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# PROBLEMS WITH MODELING OF FRACTIONAL ELECTRICAL CIRCUITS CONTAINING SUPERCAPACITORS

For proper operation, diagnostics or control, it is required to know the parameters of the supercapacitor replacement model (relationship between current and voltage at the terminals). The paper describe the bahavior of the eletrical circuit (RC) containing the supercapacitor were used the fractional derivatives of Caputo definiton and Conformable Fractional Derivative definition. Verification of the correctness of the suggested electrical circuit models was carried out a series of measurements of the system response to the given control signal. The measurement data were compared by fractionalorder derivatives: classical case, Caputo definition and CFD definition. Conducting a series of experiments with charging a supercapacitor in an RC circuit, constant control voltage from 2 V to 5 V with an exchanged external resistor, it was shown that none of the three mathematical models reflects the real behavior of the supercapacitor. It has been shown that the behavior of supercapacitor requires the use of different mathematical than fractional derivatives.

KEYWORDS: fractional order system, the electrical circuits containing supercapacitors, Caputo definition, Conformable Fractional Derivative definition.

## 1. INTRODUCTION

Supercapacitors are devices with a huge electrical capacity exceeding 1000 Farads. Supercapacitors have advantages in applications where a large amount of power is needed for a relatively short time for example in: PDA's, GPS, portable media players, hand-held devices [3, 29] and photovoltaic systems, supercapacitors can stabilize the power supply [28]. The most important applications of supercapacitors are found in transport in the KERS system, the process of recuperative braking – to receive the storage energy made when braking, which significantly increases the energy efficiency of the vehicle and reduces air pollution [17].

Currently, to simplify the structure of the replacement model of supercapacitor, while providing a very good fit characteristics of the measured and calculated for the calculations used fractional calculus. The Riemann-Liouville definition is a classic form of the fractional derivative [15, 16]. It is based on the Cauche mul-

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tiple integration rule. Another definition of an fractional derivatives is Caputo definition. It was introduced by Michele Caputo in an article from 1960 [4,5]. A further simplification of the fractional derivative is prepared by Khalil, R., Al Horani, M. Yousef. A. and Sababheh, M. in 2014 [1, 13] is Conformable Fractional Derivative definition (CFD). The autors was used the Caputo and CFD definitions in the previous papers to describe the operation of the electrical circuit containing the supercapacitor [19-25].

The paper organized as follows. The Caputo and CFD definitions were presented in section 2. Next section 3 described a model of an RC fractional electrical circuit. General description of the realize research and analysis numerical of the three mathematical models are considered in section 4. Concluding remarks are given in section 5.

## 2. DEFINITIONS FRACTIONAL DERIVATIVES

**Definition 1**. The function defined by [11]:

$$ {}^{Cap}D_t^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(\tau)}{(t-\tau)^{a+1-n}} d\tau, \tag{1} $$

is called the Caputo fractional derivative, where $n-1<\alpha<n$, $n=1,2,\dots$, $\Gamma(z)$ is the Euler gamma function and $u^{(n)}(t)=\frac{d^n u(t)}{dt^n}$.

**Definition 2.** If $n<\alpha\le n+1$, $n=0,1,2,\dots,$ then the conformable fractional derivative (CFD) of *n*- differentiable at *t* function *u* (where t > 0) is defined as [13]:

$$ {}^{CFD}D_t^{\alpha} u(t) = \lim_{\varepsilon\to 0} \frac{u^{(n)}\left(t+\varepsilon t^{n+1-\alpha}\right)-u^{(n)}(t)}{\varepsilon}. \tag{2} $$

Using the definition (5) we get a simple rule:

$$ {}^{CFD}D_t^{\alpha} u(t) = t^{n+1-\alpha} u^{(n+1)}(t), \tag{3} $$

where $u$ is $n+1$ differentiable function for $t>0$.

## 3. RC – FRACTIONAL ELECTRICAL CIRCUIT

In this paper we will consider the fractional electrical circuit shown in Figure 1 with given external resistor $R$, supercapacitor $C$ with series, internal resistance $R_C$ and source voltage $e$. Denote by $i$ the mesh current.

