DE GRUYTER
OPEN

JAISCR, 2016, Vol. 6, No. 1, pp. 13– 21
10.1515/jaiscr-2016-0002

THE BIPOLAR CHOQUET INTEGRALS BASED ON TERNARY-ELEMENT SETS

Jabbar Abbas

Department of Applied Sciences, University of Technology
Baghdad, Iraq
E-mail: jabbara1969@gmail.com

Abstract

[1]This paper first introduces a new approach for studying bi-capacities and the bipolar Choquet integrals based on ternary-element sets. In the second half of the paper, we extend our approach to bi-capacities on fuzzy sets. Then, we propose a model of bipolar Choquet integral with respect to bi-capacities on fuzzy sets, and we give some basic properties of this model.

Keywords: Capacities, Bi-capacities, Choquet integrals, Bipolar Choquet integrals, Fuzzy events

1 Introduction

The Choquet integral with respect to a capacity [4] (fuzzy measure [13, 2, 14], non-additive measure [6], or generalized measure [15]) was proposed by Murofushi and Sugeno [12]. It was introduced by Choquet [4] in potential theory with the concept of capacity. Wu and Huang [16] generalized Choquet integral on fuzzy sets.

The area of applications of capacity and Choquet integral, has been greatly expanded in the different fields, especially in the field of decision theory. However, classical capacity and Choquet integral are not suitable in some situations such as decision behaviors, in particular when the underlying scales are bipolar (see, [11]). Recently, Grabisch and Labreuche [7] proposed the concept of bi-capacity as a generalization of capacities, who consider the case where scores are expressed on a bipolar scale.

The bipolar Choquet integral with respect to bi-capacities introduced in [8] as a generalization of the Choquet integral. An exhaustive survey of bipolarity and its possible applications has been published [9]. A further generalization is that of level-dependent bipolar Choquet integral [10].

The aim of the paper is to presents a bi-capacities and its integrals on fuzzy sets. First, we present a new approach for studying bi-capacities and the bipolar Choquet integrals through introducing a notion of ternary-element sets, which is alternative approach from defined by Grabisch and Labreuche [7], [8]. It is important approach because attaching a polarity to each element is an easier than attaching a polarity to sets, and allows a simple way to prove new results on bi-capacities and its integrals as it was done for capacities. Consequently, according to this approach, we introduce bi-capacities on fuzzy sets. Then, we propose a model of bipolar Choquet integral with respect to bi-capacities on fuzzy sets, and we give some basic properties of this model.

The structure of the paper is as follows. In the next section we introduce a new approach of ca-

[1]This paper is an extended and revised version of a paper presented in the IEEE SSCI 2014 conference [1]

pacities and its integrals based on binary-element sets. Section 3 presents bi-capacities and its integrals based on ternary-element sets. In Section 4, we introduce the Choquet integral on fuzzy sets. In Section 5, we propose the bipolar Choquet integral with respect to bi-capacity on fuzzy sets. The paper finishes with some conclusions. Throughout the paper, we will consider $\mathcal{R}^+$ or $[0,1]$ to be prototypical unipolar scales, while $\mathcal{R}$ and $[-1,1]$ with 0 as neutral level will be considered as prototypical bipolar scales.

2 Capacities and its integrals based on bi-element sets

2.1 Capacities based on bi-element sets

In the context of multi-criteria decision making problem, we shall consider each single criterion has either positive effect (i.e., positively important criterion of weighted evaluation not only alone but also is interactive with other) or has negative effect (i.e., negatively important criterion). Therefore, it is necessary that we deal with the concept of the "negatively important criterion", that is, the existence of some criteria harms the evaluation of others. Thus, we can represent the criterion i as i^+ whenever i is positively important, and as i^- whenever i is negatively important, and we call this element a *binary-element* (or simply *bi-element*).

The *binary-element set* (or simply *bi-element set*) is the set which contains either i^+ or i^- for all i, $i = 1,\ldots,n$. That is, a bi-element set A is the set of the form $A := \{\tau_1,\ldots,\tau_n\}$ where $\tau_i = i^+$ or i^-, $\forall i = 1,\ldots,n$. Accodingly, we can define the characteristic function of an element i for the desired bi-element set A as follows:

$$\chi_A(i) = \begin{cases} 1 & \text{iff} \quad i^+ \in A, \\ 0 & \text{iff} \quad i^- \notin A. \end{cases}$$

For this situation, we can describe the set of all possible combinations of binary elements of n criteria (the set of all bi-element sets) given by

$$\mathcal{B} := \{\{\tau_1,\ldots,\tau_n\} \mid \forall\, \tau_i \in \{i^+, i^-\},\ i = 1,\ldots,n\},$$

which is corresponds to power set in the notation of classical set theory. The set of all bi-element sets $\mathcal{B}$ can be identified with $\{0,1\}^n$, hence $|\mathcal{B}| = 2^n$. Also, any bi-element set $A \in \mathcal{B}$ can be written as a binary alternative denoted by $(\tau_1,\ldots,\tau_n)$ with $\tau_i = 1$ if $i^+ \in A$ and $\tau_i = 0$ if $i^- \in A$, $i = 1,\ldots,n$.

We introduce the inclusion relation $\subseteq$, cardinality, complement, difference, union, and intersection of bi-element sets of $\mathcal{B}$ as follows.

- Let $\mathcal{B}$ be the set of all bi-element sets and $A,\ B \in \mathcal{B}$. Then, $A \subseteq B$ iff $\forall\, i \in X := \{1,\ldots,n\}$,
$$\text{if } i^+ \in A \text{ implies } i^+ \in B. \tag{1}$$
Note that, $X^- := \{1^-,\ldots,n^-\} \subseteq A$ and $X^+ := \{1^+,\ldots,n^+\} \supseteq A$, $\forall\, A \in \mathcal{B}$.
- The cardinality of the bi-element set $A \in \mathcal{B}$ is number of important elements i^+ in A, denoted by a^+. That is,
$$a^+ = |A| = \sum_{i=1}^{n} \chi_A(i^+), \tag{2}$$
where $\chi_A(i^+) = \begin{cases} 1 & \text{if} \quad i^+ \in A, \\ 0 & \text{if} \quad i^+ \notin A. \end{cases}$
- For any bi-element set $A \in \mathcal{B}$, we define the complement of A by $A^c = \{\ \{\tau_1{}^c,\ldots,\tau_i{}^c,\ldots,\tau_n{}^c\} \mid \forall\ \tau_i \in A\ \mathit{with}\ (i^+)^c = i^-,\ (i^-)^c = i^+\ \}$.
- Let $\mathcal{B}$ be the set of all bi-element sets and $A,\ B \in \mathcal{B}$. Then, the set difference of A whenever $A \subseteq B$ is defined by
$B \backslash A = \{\tau_k \backslash \tau_j :\ \tau_j \in A,\ \tau_k \in B,\ j = 1,\ldots,n,\ k = 1,\ldots,n\}$, with
$i^+ \backslash i^- = i^+,\ i^+ \backslash i^+ = i^-,\ i^- \backslash i^- = i^-,\ i = 1,\ldots,n.$
- Let $\mathcal{B}$ be the set of all bi-element sets and $A,\ B \in \mathcal{B}$. The union, $A \cup B$ of A and B is defined by,
$A \cup B = \{\tau_j \vee \tau_k :\ \ \tau_j \in A,\ \ \tau_k \in B,\ \ j = 1,\ldots,n,\ k = 1,\ldots,n\}$, with
$i^+ \vee i^- = i^+,\ i^+ \vee i^+ = i^+,\ i^- \vee i^- = i^-,\ i = 1,\ldots,n.$
- Let $\mathcal{B}$ be the set of all bi-element sets and $A, B \in \mathcal{B}$. The intersection, $A \cap B$ of A and B is defined by,
$A \cap B = \{\tau_j \wedge \tau_k :\ \ \tau_j \in A,\ \ \tau_k \in B,\ \ j = 1,\ldots,n,\ k = 1,\ldots,n\}$, with
$i^+ \wedge i^- = i^-,\ i^+ \wedge i^+ = i^+,\ i^- \wedge i^- = i^-,\ i = 1,\ldots,n.$

