Opuscula Math. 44, no. 1 (2024), 5–17
https://doi.org/10.7494/OpMath.2024.44.1.5 Opuscula Mathematica

UNIQUENESS FOR A CLASS p-LAPLACIAN PROBLEMS WHEN A PARAMETER IS LARGE

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Communicated by Vicenţiu D. Rădulescu

Abstract. We prove uniqueness of positive solutions for the problem

$$-\Delta_p u = \lambda f(u) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

where $1 < p < 2$ and p is close to 2, Ω is bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $f : [0,\infty) \to [0,\infty)$ with $f(z) \sim z^\beta$ at ∞ for some $\beta \in (0,1)$, and λ is a large parameter. The monotonicity assumption on f is not required even for u large.

Keywords: singular p-Laplacian, uniqueness, positive solutions.

Mathematics Subject Classification: 35J92, 35J75.

1. INTRODUCTION

In this paper, we investigate uniqueness of positive solutions to the p-Laplacian BVP

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), 1 < p < 2,\ \Omega$ is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, λ is a positive parameter, and $f : [0,\infty) \to [0,\infty)$ is p-sublinear at ∞.

It is well-known that (1.1) has a unique positive solutions for all $\lambda > 0$ if f is continuous on $[0,\infty)$ and $\frac{f(u)}{u^{p-1}}$ is strictly decreasing on $(0,\infty)$ (see the pioneering work [3] for $p = 2$ and [9,10] for its extension to $p > 1$). When the latter condition is not satisfied, there is a number of uniqueness results for (1.1) when the parameter λ is large (see e.g. [5–8, 11, 12, 15, 16] and the references therein). We are motivated by the uniqueness results in [7,8,15,16] for $p = 2$ and f smooth with $f(u) > 0$ for $u > 0$. In [15], Lin proved uniqueness of positive solutions to (1.1) when $f(u) \sim u^\beta$ for some $\beta \in (0,1), \limsup_{u\to\infty} \frac{uf'(u)}{f(u)} < 1$, and $\limsup_{u\to 0^+} u^2|f'(u)| < \infty$. The case when f is bounded was discussed in [8] and [16], where $f(u) \to C > 0$ as $u \to \infty$ and either $f(0) > 0$ or $f'(0) > 0$ in [8], and $\lim_{u\to\infty} \frac{f(u)}{u} = 0,\ \inf_{[0,\infty)} f > 0$ together with $\liminf_{u\to\infty} f(u) > \limsup_{u\to\infty} uf'(u)$ in [16]. Note that in these references, the nonlinearity f is not required to be increasing or decreasing even for u large. For $p>1$, uniqueness results for (1.1) were obtained in [5, 6, 11, 12] for λ large under the p-sublinear assumption together with some monotonicity conditions on f. In this paper, we will provide a uniqueness result in the absence of this common monotonicity requirement when $1<p<2$ and p is close to 2, $f(u)\sim u^{\beta}$ at ∞ for some $\beta\in(0,1)$ together with some natural conditions at 0 and ∞. Thus our result provides an extension of the work in [7, 8, 15, 16] from $p=2$ to $p\in(1,2)$ with $p\sim 2$, which seems to be the first in the literature. In particular, when applied to the model example $f(u)=u^{\beta}+\sin^2(u^{\beta})$, where $\beta\in(0,1)$, Theorem 1.1 below gives uniqueness of positive solutions to (1.1) provided λ is large and $p<2$ is close to 2. A calculation shows that $f(u)$ is neither increasing nor decreasing even for u large. We refer to the recent monograph [19] for the abstract results used in this paper, and to [1, 4, 18–20] for the analysis of related nonlinear problems.

We make the following assumptions:

(A_1) $f:[0,\infty)\to[0,\infty)$ is continuous and of class C^1 on $(0,\infty)$ with $f(u)>0$ for $u>0$.
(A_2) There exists a constant $\beta\in(0,1)$ such that $\lim_{u\to\infty}\frac{f(u)}{u^{\beta}}=1$.
(A_3) $\limsup_{u\to\infty}\frac{uf'(u)}{f(u)}<1$.
(A_4) $\liminf_{u\to 0^+}\frac{f(u)}{u^{p-1}}>0$.
(A_5) There exists $\alpha\in(0,1)$ such that $\limsup_{u\to 0^+}u^{\alpha+1}|f'(u)|<\infty$.