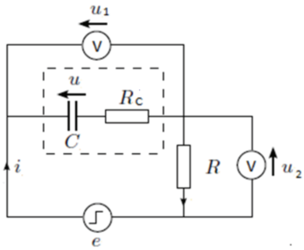


Fig. 1. *R, C, e* type electrical circuit (Source: own)

The current $i(t)$ in the fractional capacitor is related with its voltage $u(t)$ by formula [11]

$$i(t) = C_\alpha D_t^\alpha u(t) \text{ for } 0 < \alpha < 1, \tag{4}$$

where $C_\alpha$ is the pseudo-capacitance in units of $\mathrm{F/s}^{1-\alpha}$ of the fractional capacitor. $D_t^\alpha$ can be a differential operator of an incomplete order according to both definitions Caputo and Conformable Fractional Derivative definition.
When $\alpha = 1$ we have

$$i(t) = C\frac{du(t)}{dt}, \tag{5}$$

where $C$ is the capacitance of the capacitor expressed in F (farad).
Using the equation (4) and Kirchhoff's laws we may describe the transient states in the electrical circuit by the fractional differential equation

$$D_t^\alpha u(t) = \lambda u(t) - \lambda e(t), \quad 0 < \alpha < 1, \tag{6a}$$

where $e(t)$ is the input function and

$$\lambda = -\frac{1}{(R + R_C)C_\alpha}, \tag{6b}$$

is the real, negative constant. Starting point is $u(0) = u_0$.

Input control voltage is expressed by step function of magnitude $U$

$$e(t) = \begin{cases} 0 & \text{for} \quad t < 0, \\ U & \text{for} \quad t \geq 0, \end{cases} \tag{7}$$

Equation (6a) reduces to

$$D_t^\alpha u(t) = \lambda\left[u(t) - U\right], \quad 0 < \alpha < 1. \tag{8}$$

Since both the Caputo derivative and the CFD derivative from the constant function are equal to zero, we can make substitution in the formula (8) $u(t) = \tilde{u}(t) + U$ . Then we get the equation (7) without control

$$D_t^{\alpha} u(t) = \lambda\left[u(t) - U\right], \quad 0 < \alpha < 1, \tag{9}$$

with the initial condition $\tilde{u}_0 = u_0 - U$.

**Solution 1** In the case of Caputo definition, we get a solution [11] equations (9)

$$\tilde{u}_{Cap}(t) = E_{\alpha}\left(\lambda t^{\alpha}\right)\tilde{u}_0, \quad 0 < \alpha < 1, \tag{10}$$

where

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{11}$$

is the one parameter Mittag-Leffler function.

**Solution 2** For CFD definition, we get a solution [13] equation (9)

$$\tilde{u}_{CFD}(t) = e^{\lambda \frac{t^{\alpha}}{\alpha}} \tilde{u}_0, \quad 0 < \alpha < 1. \tag{12}$$

Returning to the original coordinate in equations (10) and (12), assuming zero initial condition ( discharged supercapacitor), and having (6b), we have:

$$u_{Cap}(t) = U\left[1 - E_{\alpha}\left(-\frac{t^{\alpha}}{(R + R_C)C_{\alpha}}\right)\right], \quad 0 < \alpha < 1, \tag{13}$$

$$u_{CFD}(t) = U\left[1 - e^{-\frac{t^{\alpha}}{\alpha(R + R_C)C_{\alpha}}}\right], \quad 0 < \alpha < 1. \tag{14}$$

**Solution 3** Substituting $\alpha = 1$ into (13) or (14), we obtain classical case

$$u_{Class}(t) = U\left[1 - e^{-\frac{t}{(R + R_C)C}}\right], \tag{15}$$

where $C$ is capacitance of supercapacitor.