The following definition is equivalent definition of capacities based on notion of bi-element sets.
Definition 1 *Let $\mathcal{B}$ be the set of all bi-element sets.*

A set function (i) $\mu(X^-) = \mu(\{1^-, \ldots, n^-\}) = 0$ *and* $\mu(X^+) = \mu(\{1^+, \ldots, n^+\}) = 1$.
(ii) $\forall A,\ B \in \mathcal{B},\ A \subseteq B$ *implies* $\mu(A) \leq \mu(B)$.

We interpret a value $\mu(A)$ of a capacity, as the "degrees of importance" of bi-element set $A \in \mathcal{B}$. A capacity $\mu(A)$ in the context of multicriteria decision aid, when alternatives are evaluated on the [0, 1] unipolar scale, assigns to every $A \in \mathcal{B}$ a number in [0, 1], which is the overall score of the binary alternative $(\tau_1, \ldots, \tau_n)$ with $\tau_i = 1$ (full satisfaction) if $i^+ \in A$, i.e. on positively important elements in A, and $\tau_i = 0$ (full non satisfaction) if $i^- \in A$, i.e. according to the negatively important elements in A. For example, in evaluating students in a high school with respect to three subjects: Mathematics (M), Physics (P) and Literature (L). The alternative consists of observing the subjects where a student has qualified. The possible outcomes of the alternative are the set of all possible combinations of binary elements of criteria M, P, and L. That is, $\mathcal{B} = \{\{M^+,P^-,L^-\},\{M^-,P^+,L^-\},\{M^-,P^-,L^+\},$ $\{M^+,P^+,L^-\},\{M^+,P^-,L^+\},\{M^-,P^+,L^+\},$ $\{M^+,P^+,L^+\},\{M^-,P^-,L^-\}\}$. For instance, the meaning of the bi-element set $\{M^+,P^+,L^-\}$ is as follows: the alternative on the set of subjects M, P, and L in which Mathematics and Physics are positively important while Literature is negatively important. Thus, the capacity $\mu(\{M^+,P^+,L^-\})$ is interpreted as the overall assessment of the bi-element set $\{M^+,P^+,L^-\}$.

2.2 The Choquet integral based on the bi-element sets

In this subsection, we introduce an equivalent model of Choquet integral with respect to capacities based on the bi-element sets. We assume that to each alternative in multi-criteria decision making problem is described by a vector $\mathbf{x} = (x_{1^+}, \ldots, x_{i^+}, \ldots, x_{n^+})$, $x_{i^+} \in \mathcal{R}$ with $i \in \{1, \ldots, n\}$, and we consider a bi-element set $X^+ = \{i^+ |\ x_{i^+} \in \mathcal{R}$ for each $i \in \{1, \ldots, n\}\}$. Thus, we define the Choquet integral with respect to capacity of real input $\mathbf{x}$ as follows.

Definition 2 *Let* $\mathcal{B}(X)$ *be the set of all bi-element sets and* $\mu : \mathcal{B}(X) \to [0,\ 1]$, *be a capacity based on bi-element set. Then Choquet integral of* $\mathbf{x}$ *with respect to* μ *is given by*

$$Ch_\mu(\mathbf{x}) = \sum_{i=1}^{n} [x_{\sigma(i^+)} - x_{\sigma((i+1)^+)}]\ \mu(\{A_{\sigma(i^+)}\}), \quad (3)$$

where $A_{\sigma(i^+)} = \{\sigma(1^+), \cdots, \sigma(i^+), \sigma((i+1)^-), \sigma((i+2)^-), \cdots\}$ *is bi-element set* $\subseteq X^+$, *and* σ *is a permutation on* X^+ *so that* $x_{\sigma(1^+)} \geq \cdots \geq x_{\sigma(n^+)}$ *with the convention* $x_{\sigma((n+1)^+)} := 0$.

An equivalent expression for the equation (3) is

$$Ch_\mu(\mathbf{x}) = \sum_{i=1}^{n} x_{\sigma(i^+)} [\mu(\{A_{\sigma(i^+)}\}) - \mu(\{A_{\sigma((i-1)^+)}\})] \quad (4)$$

with the same notation above and $\mu(\{A_0\}) := 0$.

From the definition of the Choquet integral with respect to capacity based on bi-element set, we immediately see the following property.

Proposition 1 *Let* $\mathcal{B}(X)$ *be the set of all bi-element sets and* $\mu : \mathcal{B}(X) \to [0,\ 1]$, *be a capacity based on bi-element set. Then Choquet integral of* $\mathbf{x}$ *with respect to* μ *is given by*

$$Ch_\mu(\mathbf{x}) = \sum_{i=1}^{n} [x_{\sigma(i^+)} - x_{\sigma((i-1)^+)}]\ \mu(\{A_{\sigma(i^+)}\}), \quad (5)$$

or as

$$Ch_\mu(\mathbf{x}) = \sum_{i=1}^{n} x_{\sigma(i^+)} [\mu(\{A_{\sigma(i^+)}\}) - \mu(\{A_{\sigma((i+1)^+)}\})] \quad (6)$$

where $A_{\sigma(i^+)} = \{\cdots, \sigma((i-2)^-), \sigma((i-1)^-), \sigma(i^+), \cdots, \sigma(n^+)\}$ *is bi-element set* $\subseteq X^+$, *and* σ *is a permutation on* X^+ *so that* $x_{\sigma(1^+)} \leq \cdots \leq x_{\sigma(n^+)}$ *with the convention* $x_{\sigma((0)^+)} := 0$ *and* $\mu(\{A_{(n+1)^+}\}) := 0$.

For the sake of clarity, let us give the following numerical example.

Example 1: For $n = 3$, let us consider $\mathbf{x} = (4, 6, -3)$. Applying the Choquet integral with respect to the capacity based on bi-element sets (Formula (3)) we obtain $Ch_\mu(4, 6, -3) = (6 - 4)\ \mu(\{2^+, 1^-, 3^-\}) + (4 - (-3))\ \mu(\{2^+, 1^+, 3^-\}) + (-3 - 0)\ \mu(\{2^+, 1^+, 3^+\}) = 2\ \mu(\{2^+, 1^-, 3^-\}) + 7\ \mu(\{2^+, 1^+, 3^-\}) - 3\mu(\{2^+, 1^+, 3^+\})$.