By a positive solution of (1.1), we mean a function $u\in C^{1,\nu}(\bar{\Omega})$ for some $\nu\in(0,1)$ with $u>0$ in Ω and satisfying (1.1) in the weak sense.

Our main result is the following.

Theorem 1.1. *Let $1<p<2$ and (A_1)–(A_5) hold. Then if p is sufficiently close to 2, there exists a constant $\lambda_0>0$ such that (1.1) has a unique positive solution for $\lambda>\lambda_0$.*

Remark 1.2. (i) Theorem 1.1 is not true for $\lambda>0$ small. Indeed, let $\alpha,\beta\in(0,1)$ and

$$f(u)=\begin{cases}u^{p-1}e^{a(1-u)} & \text{for } u\in(0,1),\\ u^{\beta} & \text{for } u\geq 1,\end{cases}$$

where $a=p-1-\beta$. Note that $a>0$ if p is sufficiently close to 2. Then (A_1)–(A_5) hold. Suppose u is a positive solution of (1.1) with $\lambda<\lambda_1 e^{\beta-1}$, where λ_1 denotes the first eigenvalue of $-\Delta_p$ with Dirichlet boundary condition. Since $a\leq 1-\beta$, $f(u)\leq e^{1-\beta}u^{p-1}$ for all $u\geq 0$. Hence, multiplying the equation in (1.1) by u and integrating, we get

$$\int\limits_{\Omega}|\nabla u|^p dx\leq \lambda e^{1-\beta}\int\limits_{\Omega}u^p dx<\lambda_1\int\limits_{\Omega}u^p dx,$$

a contradiction with

$$\lambda_1=\inf_{\substack{v\in W_0^{1,p}(\Omega)\\ v\neq 0}}\frac{\int_{\Omega}|\nabla v|^p dx}{\int_{\Omega}|v|^p dx}.$$

Hence, (1.1) has no positive solution for λ small.

(ii) Theorem 1.1 gives uniqueness of positive solutions to (1.1) when

$$\limsup_{u\to\infty} \frac{uf'(u)}{f(u)} < p-1, \tag{1.2}$$

where $p \in (1,2)$ and is sufficiently close to 2 without requiring any monotonicity of f. We believe that without any monotonicity assumption, uniqueness for (1.1) for λ large under conditions (1.2) and (A_1), (A_2), (A_4), (A_5) for other values of p is an open question. Note that a uniqueness result under these conditions together with the additional assumption that f is nondecreasing on $[0,\infty)$ was obtained in [12].

2. PRELIMINARIES

In what follows, we denote by $d(x)$ the distance from x to the boundary $\partial\Omega$. Let λ_1 be the first eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions, and ϕ_1 the corresponding positive normalized eigenfunction, i.e. $\|\phi_1\|_\infty = 1$.

Lemma 2.1. *Let $h : [0,\infty) \to [0,\infty)$ be nondecreasing and D be an open set in Ω. Suppose there exists $q \in (0, p-1)$ such that $u^{-q}h(u)$ is nonincreasing on $(0,\infty)$ and $\liminf_{u\to 0^+} u^{1-p}h(u) > 0$. Let $g : \Omega \to [0,\infty)$ be bounded in Ω. Then the problem*

$$-\Delta_p u = \begin{cases} h(u) & \text{in } D, \\ g(x) & \text{in } \Omega\backslash D, \end{cases} \qquad u = 0 \text{ on } \partial\Omega \tag{2.1}$$

has a positive solution $\phi_D \in C^1(\bar\Omega)$ with $\inf_\Omega \frac{\phi_D}{d} > 0$. Furthermore,

(i) *$\phi_D \to \omega_p$ in $C^1(\bar\Omega)$ as $|\Omega\backslash D| \to 0$, where ω_p is the solution of*

$$-\Delta_p u = h(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{2.2}$$

and $|A|$ denotes the Lebesgue measure of A;

(ii) *Let $h(u) = u^\beta$ for some $\beta \in (0,1)$. Then $\omega_p \to \omega_2$ in $C^1(\bar\Omega)$ as $p \to 2$, $p < 2$.*

Proof. We first show that the problem

$$\begin{cases} -\Delta_p u = h(u) + g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

has a positive solution by the method of sub- and supersolutions.