## 4. NUMERICAL ANALYSIS

The measurements were carried out each time with a discharged supercapacitor. The exchanged external resistors were resistances $R \in \{10.0\,\Omega, 20.7\,\Omega, 51.7\,\Omega, 98.7\,\Omega\}$. The parameter of supercapacitor produced by Panasonic EECS0HD334H of nominal capacity 0.33 F and measured internal resistance $R_C = 28.26\,\Omega$. The magnitudes of step, input voltage are $U \in \{2\,V,\ 3\,V,\ 4\,V,\ 5\,V\}$.The measurements rated were taken with a sampling time of 50 ms. The parameter values $C$, $C_{\alpha}$ and $\alpha$, obtained as a result of the optimization procedure, using the Least Squares Fitting method, formulas (13)-

(15), experimental data. Parameter obtained from fitting method for a determined value of the control voltage, a selected external resistor and a fixed supercapacitor model are presented in the table 1.

Conducting a series of experiments with charging a supercapacitor in an RC circuit, constant control voltage from 2 V to 5 V with an exchanged external resistor, it was shown that none of the three mathematical models reflects the actual behavior of the supercapacitor. This is demonstrated by the workability of the C, $C_\alpha$, $\alpha$ parameters values, the given in table 1.

Table 1. Parameter obtained from fitting method for a determined value of control voltage and a selected external resistor with a selected fractional derivative definitione.

| *U* [V] | *R* [Ω] | Classical case | | CFD | | Caputo | |
|---|---|---|---|---|---|---|---|
| | | $\alpha$ | *C* [F] | α | $C_\alpha$[F/s1-α] | $\alpha$ | $C_\alpha$[F/s1-α] |
| 2 | 10.0 | 1 | 0.103 | 0.355 | 0.1000 | 0.595 | 0.0363 |
| | 20.7 | 1 | 0.121 | 0.372 | 0.0885 | 0.600 | 0.0363 |
| | 51.7 | 1 | 0.177 | 0.438 | 0.0765 | 0.638 | 0.0435 |
| | 98.7 | 1 | 0.215 | 0.545 | 0.0756 | 0.753 | 0.0689 |
| 3 | 10.0 | 1 | 0.111 | 0.358 | 0.1026 | 0.608 | 0.0392 |
| | 20.7 | 1 | 0.128 | 0.378 | 0.0900 | 0.615 | 0.0389 |
| | 51.7 | 1 | 0.182 | 0.443 | 0.0776 | 0.653 | 0.0462 |
| | 98.7 | 1 | 0.222 | 0.543 | 0.0766 | 0.752 | 0.0703 |
| 4 | 10.0 | 1 | 0.118 | 0.366 | 0.1044 | 0.624 | 0.0426 |
| | 20.7 | 1 | 0.136 | 0.387 | 0.0920 | 0.630 | 0.0424 |
| | 51.7 | 1 | 0.185 | 0.451 | 0.0785 | 0.675 | 0.0503 |
| | 98.7 | 1 | 0.228 | 0.541 | 0.0775 | 0.756 | 0.0730 |
| 5 | 10.0 | 1 | 0.124 | 0.371 | 0.1057 | 0.635 | 0.0450 |
| | 20.7 | 1 | 0.141 | 0.390 | 0.0930 | 0.640 | 0.0445 |
| | 51.7 | 1 | 0.187 | 0.446 | 0.0785 | 0.670 | 0.0497 |
| | 98.7 | 1 | 0.232 | 0.538 | 0.0780 | 0.760 | 0.0750 |

In sixteen measurement series, a discrepancy of the derivative values was observed - from 0.595 to 0.760 for the Caputo definition and 0.355-0.545 for the CFD definition. Obtained α and $C_\alpha$ depend on what voltages the system was supplied with and what was the value of external resistivity.

In figure 2 presents the measurements voltage across the supercapacitor and the best fit curves based on classical case solution (15). In figure 3 and figure 4 CFD solution (14) and in figure 5 and figure 6 Caputo solution (13) for resistances $R_1 = 10.0\Omega$, $R_2 = 20.7\Omega$, $R_3 = 51.7\Omega$, $R_4 = 98.7\Omega$.