3 Bi-capacities and its integrals based on ter-element sets

3.1 Bi-capacities based on ter-element sets

In this subsection, we will extend the scale of capacities to the bipolar scale $[-1, 1]$ and we generalize the concept of binary-element sets to ternary-element sets, then we give equivalent definitions of bi-capacities based on ternary-element sets. So we are going to work on the scale [-1, 1] and we will say that [0,1] is the degree of satisfaction (or "positive part") and [-1,0] the degree of non satisfaction (or "negative part"), where 1 is the full satisfaction and -1 the full non satisfaction. In this construction an interesting point is 0. It is the middle point between the full satisfaction and the full non-satisfaction. So we will consider that it represents the "neutral part".

The idea is to extend the notion of bi-element sets, and define a concept that gathers all combinations of positive, negative, and neutral values on the criteria. For this situation, we assume that, for every criterion i has either positive effect (i.e., i is positively important criterion), or has negative effect (i.e., i is negatively important criterion), or has no effect (i.e., i is criterion at neutral level). Hence, we represent the element i as i^+ whenever i is positively important, as i^- whenever i is negatively important, and as $i^\emptyset$ whenever i is neutral, and we call this element a *ternary-element* (or simply *ter-element*). The *ternary-element set* (or simply *ter-element set*) is the set which contains only out of i^+, i^-, and $i^\emptyset$ for all i, $i = 1, \cdots, n$. Thus, in our model we consider the set of all possible combinations of ternary elements of n criteria given by

$$\mathcal{T} := \{\{\tau_1, \cdots, \tau_n\} \mid \forall\, \tau_i \in \{i^+, i^-, i^\emptyset\},\ i = 1, \cdots, n\},$$

which corresponds to $\mathcal{Q}$ in the notation of classical bi-capacities ([7]).
Note that, $\mathcal{T}$ can be identified with $\{-1, 0, 1\}^n$, hence $|\mathcal{T}| = 3^n$. Also, simply remark that for any ter-element set $A \in \mathcal{T}$, A is equivalent to a ternary alternative $(\tau_1, \ldots, \tau_n)$ with $\tau_i = 1$ if $i^+ \in A$, $\tau_i = 0$ if $i^\emptyset \in A$, and $\tau_i = -1$ if $i^- \in A$; $\forall\, i = 1, \ldots, n$.

We introduce the order relation $\sqsubseteq$ between ter-element sets of $\mathcal{T}$ as follows.

Definition 3 *Let $\mathcal{T}$ be the set of all ter-element sets and $A, B \in \mathcal{T}$. Then, $A \sqsubseteq B$ iff $\forall\, i \in X$,*

"if $i^+ \in A$ implies $i^+ \in B$", and

$$\text{"if } i^\emptyset \in A \text{ implies } i^+ \text{ or } i^\emptyset \in B\text{".} \quad (7)$$

Note that, $X^- = \{1^-, \ldots, n^-\} \sqsubseteq A$ and $X^+ = \{1^+, \ldots, n^+\} \sqsupseteq A$, $\forall\, A \in \mathcal{T}$.

Based on the notion of ter-element sets, the following definition gives an equivalent definition of bi-capacities.

Definition 4 *Let $\mathcal{T}$ be the set of all ter-element sets. A set function $\nu : \mathcal{T} \to [-1, 1]$, is called bi-capacity if it satisfies the following requirements:*
(i) $\nu(X^-) = \nu(\{1^-, \cdots, n^-\}) = -1$, $\nu(X^\emptyset) = \nu(\{1^\emptyset, \cdots, n^\emptyset\}) = 0$, *and* $\nu(X^+) = \nu(\{1^+, \cdots, n^+\}) = 1$,
(ii) $\forall\, A, B \in \mathcal{T}$, $A \sqsubseteq B$ *implies* $\nu(A) \le \nu(B)$.

Bi-capacity based on ter-element set $\nu(A)$, $\forall\, A \in \mathcal{T}$ is exactly the overall score or utility of the ternary alternative $(\tau_1, \ldots, \tau_n)$ with $\tau_i = 1$ (full satisfaction) if $i^+ \in A$, i.e. on positively important elements in A, $\tau_i = -1$ (full non satisfaction) if $i^- \in A$, i.e. on negatively important elements in A, and $\tau_i = 0$ (neutral) if $i^\emptyset \in A$, i.e. according to the neutral elements in A.

3.2 The order $\subseteq$ on $\mathcal{T}$

Bi-capacities are functions defined on the structure of the underlying partially ordered set [5]. There are several orders on the structure $\mathcal{Q}$ have introduced by Grabisch and Labreuche [7] and Bilbao et al. [3]. In this subsection, we introduce an order on the structure $\mathcal{T}$ different from the order ($\sqsubseteq$) described in Subsection 3.1 (Definition 3). We consider the following definition of an order on $\mathcal{T}$ which is equivalent to Bilbao order on bi-cooperative game [3]. For convenience, we denote by $\subseteq$ the order relation defined on $\mathcal{T}$ as in the equation (1), and we will use the order $\subseteq$ on $\mathcal{T}$ to establish our next results of this research.

Definition 5 *Let $\mathcal{T}$ be the set of all ter-element sets and $A, B \in \mathcal{T}$. Then, $A \subseteq B$ iff $\forall\, i \in X$,*

"if $i^+ \in A$ implies $i^+ \in B$",

$$\text{and "if } i^- \in A \text{ implies } i^- \in B\text{".} \quad (8)$$

Furthermore, in this order:

- the number of positively important elements i^+ of the ter-element set $A \in \mathcal{T}$, denoted by a^+, is defined as $a^+ = \sum_{i=1}^{n} \chi_A(i^+)$, where,

$$\chi_A(i^+) = \begin{cases} 1 & \text{if } i^+ \in A, \\ 0 & \text{if } i^+ \notin A. \end{cases}$$

- the number of negatively important elements i^- of the ter-element set $A \in \mathcal{T}$, denoted by a^-, is defined as $a^- = \sum_{i=1}^{n} \chi_A(i^-)$, where,

$$\chi_A(i^-) = \begin{cases} 1 & \text{if} \quad i^- \in A, \\ 0 & \text{if} \quad i^- \notin A. \end{cases}$$

– the cardinality of the ter-element set $A \in \mathcal{T}$ is

$$a = |A| = a^+ + a^-. \qquad (9)$$

– for any ter-element set $A \in \mathcal{T}$, we define the complement of A by
$A^c = \{ \{\tau_1{}^c, \ldots, \tau_i{}^c, \ldots, \tau_n{}^c\} \mid \forall\ \tau_i \in A$ with $(i^+)^c = i^-$, $(i^-)^c = i^+$, and $(i^\emptyset)^c = i^\emptyset$ $\}$.

