{s_2\tau} + s_3(s_1-s_2)e^{s_3\tau}],$$
$$B_2 = \left[s_1(s_3^2-s_2^2)e^{s_1\tau} + s_2(s_1^2-s_3^2)e^{s_2\tau} + s_3(s_2^2-s_1^2)e^{s_3\tau}\right], \tag{38}$$
$$B_1 = s_1 s_2 s_3 \left[(s_2-s_3)e^{s_1\tau} + (s_3-s_1)e^{s_2\tau} + (s_1-s_2)e^{s_3\tau}\right].$$

We obtain

$$\frac{c_3}{c_1} = \frac{B_1(a_1 a_2 + a_3) - B_2 a_2^2}{B_2 a_2 - B_3(a_1 a_2 + a_3)}, \tag{39}$$

$$\frac{c_2}{c_1} = \frac{-a_2 B_1 + a_2^2 B_3}{B_2 a_2 - B_3(a_1 a_2 + a_3)}, \tag{40}$$

where τ is determined by (29).

It is clear from Eq. (34) that $c_2 = 0$ gives $\tau_2 = 0$.

Similarly to this we see from Eq. (35) that $c_3 = 0$ gives $\tau_2 = -\frac{a_2}{a_3 + a_1 a_2} < 0$ because a_1, a_2, a_3 are positive.

Finally from Eq. (36) we obtain that $c_2 = 0$ gives $\tau_2 = 0$, and putting $c_3 = 0$ we obtain $\tau_2 = -\dfrac{a_2}{a_3 + a_1 a_2} < 0$.

We conclude that for $\tau_2 > 0$ all three initial conditions c_1, c_2, c_3 must be different from zero, $\dfrac{c_2}{c_1} > 0$ and $\dfrac{c_3}{c_1} < -a_2$.

If we require concrete value of τ_2 then $\dfrac{c_2}{c_1}$ is determined by (34) and (40) and $\dfrac{c_3}{c_1}$ is obtained from (39) by putting $\tau_2 = \dfrac{c_2}{c_1 a_2}$ into (38).

Remark 1. If the necessary conditions (29) or (33) are not fulfilled then it is possible to obtain $\tau_1^* > 0$ and $\tau_2^* > 0$, but these are not optimal times.

Remark 2. Putting $\left.\dfrac{dx}{dt}\right|_{\tau_p} = 0$ and $\left.\dfrac{d^2x}{dt^2}\right|_{\tau_p} = 0$ it is easy to obtain the condition for flexing point $c_1 c_3 - c_2^2 = 0$.

3.4. Numerical examples. Example 1. Let us consider the 3-rd order equation

$$\frac{d^3x(t)}{dt^3} + 6\frac{d^2x(t)}{dt^2} + 11\frac{dx(t)}{dt} + 6x(t) = 0. \tag{1e}$$

Here $a_1 = 6$, $a_2 = 11$, $a_3 = 6$ and $s_1 = -1$, $s_2 = -2$, $s_3 = -3$.

From the condition (29) we have

$$\tau = \frac{a_2}{a_3} = \frac{11}{6}. \tag{2e}$$

From the condition (33) we have

$$121c_1 + 72c_2 + 11c_3 = 0$$

$$\text{If} \quad c_1 \neq 0, \quad \text{then} \quad 72\frac{c_2}{c_1} + 11\frac{c_3}{c_1} = -121. \tag{3e}$$

From Eq. (37) we obtain the second condition

$$\left(\frac{c_3}{c_1} + 5\frac{c_2}{c_1} + 6\right) e^{2\tau} - 4\left(\frac{c_3}{c_1} + 4\frac{c_2}{c_1} + 3\right) e^{\tau} + 3\left(\frac{c_3}{c_1} + 3\frac{c_2}{c_1} + 2\right) = 0. \tag{4e}$$

For $\tau = \frac{11}{6}$ we find from (39) and (40) that $\frac{c_3}{c_1} = 8.829826$, $\frac{c_2}{c_1} = -3.0295567$ and $\tau_2 = \frac{c_2}{c_1 a_2} < 0$.

Example 2. We assume $c_2 = 0$, then $\tau_1 = 0$. The condition (2e) is not fulfilled.

From (3e) we obtain

$$\frac{c_3}{c_1} = -11. \tag{5e}$$

From (4e) we have

$$5e^{2\tau_2} - 32e^{\tau_2} + 27 = 0 \tag{6e}$$

what yields $\tau_2 = 1.686399$, which is not extremal time.