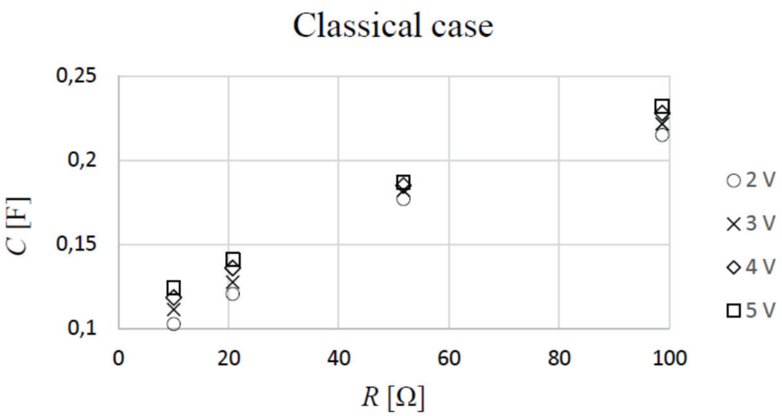


Fig. 2. Capacities obtained in classic case definition for the supercapacitor 0.33 F

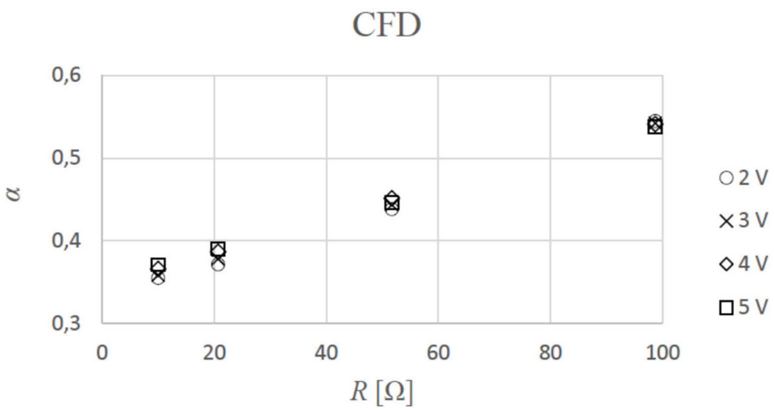


Fig. 3. Order of the derivative obtained for CFD definition for the supercapacitor 0.33 F

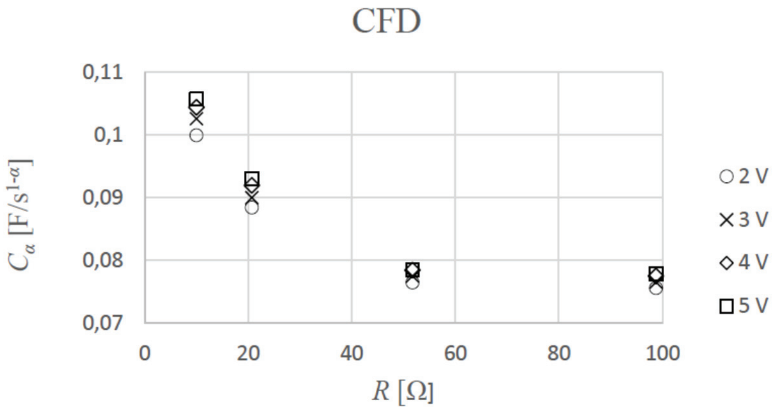


Fig. 4. Pseudo capacity obtained for the CFD definition for the supercapacitor 0.33 F

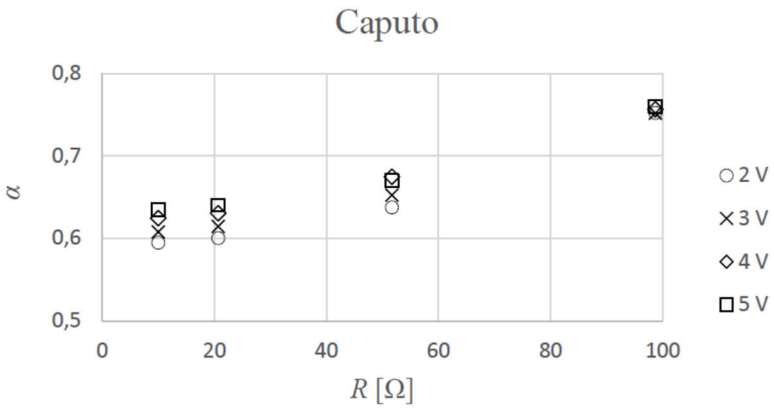


Fig. 5. Order of the derivative obtained for the Caputo definition for the supercapacitor 0.33 F