– for all $A,\ B \in \mathcal{T}$, the set difference of A whenever $A \subseteq B$ is defined by $B \backslash A\ = \{\tau_k \backslash \tau_j :\ \tau_j \in A,\ \tau_k \in B,\ j = 1, \ldots, n,\ k = 1, \ldots, n\}$, with $i^+ \backslash i^\emptyset = i^+,\ i^- \backslash i^\emptyset = i^-,\ i^\emptyset \backslash i^\emptyset = i^\emptyset,\ i^+ \backslash i^+ = i^\emptyset,\ i^- \backslash i^- = i^\emptyset,\ i = 1, \ldots, n.$

– for all $A,\ B \in \mathcal{T}$, the union, $A \cup B$ of A and B is defined by,
$A \cup B = \{\tau_j \vee \tau_k :\ \ \tau_j \in A,\ \ \tau_k \in B,\ \ j = 1, \ldots, n,\ k = 1, \ldots, n\}$, with $i^+ \vee i^- = i^+,\ i^+ \vee i^\emptyset = i^+,\ i^- \vee i^\emptyset = i^-,\ \ i = 1, \ldots, n.$

– for all $A, B \in \mathcal{T}$, the intersection, $A \cap B$ of A and B is defined by,
$A \cap B = \{\tau_j \wedge \tau_k :\ \ \tau_j \in A,\ \ \tau_k \in B,\ \ j = 1, \ldots, n,\ k = 1, \ldots, n\}$, with $i^+ \wedge i^- = i^-,\ i^+ \wedge i^\emptyset = i^\emptyset,\ i^- \wedge i^\emptyset = i^\emptyset,\ \ i = 1, \ldots, n.$

3.3 The bipolar Choquet integral based on the ter-element sets

In this subsection, we present bipolar Choquet integral model with respect to bi-capacity based on the ter-element set of real input **x**. The basic idea underlying this model is for an input vector $\mathbf{x} = (x_{\tau_1}, \ldots, x_{\tau_i}, \ldots, x_{\tau_n}),\ x_{\tau_i} \in \mathcal{R}$ with $i \in \{1, \ldots, n\}$, we consider a ter-element set $X^* := \{\tau_1, \ldots, \tau_n\}$ with $\tau_i = i^+$ if $x_{\tau_i} > 0$, $\tau_i = i^-$ if $x_{\tau_i} < 0$, and $\tau_i = i^\emptyset$ if $x_{\tau_i} = 0$; $\forall\, i = 1, \ldots, n$. Thus, we define the bipolar Choquet integral with respect to bi-capacity based on ter-element set of real input **x** as follows.

Definition 6 *Let $\mathcal{T}(X)$ be the set of all ter-element sets and $\nu : \mathcal{T}(X) \to [-1,\ 1]$, be a bi-capacity based on ter-element set. Then, the bipolar Choquet integral of* **x** *with respect to* ν *is given by*

$$Ch_\nu(\mathbf{x}) = \sum_{i=1}^{n} [|x_{\sigma(\tau_i)}| - |x_{\sigma(\tau_{i+1})}|]\ \nu(\{A_{\sigma(\tau_i)}\}), \quad (10)$$

where $\tau_i \in \{i^+, i^-, i^\emptyset\}$, $A_{\sigma(\tau_i)} = \{\sigma(\tau_1), \cdots, \sigma(\tau_i), \sigma((i+1)^\emptyset), \sigma((i+2)^\emptyset), \cdots\}$ *is ter-element set* $\subseteq X^*$, *and* σ *is a permutation on* X^* *so that* $|x_{\sigma(\tau_1)}| \geq \cdots \geq |x_{\sigma(\tau_n)}|$ *with the convention* $x_{\sigma(\tau_{n+1})} := 0$.

An equivalent expression for the equation (9) is

$$Ch_\nu(\mathbf{x}) = \sum_{i=1}^{n} |x_{\sigma(\tau_i)}| [\nu(\{A_{\sigma(\tau_i)}\}) - \nu(\{A_{\sigma(\tau_{(i+1)})}\})] \quad (11)$$

with the same notation above and $\nu(\{A_0\}) := 0$.

From the definition of the bipolar Choquet integral with respect to the bi-capacity based on ter-element set, we immediately see the following property.

Proposition 2 *Let $\mathcal{T}(X)$ be the set of all ter-element sets and $\nu : \mathcal{T}(X) \to [-1,\ 1]$, be a bi-capacity based on ter-element set. Then, the bipolar Choquet integral of* **x** *with respect to* ν *is given by*

$$Ch_\nu(\mathbf{x}) = \sum_{i=1}^{n} [|x_{\sigma(\tau_i)}| - |x_{\sigma(\tau_{i-1})}|]\ \nu(\{A_{\sigma(\tau_i)}\}), \quad (12)$$

or as

$$Ch_\nu(\mathbf{x}) = \sum_{i=1}^{n} |x_{\sigma(\tau_i)}| [\nu(\{A_{\sigma(\tau_i)}\}) - \nu(\{A_{\sigma(\tau_{(i-1)})}\})] \quad (13)$$

where $\tau_i \in \{i^+, i^-, i^\emptyset\}$, $A_{\sigma(\tau_i)} = \{\cdots, \sigma((i-2)^\emptyset), \sigma((i-1)^\emptyset), \sigma(\tau_i), \cdots, \sigma(\tau_n)\}$ *is ter-element set* $\subseteq X^*$, *and* σ *is a permutation on* X^* *so that* $|x_{\sigma(\tau_1)}| \leq \cdots \leq |x_{\sigma(\tau_n)}|$ *with the convention* $x_{\sigma(\tau_0)} := 0$ *and* $\nu(\{A_{n+1}\}) := 0$.

The following numerical example illustrates the use of the bipolar Choquet integral with respect to bi-capacity based on the ter-element sets.

Example 2: [Example 1 continued] Applying the bipolar Choquet integral with respect to bi-capacity based on the ter-element sets (Formula (10)) we obtain $Ch_\nu(4, 6, -3) =$
$(6-4)\ \nu(\{2^+, 1^\emptyset, 3^\emptyset\}) + (4-3)\ \nu(\{2^+, 1^+, 3^\emptyset\}) + (3-0)\ \nu(\{2^+, 1^+, 3^-\})$
$= 2\ \ \nu(\{2^+, 1^\emptyset, 3^\emptyset\}) + \ \ \nu(\{2^+, 1^+, 3^\emptyset\}) + 3\, \nu(\{2^+, 1^+, 3^-\}).$

Proposition 3 *For positive input vectors the bipolar Choquet integral with respect to bi-capacity and Choquet integral with respect to capacity coincide.*

Proof: Clear from the definition of the bipolar Choquet integral with respect to the bi-capacity based on the ter-element sets.

4 Choquet integral on fuzzy sets

In this section, we propose a new approach for studying Choquet integral on fuzzy sets through introducing a notion of fuzzy binary-element sets, which is alternative approach from defined by Wu and Huang [16].