Example 3. We assume that $c_2 = 0$, and that the condition (2e) holds, i.e. $\tau = \frac{11}{6}$, but the condition (33) $\frac{c_3}{c_1} = -11$ is not fulfilled because from (4e) we obtain

$$\frac{3\frac{c_3}{c_1} + 6}{\frac{c_3}{c_1} + 6} = e^{\frac{11}{6}} \tag{7e}$$

and $\frac{c_3}{c_1} = -9.687 \neq -11$.

Hence $\tau = \frac{11}{6}$ is not extremal, because the necessary condition $\frac{c_3}{c_1} = -11$ is not fulfilled.

Example 4. We assume $c_1 = 0$, $c_2 = 1$, then from (33) we have $c_3 = -\frac{72}{11}c_2$. We have expected $\tau_1 = \frac{11}{6}$, but the necessary condition (28) is not fulfilled because:

$$\frac{dx}{d\tau} = 0 \quad \text{gives} \quad \tau_1 = 0.264, \quad \tau_2 = 1.665.$$

4. Task 3

It is very difficult problem to determine the values of roots $s_1, s_2, ..., s_n$, which fulfill the necessary condition $\tau = \frac{a_{n-1}}{a_n}$ and $D_n = 0$.

The solution of algebraic equation of degree higher than n=4 is possible only using some additional assumption [3]. For that reason we use the properties of symmetrical algebraic equations. From the theoretical point of view such equations can be solved up to 9-th degree, which is satisfactory for practical applications.

We illustrate the method by example of 3-rd order equation.

$$\frac{d^3x(t)}{dt^3} + a_1\frac{d^2x(t)}{dt^2} + a_2\frac{dx(t)}{dt} + a_3x(t) = 0 \tag{41}$$

with the initial conditions: $x(0) = c_1$, $x^{(1)}(0) = c_2$, $x^{(2)}(0) = c_3$.

The characteristic Eq. (41) is:

$$s^3 + a_1s^2 + a_2s + a_3 = 0. \tag{42}$$

We assume that the roots of (42) $s_1 \neq s_2 \neq s_3$ are real and negative.

We want to have simple analytic formulae for s_1, s_2, s_3 by using some symmetrization of Eq. (42).

First step.
We put $s = \sqrt[3]{a_3} \cdot z$, $a_3 > 0$ (43)
Then we obtain the equation:

$$z^3 + \frac{a_1}{\sqrt[3]{a_3}}z^2 + \frac{a_2}{\sqrt[3]{a_3^2}}z + 1 = 0. \tag{44}$$

Second step.
We denote

$$b_1 = \frac{a_1}{\sqrt[3]{a_3}}, \qquad b_2 = \frac{a_2}{\sqrt[3]{a_3^2}} \tag{45}$$

and assume that

$$b_1 = b_2 \quad \text{or} \quad a_1 = \frac{a_2}{\sqrt[3]{a_3}}. \tag{46}$$

Then the Eq. (44) takes a form

$$z^3 + b_1z^2 + b_1z + 1 = 0 \tag{47}$$

which is symmetric.

We observe that the extremal time for Eq. (41) is

$$\tau_1 = \frac{a_2}{a_3} \tag{48}$$

with the necessary condition

$$D_3 = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & -a_2 & a_3 \\ 1 & 0 & -a_2 \end{vmatrix} = 0, \tag{49}$$

so

$$D_3 = a_2^2c_1 + (a_1a_2 + a_3)c_2 + a_2c_3 = 0. \tag{50}$$

After symmetrization we have for Eq. (47) that

$$\tau_1 = b_1. \tag{51}$$

Now the condition (50) has the form

$$D_3 = \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & -b_1 & 1 \\ 1 & 0 & -b_1 \end{vmatrix} = b_1^2(c_1 + c_2) + b_1c_3 + c_2 = 0. \tag{52}$$

Using (51) we have

$$\tau_1^2(c_1 + c_2) + \tau_1c_3 + c_2 = 0. \tag{53}$$

In the well-known way [3] we calculate the roots of Eq. (47)

$$\begin{aligned} z_1 &= -1, \\ z_2 &= \frac{1}{2}\left[1 - \tau_1 + \sqrt{(1 - \tau_1)^2 - 4}\right], \\ z_3 &= \frac{1}{2}\left[1 - \tau_1 - \sqrt{(1 - \tau_1)^2 - 4}\right]. \end{aligned} \tag{54}$$

The necessary condition for extremum of $z(\tau)$ is:

$$\begin{aligned} \left.\frac{dz}{dt}\right|_{\tau} &= z_1\left[c_3 - c_2\left(z_2 + z_3\right) + c_1z_2z_3\right]\left(z_2 - z_3\right)e^{z_1\tau_1} \\ &+ z_2\left[c_3 - c_2\left(z_3 + z_1\right) + c_1z_3z_1\right]\left(z_3 - z_1\right)e^{z_2\tau_1} \\ &+ z_3\left[c_3 - c_2\left(z_1 + z_2\right) + c_1z_1z_2\right]\left(z_1 - z_2\right)e^{z_3\tau_1} = 0. \end{aligned} \tag{55}$$

After substitution of (54) into Eq. (55) we obtain two linear Eqs. (53) and (55) for calculation of $\frac{c_2}{c_1}$ and $\frac{c_3}{c_1}$, $c_1 \neq 0$, which are valid for desired τ.