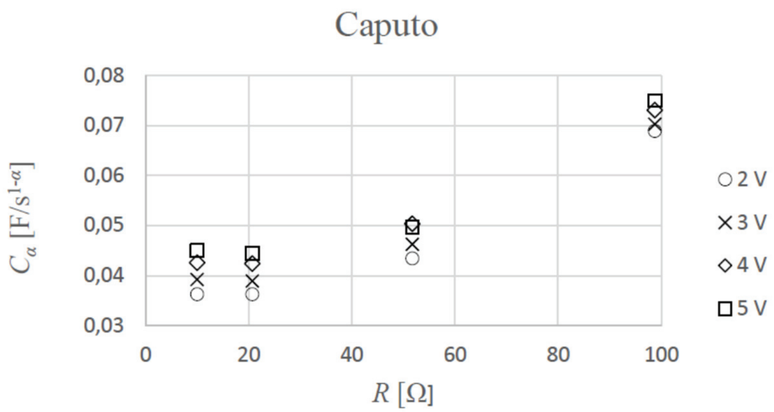


Fig. 6. Pseudo capacity obtained for the Caputo definition for the supercapacitor 0.33 F

The diagrams presents the charging of the supercapacitor in the RC circuit with constant control voltage from 2 V to 5 V at the replaced external resistor, obtaining values of α and Cα depending on the resistances 10 Ω, 21 Ω, 52 Ω, 99 Ω. As a result, the alpha derivative order (0.6 to 0.76) is not convergent regardless of the calculation method. The pseudo capacity values fluctuated significantly from 0.036 to 0.076. This means that using constant control voltage and resistance parameter of supercapacitor model is incorrect.

Another method was also used to determine the parameters describing the supercapacitor. It consists in the application of the optimization procedure, for measurement data at a fixed control voltage, four measurement series were performed with subsequent external resistances removed. Results are presented in the table 2.

Table 2. The parameters obtained from fitting method for a fixed control voltage and four measurement series.

| | Classical case | | CFD | | Caputo | |
|---|---|---|---|---|---|---|
| e [V] | $\alpha$ | C | $\alpha$ | $C_\alpha$ | $\alpha$ | $C_\alpha$ |
| 2 | 1 | 0.1655 | 0.5249 | 0.08206 | 0.6602 | 0.04563 |
| 3 | 1 | 0.1722 | 0.5260 | 0.08398 | 0.6673 | 0.04789 |
| 4 | 1 | 0.1779 | 0.5301 | 0.08593 | 0.6811 | 0.05130 |
| 5 | 1 | 0.1815 | 0.5314 | 0.08703 | 0.6854 | 0.05279 |

In the second method optimization procedure based on data from four measurement series, describe to different external resistors, the differences of approximated parameters α, $C_\alpha$ and C are much smaller than in Table 1.

## 5. CONCLUSION

The study compares three mathematical models of supercapacitors. The first of them assumed that the supercapacitor consists of series connected internal resistances $R_C$ and a classic capacitor with capacity $C$. The next two supercapacitor models assume the existence of a series of internal resistances $R_C$ and a pseudo-capacitive $C$ element, described α by one of the fractional definition. None of these three models reflects the actual behavior of the supercapacitor. This is evidenced by the derivative values of $C$, $C_\alpha$, $\alpha$. In sixteen measurement series, the derivative order has a large variance. Also, the pseudo capacity values are not valid. Such results indicate that the supercapacitor model assumes invariance of parameters $\alpha, C_\alpha$ is incompatible with the results of the experiments. This effect may be caused by the fact that the supercapacitor is not a linear element.

In many papers, supercapacitors are part of the incomplete order used to build electrical circuits. Research included in this work negatively verified the assumption that supercapacitors are elements of the incomplete order, at the same time pointing to the occurrence of non-linear phenomena during charging of the supercapacitor. Many authors in their papers assume the correctness of the supercapacitor model based on the Caputo definition, not knowing about the problems with describe the behavior of the supercapacitor.

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