4.1 Fuzzy bi-element sets

For any bi-element set $A \in \mathcal{B}$, the characteristic function $\mathcal{M}_A(i)$ can be defined by $\mathcal{M}_A(i) = 1$ iff $i^+ \in A$ and $\mathcal{M}_A(i) = 0$ iff $i^- \in A$. A fuzzy bi-element set is any bi-element set that allows its members to have different degree of membership, called membership function, in the interval [0, 1]. We then define a *fuzzy bi-element set* as follows.

Definition 7 *A fuzzy bi-element set $\tilde{A}$ is the set $\tilde{A} = \{(\tau_i, \mathcal{M}_{\tilde{A}}(i)) |\, \tau_i \in \{i^+, i^-\},\ i = 1, \cdots, n\}$, where the value $\mathcal{M}_{\tilde{A}}(i)$ is called degree of membership of i in $\tilde{A}$ with $\mathcal{M}_{\tilde{A}}(i) \in (0,1]$ iff $\tau_i = i^+$ and $\mathcal{M}_{\tilde{A}}(i) = 0$ iff $\tau_i = i^-$.*

We denote the set of all fuzzy bi-element sets by $\tilde{\mathcal{B}}$, and we introduce the complement, inclusion relation, union, and intersection between fuzzy bi-element sets of $\tilde{\mathcal{B}}$ as follows.

- Let $\tilde{\mathcal{B}}$ be the set of all fuzzy bi-element sets and $\tilde{A} \in \tilde{\mathcal{B}}$. The complement, $\tilde{A}^c$ of $\tilde{A}$ is defined by the following membership function, $\mathcal{M}_{\tilde{A}^c}(i) = 1 - \mathcal{M}_{\tilde{A}}(i)$, where the value $\mathcal{M}_{\tilde{A}}(i)$, $\tau_i \in \{i^+, i^-\}$, is called degree of membership of i in $\tilde{A}$ with $\mathcal{M}_{\tilde{A}}(i) \in [0,1]$.

- Let $\tilde{\mathcal{B}}$ be the set of all fuzzy bi-element sets and $\tilde{A},\ \tilde{B} \in \tilde{\mathcal{B}}$. Then, $\tilde{B} \subseteq \tilde{A}$ holds iff $\mathcal{M}_{\tilde{B}}(i) \leq \mathcal{M}_{\tilde{A}}(i)$, for each $\tau_i \in \{i^+, i^-\}$.

- Let $\tilde{\mathcal{B}}$ be the set of all fuzzy bi-element sets and $\tilde{A}, \tilde{B} \in \tilde{\mathcal{B}}$. The union, $\tilde{A} \cup \tilde{B}$ of $\tilde{A}$ and $\tilde{B}$ is defined by, $\mathcal{M}_{\tilde{A}\cup\tilde{B}}(i) = \mathcal{M}_{\tilde{A}}(i) \vee \mathcal{M}_{\tilde{B}}(i)$ for each $\tau_i \in \{i^+, i^-\}$.

- Let $\tilde{\mathcal{B}}$ be the set of all fuzzy bi-element sets and $\tilde{A}, \tilde{B} \in \tilde{\mathcal{B}}$. The intersection, $\tilde{A} \cap \tilde{B}$ of $\tilde{A}$ and $\tilde{B}$ is defined by, $\mathcal{M}_{\tilde{A}\cap\tilde{B}}(i) = \mathcal{M}_{\tilde{A}}(i) \wedge \mathcal{M}_{\tilde{B}}(i)$ for each $\tau_i \in \{i^+, i^-\}$.

4.2 Capacity on fuzzy sets

Let us consider algebras of fuzzy bi-element sets as a base for defining capacities on fuzzy sets, and we shall always denote a fuzzy algebra of fuzzy bi-element sets by $\tilde{\mathcal{B}}$.

Definition 8 *A fuzzy algebra $\tilde{\mathcal{B}}$ is a nonempty subclass of $\tilde{\mathcal{B}}$ with the following properties:*
(i) $\{1^-, \cdots, n^-\}$ and $\{1^+, \cdots, n^+\} \in \tilde{\mathcal{B}}$
(ii) If $\tilde{A}, \tilde{B} \in \tilde{\mathcal{B}}$, then $\tilde{A} \cup \tilde{B} \in \tilde{\mathcal{B}}$,
(iii) If $\tilde{A} \in \tilde{\mathcal{B}}$ then $\tilde{A}^c \in \tilde{\mathcal{B}}$.

Based on the above definitions, we define *capacity on fuzzy sets* as follows.

Definition 9 *A fuzzy set function $\tilde{\mu} : \tilde{\mathcal{B}} \to [0, \infty]$, is called capacity on fuzzy sets if it satisfies the following requirements:*
$\tilde{\mu}(\{1^-, \cdots, n^-\}) = 0$,
$\tilde{\mu}(\tilde{B}) \leq \tilde{\mu}(\tilde{A})$ whenever $\tilde{B}, \tilde{A} \in \tilde{\mathcal{B}}$ with $\tilde{B} \subseteq \tilde{A}$.

4.3 Choquet Integral based on bi-element sets on fuzzy sets

In this subsection, we can generalize the definition of Choquet integral based on bi-element sets (Definition 2) to fuzzy capacities on fuzzy sets. The Choquet integral based on bi-element sets of real input $\mathbf{x}$ with respect to capacity on fuzzy sets $\tilde{\mu}$ we will denote by $C_{\tilde{\mu}}(\mathbf{x})$, and we define it as follows.

Definition 10 *Let $\tilde{\mu}$ be a capacity on fuzzy set. The Choquet integral of $\mathbf{x}$ with respect to $\tilde{\mu}$ is given by*

$$C_{\tilde{\mu}}(\mathbf{x}) = \sum_{i=1}^{n} [x_{\sigma(i^+)} - x_{\sigma((i+1)^+)}]\tilde{\mu}(\{\tilde{A}_{\sigma(i^+)}\}), \quad (14)$$

where $\tilde{A}_{\sigma(i^+)} = \{(\sigma(1^+), \mathcal{M}_{A_{\sigma(1^+)}}(\sigma(1^+))), \cdots, (\sigma(i^+), \mathcal{M}_{A_{\sigma(i^+)}}(\sigma(i^+))), (\sigma((i+1)^-), 0), \cdots, (\sigma(n^-), 0)\}$, *and σ is a permutation on $\tilde{X}^+$ so that $x_{\sigma(1^+)} \geq \cdots \geq x_{\sigma(n^+)}$ with the convention $x_{\sigma((n+1)^+)} := 0$. An equivalent expression for the equation (14) is*

$$C_{\tilde{\mu}}(\mathbf{x}) = \sum_{i=1}^{n} x_{\sigma(i^+)} [\tilde{\mu}(\{\tilde{A}_{\sigma(i^+)}\}) - \tilde{\mu}(\{\tilde{A}_{\sigma((i-1)^+)}\})] \quad (15)$$

with the same notation above and $\tilde{\mu}(\{\tilde{A}_0\}) := 0$.

For the sake of clarity, let us give the following numerical example.