In Table 1 calculated values of $\frac{c_2}{c_1}$, $\frac{c_3}{c_1}$ and extremal value $\frac{x_e}{c_1}$ for desired τ are given.

Table 1
Calculated values of $\frac{c_2}{c_1}$, $\frac{c_3}{c_1}$ and $\frac{x_e}{c_1}$ for desired τ

τ	$\frac{c_2}{c_1}$	$\frac{c_3}{c_1}$	$\frac{x_e}{c_1}$
4	−1.859175078	3.901494083	−0.02011293436
5	−2.426487984	7.617737519	−0.01001620229
6	−3.022879434	12.64108984	−0.004615215005
7	−3.602067672	18.72905480	−0.00207307644
8	−4.157408762	25.77894619	−0.000912363558
9	−4.692639896	33.75516350	−0.000393838628
10	−5.213693094	42.65830025	−0.000167021302

In Fig. 1 dependency of τ on $\frac{c_2}{c_1}$ and $\frac{c_3}{c_1}$ is illustrated and in Fig. 2 dependency of the extremal value $\frac{x_e}{c_1}$ on τ.

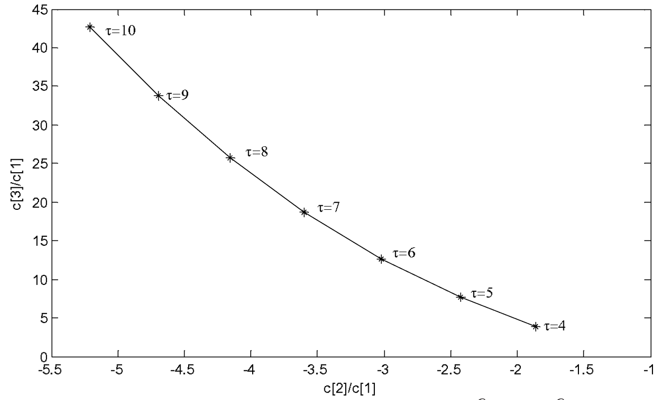

Fig. 1. Extremal timeτas a function of $\frac{c_2}{c_1}$ and $\frac{c_3}{c_1}$

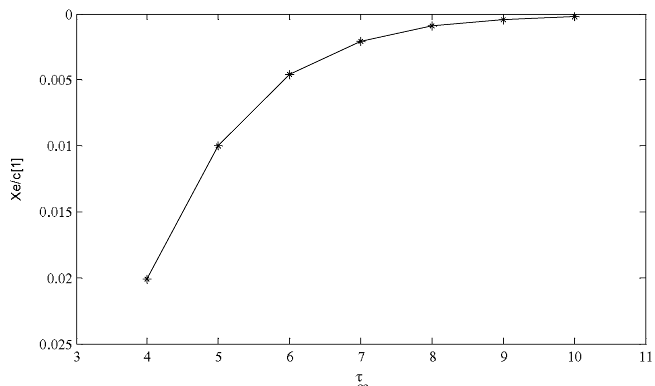

Fig. 2. Extremal value of $\frac{x_e}{c_1}$ as a function of τ

After returning to the variable s we have:

$$a_1 = b_1 \sqrt[3]{a_3},$$

$$a_2 = \frac{b_1}{\sqrt[3]{a_3}}, \tag{56}$$

$$\tau_1 = \frac{b_1}{\sqrt[3]{a_3^2}} = \frac{\tau_{1z}}{\sqrt[3]{a_3^2}},$$

where $\tau_{1z} = \frac{a_2}{a_3}$.

5. Conclusions

Problem 1. In this problem no parametric optimization is possible.

Problem 2. The application of the symmetrical equations enables the determination of the roots of the characteristic equations. These roots fulfill the necessary and satisfactory conditions of the existence of the dynamic error $x(\tau)$ and the extremal value of τ which are determined in the article. This enables the parametric optimization of the system.

Problem 3. The analytic formulae for the roots s_i , the time τ and extremal value of $x(\tau)$ which are obtained in this article can be used to design the control system with the prescribed value of τ and $x(\tau)$ as a function of the initial conditions c_i. The known method of the zeros and poles location [4] can be applied in the final step of the design. These results will be used to develop the method for the more general systems [7, 8].

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