Example 3: [Example 1 continued] For $n=3$, let us consider $\mathbf{x}=(4,\ 6,\ -3)$. The bipolar Choquet integral of $\mathbf{x}$ with respect to $\tilde{\mu}$ is calculated as (Formula (14)): $C_{\tilde{\mu}}(\mathbf{x})=$
$(6-4)\,\tilde{\mu}(\{(2^+,\mathcal{M}_{A_{2+}}(2^+)),(1^-,0),(3^-,0)\})+$
$(4-3)\,\tilde{\mu}(\{(2^+,\mathcal{M}_{A_{2+}}(2^+)),(1^+,\mathcal{M}_{A_{1+}}(1^+)),(3^-,0)\})$
$+(3-0)\tilde{\mu}(\{(2^+,\mathcal{M}_{A_{2+}}(2^+)),(1^+,\mathcal{M}_{A_{1+}}(1^+)),$
$(3^+,\mathcal{M}_{A_{3+}}(3^+))\})$.

Note that the definition 10 remains valid if $\tilde{\mu}$ is a capacity on crisp set and the Choquet integral with respect to capacity on fuzzy sets reduces to Choquet integral with respect to capacity on crisp set.

5 Bipolar Choquet Integral on fuzzy sets

5.1 Fuzzy ter-element sets

For any ter-element set $A\in\mathcal{T}$, the characteristic function $\mathcal{M}_A(i)$, can be defined by $\mathcal{M}_A(i)=1$ iff $i^+\in A$, $\mathcal{M}_A(i)=-1$ iff $i^-\in A$, and $\mathcal{M}_A(i)=0$ iff $i^\phi\in A$. A fuzzy ter-element set is any ter-element set that allows its members to have different degree of membership, called membership function $\mathcal{M}_A(i)$, in the interval $[-1,\ 1]$. We then define a *fuzzy ter-element set* as follows.

Definition 11 *A fuzzy ter-element set* $\tilde{A}$ *is the set* $\tilde{A}=\{(\tau_i,\mathcal{M}_{\tilde{A}}(i))|\ \tau_i\in\{i^+,i^-,i^\phi\},\quad i=1,\cdots,n\}$, *where the value* $\mathcal{M}_{\tilde{A}}(i)$ *is called degree of membership of* i *in* $\tilde{A}$ *with* $\mathcal{M}_{\tilde{A}}(i)\in[-1,0)$ *iff* $\tau_i=i^-$, $\mathcal{M}_{\tilde{A}}(i)\in(0,1]$ *iff* $\tau_i=i^+$, and $\mathcal{M}_{\tilde{A}}(i)=0$ *iff* $\tau_i=i^\phi$.

We denote the set of all fuzzy ter-element sets by $\tilde{\mathcal{T}}$, and we introduce the complement, inclusion relation, union, and intersection between fuzzy ter-element sets of $\tilde{\mathcal{T}}$ as follows.

- Let $\tilde{\mathcal{T}}$ be the set of all fuzzy ter-element sets and $\tilde{A}\in\tilde{\mathcal{T}}$. The complement, $\tilde{A}^c$ of $\tilde{A}$ is defined by the following membership function,
$$\mathcal{M}_{\tilde{A}^c}(i)=\begin{cases}1-\mathcal{M}_{\tilde{A}}(i) & if\ \tau_i=i^+,\\ 0 & if\ \tau_i=i^\phi,\\ -1-\mathcal{M}_{\tilde{A}}(i) & if\ \tau_i=i^-.\end{cases}$$
- Let $\tilde{\mathcal{T}}$ be the set of all fuzzy ter-element sets and $\tilde{A},\ \tilde{B}\in\tilde{\mathcal{T}}$. Then, $\tilde{B}\subseteq\tilde{A}$ holds iff $\mathcal{M}_{\tilde{B}}(i)\leq\mathcal{M}_{\tilde{A}}(i)$, for each $\tau_i\in\{i^+,i^-,i^\phi\}$.
- Let $\tilde{\mathcal{T}}$ be the set of all fuzzy ter-element sets and $\tilde{A},\tilde{B}\in\tilde{\mathcal{T}}$. The union, $\tilde{A}\cup\tilde{B}$ of $\tilde{A}$ and $\tilde{B}$ is defined by, $\mathcal{M}_{\tilde{A}\cup\tilde{B}}(i)=$ $\mathcal{M}_{\tilde{A}}(i)\vee\mathcal{M}_{\tilde{B}}(i)$ for each $\tau_i\in\{i^+,i^-,i^\phi\}$.
- Let $\tilde{\mathcal{T}}$ be the set of all fuzzy ter-element sets and $\tilde{A},\tilde{B}\in\tilde{\mathcal{T}}$. The intersection, $\tilde{A}\cap\tilde{B}$ of $\tilde{A}$ and $\tilde{B}$ is defined by, $\mathcal{M}_{\tilde{A}\cap\tilde{B}}(i)=$ $\mathcal{M}_{\tilde{A}}(i)\wedge\mathcal{M}_{\tilde{B}}(i)$ for each $\tau_i\in\{i^+,i^-,i^\phi\}$.

For example, when $n=3$, the $\tilde{A}$ and $\tilde{B}$ are fuzzy ter-element sets defined by the following membership functions

$$\mathcal{M}_{\tilde{A}}(i)=\begin{cases}0.4 & \text{if}\quad\tau_i=1^+,\\ -0.5 & \text{if}\quad\tau_i=2^-,\\ 0 & \text{if}\quad\tau_i=3^\phi\end{cases}\quad\text{and}$$

$$\mathcal{M}_{\tilde{B}}(i)=\begin{cases}0.6 & \text{if}\quad\tau_i=1^+,\\ -0.3 & \text{if}\quad\tau_i=2^-,\\ 0.2 & \text{if}\quad\tau_i=3^+.\end{cases}$$

Then, $\tilde{A}\subset\tilde{B}$,

$$\mathcal{M}_{\tilde{A}\cup\tilde{B}}(i)=\begin{cases}0.6 & \text{if}\quad i=1^+,\\ -0.3 & \text{if}\quad i=2^-,\\ 0.2 & \text{if}\quad i=3^+,\end{cases}\quad\text{and}$$

$$\mathcal{M}_{\tilde{A}\cap\tilde{B}}(i)=\begin{cases}0.4 & \text{if}\quad i=1^+,\\ -0.5 & \text{if}\quad i=2^-,\\ 0 & \text{if}\quad i=3^\phi.\end{cases}$$

5.2 Bi-capacity on fuzzy sets

Let us consider algebras of fuzzy ter-element sets as a base for defining bi-capacities on fuzzy sets. We shall always denote a fuzzy algebra by $\tilde{\mathcal{T}}$.

Definition 12 *A fuzzy algebra* $\tilde{\mathcal{T}}$ *is a nonempty subclass of* $\tilde{\mathcal{T}}$ *with the following properties:*
(i) $\{1^-,\cdots,n^-\}$, $\{1^\phi,\cdots,n^\phi\}$, and $\{1^+,\cdots,n^+\}\in\tilde{\mathcal{T}}$,
(ii) If $\tilde{A},\tilde{B}\in\tilde{\mathcal{T}}$, then $\tilde{A}\cup\tilde{B}\in\tilde{\mathcal{T}}$,
(iii) If $\tilde{A}\in\tilde{\mathcal{T}}$ then $\tilde{A}^c\in\tilde{\mathcal{T}}$.

Based on the above definitions, we define *bi-capacity on fuzzy sets* as follows.

Definition 13 *A fuzzy set function* $\tilde{\nu}:\tilde{\mathcal{T}}\to\mathcal{R}$, *is called bi-capacity on fuzzy sets if it satisfies the following requirements:*
(i) $\tilde{\nu}(\{1^\phi,\cdots,n^\phi\})=0$,
(ii) $\tilde{\nu}(\tilde{B})\leq\tilde{\nu}(\tilde{A})$ *whenever* $\tilde{B},\tilde{A}\in\tilde{\mathcal{T}}$ *with* $\tilde{B}\subseteq\tilde{A}$.

5.3 Bipolar Choquet Integral on fuzzy sets

In this subsection, we can generalize the definition of bipolar Choquet integral (Definition 6) to bi-capacities on fuzzy sets. For an input vector $\mathbf{x} = (x_1, \cdots, x_i, \cdots, x_n)$, $x_i \in \mathcal{R}$ with $i \in \{1, \ldots, n\}$, we consider a ter-element set $\tilde{A^*} = \{(\tau_i, \mathcal{M}_{\tilde{A^*}}(i))\}$ with $\tau_i = i^+$ if $x_i > 0$, $\tau_i = i^-$ if $x_i < 0$, and $\tau_i = i^\emptyset$ if $x_i = 0$; $\forall\ i = 1, \ldots, n$. The bipolar Choquet integral model of real input $\mathbf{x}$ with respect to bi-capacity on fuzzy sets $\tilde{\nu}$ we will denote by $C_{\tilde{\nu}}(\mathbf{x})$, and we define it as follows.

Definition 14 *Let $\tilde{\nu}$ be a bi-capacity on fuzzy set. The bipolar Choquet integral of $\mathbf{x}$ with respect to $\tilde{\nu}$ is given by*

$$C_{\tilde{\nu}}(\mathbf{x}) = \sum_{j=1}^{n} [|x_{\sigma(\tau_i)}| - |x_{\sigma(\tau_{i+1})}|]\ \tilde{\nu}(\{\tilde{A}_{\sigma(\tau_i)}\}), \quad (16)$$

where $\tau_i \in \{i^+,\ i^-, i^\phi\}$, $\tilde{A}_{\sigma(\tau_i)} = \{(\sigma(\tau_1), \mathcal{M}_{A_{\sigma(\tau_1)}}(\sigma(\tau_1))), \cdots, (\sigma(\tau_i), \mathcal{M}_{A_{\sigma(\tau_i)}}(\sigma(\tau_i))), (\sigma((i+1)^\phi), 0), \cdots, (\sigma(n^\phi), 0)\}$ and σ is a permutation on X so that $|x_{\sigma(\tau_1)}| \geq \cdots \geq |x_{\sigma(\tau_n)}|$ *with the convention* $x_{\sigma(\tau_{n+1})} := 0$.

An equivalent expression for the equation (16) is $BC_{\tilde{\nu}}(\mathbf{x}) =$

$$\sum_{i=1}^{n} |x_{\sigma(\tau_i)}|\ [\tilde{\nu}(\{\tilde{A}_{\sigma(\tau_i)}\}) - \tilde{\nu}(\{\tilde{A}_{\sigma(\tau_{i-1})}\})], \quad (17)$$

with the same notation above and $\tilde{\nu}(\{\tilde{A}_0\}) := 0$.

For the sake of clarity, let us give the following numerical example.

Example 4: [Example 1 continued] For $n = 3$, let us consider $\mathbf{x} = (4,\ 6,\ -3)$. The bipolar Choquet integral of $\mathbf{x}$ with respect to $\tilde{\nu}$ is calculated as (Formula (16)): $C_{\tilde{\nu}}(\mathbf{x}) =$
$(6-4)\tilde{\nu}(\{(2^+, \mathcal{M}_{A_{2^+}}(2^+)), (1^\phi, 0), (3^\phi, 0)\}) +$
$(4-3)\tilde{\nu}(\{(2^+, \mathcal{M}_{A_{2^+}}(2^+)), (1^+, \mathcal{M}_{A_{1^+}}(1^+)), (3^\phi, 0)\})$
$+ (3-0)\ \tilde{\nu}(\{(2^+, \mathcal{M}_{A_{2^+}}(2^+)), (1^+, \mathcal{M}_{A_{1^+}}(1^+)), (3^-, \mathcal{M}_{A_{3^-}}(3^-))\})$.

Note that the definition 14 remains valid if $\tilde{\nu}$ is a bi-capacity and the bipolar Choquet integral with respect to bi-capacity on fuzzy sets reduces to bipolar Choquet integral with respect to bi-capacity.

We now state some properties satisfied by the proposed approach. The first shows the bipolar Choquet integral with respect to bi-capacity on fuzzy sets of ternary alternatives $(1_{\tilde{A}}, -1_{\tilde{A}}, 0_{\tilde{A}})$ equals $\tilde{\nu}(\tilde{A})$ as shown by the following result.

Proposition 4 *For any bi-capacity on fuzzy sets ($\tilde{\nu}$) on $\tilde{\mathcal{T}}$,*

$$C_{\tilde{\nu}}(1_{\tilde{A}}, -1_{\tilde{A}}, 0_{\tilde{A}}) = \tilde{\nu}(A),\ \forall\, A \in \mathcal{T}(X),\ \tilde{A} \in \tilde{\mathcal{T}}.$$

Proof: For ternary vector $(1_{\tilde{A}}, -1_{\tilde{A}}, 0_{\tilde{A}})$,

$$|x_{\sigma(\tau_i)}| = 1 \text{ or } |x_{\sigma(\tau_i)}| = 0,\ \forall\ \tau_i \in \{i^+,\ i^-, i^\phi\}$$

and $\tilde{\nu}(\{\tilde{A}_{\sigma(\tau_i)}\}) =$

$$\tilde{\nu}(\{(i^+, \mathcal{M}_A(i^+)), (i^-, \mathcal{M}_A(i^-), (i^\phi, 0)\}) =$$

$$\tilde{\nu}(\{(i^+, 1), (i^-, -1), (i^\phi, 0)\}) = \tilde{\nu}(\tilde{A}) = \tilde{\nu}(A).$$

Therefore, from the bipolar Choquet integral with respect to bi-capacity on fuzzy sets (Formula (16)), we have, $C_{\tilde{\nu}}(1_{\tilde{A}}, -1_{\tilde{A}}, 0_{\tilde{A}}) =$

$$\sum_{i=1}^{n} [|x_{\sigma(\tau_i)}| - |x_{\sigma(\tau_{i+1})}|]\ \tilde{\nu}(\{\tilde{A}_{\sigma(\tau_i)}\})$$

Thus,

$$C_{\tilde{\nu}}(1_{\tilde{A}}, -1_{\tilde{A}}, 0_{\tilde{A}}) = \tilde{\nu}(A),\ \ \forall\, A \in \mathcal{T}(X),\ \tilde{A} \in \tilde{\mathcal{T}}.$$

The next property shows that the bipolar Choquet integral with respect to bi-capacity on fuzzy sets satisfy the monotonicity.

Proposition 5 *(Monotonicity) For any bi-capacity on fuzzy sets $\tilde{\nu}$ on $\tilde{\mathcal{T}}$, $\forall\, \mathbf{x}$, $\mathbf{x}' \in \mathcal{R}$, if $x_{\tau_i} \geq x'_{\tau_i}$, $\forall\ \tau_i \in \{i^+,\ i^-, i^\phi\}$, then $C_{\tilde{\nu}}(\mathbf{x}) \geq C_{\tilde{\nu}}(\mathbf{x}')$.*

Proof: First, we assume that for any $\tau_i \in \{i^+,\ i^-, i^\phi\}$, $x_{\tau_i} > x'_{\tau_i}$ and $\forall\ k \in \{1, \cdots, i-1, i+1, \cdots, n\}$, $x_{\tau_k} = x'_{\tau_k}$. Also, we assume that for all elements of X, the order of each element is the same, i.e., $|x_{\sigma(\tau_1)}| \geq \cdots \geq |x_{\sigma(\tau_n)}|$ and $|x'_{\sigma(\tau_1)}| \geq \cdots \geq |x'_{\sigma(\tau_n)}|$. Firstly, we prove the monotonicity of this case.

Using the equivalent expression of bipolar Choquet integral with respect to bi-capacity on fuzzy sets (Formula (17)), we have, $C_{\tilde{\nu}}(\mathbf{x}) =$

$$\sum_{i=1}^{n} |x_{\sigma(\tau_i)}| [\tilde{\nu}(\{\tilde{A}_{\sigma(\tau_i)}\}) - \tilde{\nu}(\{\tilde{A}_{\sigma(\tau_{i-1})}\})] \quad (18)$$

also, $C_{\tilde{\nu}}(\mathbf{x}') =$

$$\sum_{i=1}^{n} |x'_{\sigma(\tau_i)}| [\tilde{\nu}(\{\tilde{A}_{\sigma(\tau_i)}\}) - \tilde{\nu}(\{\tilde{A}_{\sigma(\tau_{i-1})}\})] \quad (19)$$

$\tilde{A}_{\sigma(\tau_i)}$ and $\tilde{A}_{\sigma(\tau_{i-1})}$ are the fuzzy ternary-element sets with $\tilde{A}_{\sigma(\tau_{i-1})} \subseteq \tilde{A}_{\sigma(\tau_i)}$. Hence, $\tilde{\nu}(\{\tilde{A}_{\sigma(\tau_i)}\}) - \tilde{\nu}(\{\tilde{A}_{\sigma(\tau_{i-1})}\}) \geq 0$.

Now, since $x_{\sigma(\tau_i)} \geq x'_{\sigma(\tau_i)}$, $\forall\ \tau_i \in \{i^+,\ i^-, i^\phi\}$, it is clear that $C_{\tilde{v}}(\mathbf{x}) \geq C_{\tilde{v}}(\mathbf{x}')$. Therefore, if $x_{\tau_i} > x'_{\tau_i}$ then $C_{\tilde{v}}(\mathbf{x}) \geq C_{\tilde{v}}(\mathbf{x}')$ is proved within the range that the order of values of each element of $\mathbf{x}$ and $\mathbf{x}'$ does not change. Thus, by repeating the above procedures two times at the point of the change of the order, if $x_{\tau_i} > x'_{\tau_i}$ then $C_{\tilde{v}}(\mathbf{x}) \geq C_{\tilde{v}}(\mathbf{x}')$ can be proved even in the range with the change of the order. By applying this procedure successively for each i, the proposition can be proved.

Conclusion

In this paper, we defined a bi-capacity based on ter-element set, satisfying properties similar to the classical definition of bi-capacities. According to this definition, we have presented an approach for studying bi-capacities on fuzzy sets. Then, we have proposed a model of bipolar Choquet integral on fuzzy sets, and we gave some basic properties of this model. The applications of the bipolar Choquet integral on fuzzy sets in many actual fields, such as decision making, pattern recognition, and medical diagnosis, are open questions for future research.

References

[1] Abbas J., Bipolar Choquet integral of fuzzy events, IEEE SSCI conference on MCDM, Florida, USA, pp.116- 123, 2014.

[2] Abbas J., Logical twofold integral, Engineering & Techology Journal, Vol. 28, No. 3; 2010.

[3] Bilbao, J.M., Fernandez, J.R., Jimenez Losada, A. and Lebron, E., Bicooperative Games, First World Congress of the Game Theory Society (Games 2000) July 24-28, Bilbao, Spain, 2000.

[4] Choquet G., Theory of capacities, Ann. Inst. Fourier, Vol. 5, pp. 131-295, 1953.

[5] Davey B.A., Priestley H.A., Introduction to Lattices and Orders, Cambridge University Press, Cambridge, 1990.

[6] Denneberg D., Non-additive measure and integral, Kluwer Academic Publisher, 1994.

[7] Grabisch, M., Labreuche C., Bi-capacities I: Definition, Mobius transform and interaction, Fuzzy Sets and Systems 151, pp. 211- 236, 2005.

[8] Grabisch M. and Labreuche C., Bi-capacities II: The Choquet integral, Fuzzy Sets and Systems 151, pp. 237-259, 2005.

[9] Greco S., Grabisch M., Pirlot M., Bipolar and bivariate models in multicriteria decision analysis: descriptive and constructive approaches, Int. J. Intell. Syst. 23, 930-969, 2008.

[10] Greco S., Matarazzo B., Giove S., The Choquet integral with respect to a level dependent capacity, Fuzzy Sets Syst. 175, 1-35, 2011.

[11] Labreuche Ch. and Grabsch M., Generalized Choquet like-aggregation for handling ratio scales, European J. Oper. Res.,172, pp. 931-955, 2006.

[12] Murofushi T. and Sugeno M., An interpretation of fuzzy measure and the Choquet integral as an integral with respect to a fuzzy measure, Fuzzy Sets and Systems, Vol. 29, pp. 201-227, 1989.

[13] Sugeno M., Theory of fuzzy integrals and its applications, Ph.D. Thesis, Tokyo Institute of Technology, 1974.

[14] Wang Z. and Klir G.J., Fuzzy measure theory, Plenum Press, New york, 1992.

[15] Wang Z. and Klir G.J., Generalized measure theory, Springer Science+ Business Media, LLC, 2009.

[16] Wu C. X. and Huang Y., Choquet integrals on fuzzy sets, IEEE International Conference on Machine Learning and Cybernetics,(4), PP.1438-2552, 2005.

Jabbar Abbas Ghafil received the Ph.D. degree in mathematics from the University of Pune, India, in 2007. He is currently an Assistant Professor of applied mathematics with the Department of Applied Sciences, University of Technology, Baghdad, Iraq. His research interests include foundations of fuzzy measures, aggregation functions, and their applications in multicriteria decision making.