Clearly the function ω_p defined in (2.2) is a subsolution of (2.3). Note that the existence and uniqueness of ω_p follows from [9,10].

Let $\psi \in C^1(\bar\Omega)$ satisfy

$$-\Delta_p \psi = 1 \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega. \tag{2.4}$$

Then

$$-\Delta_p(M\psi) = M^{p-1} \geq M^q h(\psi) + g(x) \geq h(M\psi) + g(x) \text{ in } \Omega$$

for M large since h is nondecreasing with $u^{-q}h(u)$ decreasing, $q < p-1$, and g is bounded in Ω. Thus $M\psi$ is a supersolution of (2.3) with $M\psi \geq \omega_p$ in Ω for M large. Hence, (2.3) has a solution $\bar{\psi} \in C^1(\bar{\Omega})$ with $\omega_p \leq \bar{\psi} \leq M\psi$ in Ω. Next, we show that the problem

$$-\Delta_p u = \begin{cases} h(u) & \text{in } D, \\ 0 & \text{in } \Omega \backslash D, \end{cases} \qquad u = 0 \text{ on } \partial\Omega \tag{2.5}$$

has a positive solution. Let ψ_0 be the solution of

$$-\Delta_p u = \begin{cases} \lambda_1 \phi_1^{p-1} & \text{in } D, \\ 0 & \text{in } \Omega \backslash D, \end{cases} \qquad u = 0 \text{ on } \partial\Omega.$$

By the strong maximum principle [22], $\inf_\Omega \frac{\psi_0}{\phi_1} \geq m_1$ for some $m_1 \in (0,1)$.

Since $\liminf_{u\to 0^+} u^{1-p}h(u) > 0$, $\inf_{u\in(0,1]} u^{1-p}h(u) = m_0 > 0$. Hence

$$\begin{aligned} h(\varepsilon\psi_0) &\geq h(\varepsilon m_1\phi_1) \geq (\varepsilon m_1)^q h(\phi_1) \geq (\varepsilon m_1)^q m_0 \phi_1^{p-1} \\ &\geq \lambda_1(\varepsilon\phi_1)^{p-1} = -\Delta_p(\varepsilon\psi_0) \ \text{ in } D \end{aligned}$$

for ε small. Thus $\varepsilon\psi_0$ is a subsolution of (2.5). Since ω_p is a supersolution of (2.5) with $\omega_p \geq \varepsilon\psi_0$ in Ω for ε small, it follows that (2.5) has a solution ψ_1 with $\varepsilon\psi_0 \leq \psi_1 \leq \omega_p$ in Ω. Clearly ψ_1 and $\bar{\psi}$ are sub- and supersolution of (2.1) respectively with $\psi_1 \leq \omega_p \leq \bar{\psi}$ in Ω, and the existence of a solution $\phi_D \in C^1(\bar{\Omega})$ with $\inf_\Omega \frac{\phi_D}{d} > 0$ follows.

(i) Let $M > 0$ be such that

$$g(x) \leq M$$

for $x \in \Omega$. Then

$$-\Delta_p(\phi_D) \leq h(\|\phi_D\|_\infty) + M \ \text{ in } \Omega,$$

which implies by the maximum principle that

$$-\Delta_p \left(\frac{\phi_D}{(h(\|\phi_D\|_\infty) + M)^{\frac{1}{p-1}}} \right) \leq 1 \text{ in } \Omega.$$

This implies $\phi_D \in C^{1,\nu}(\bar{\Omega})$ for some $\nu \in (0,1)$ and there exists a constant $M_1 > 0$ independent of ϕ_D such that

$$|\phi_D|_{C^{1,\nu}} \leq M_1(h(\|\phi_D\|_\infty) + M)^{\frac{1}{p-1}} \leq M_1(h(|\phi_D|_{C^{1,\nu}}) + M)^{\frac{1}{p-1}}.$$

In particular,

$$\frac{h(|\phi_D|_{C^{1,\nu}}) + M}{|\phi_D|_{C^{1,\nu}}^{p-1}} \geq \frac{1}{M_1^{p-1}}.$$

Since $\lim_{t\to\infty} \frac{h(t)+M}{t^{p-1}} = 0$, there exists a constant $M_2 > 0$ independent of D such that $|\phi_D|_{C^{1,\nu}} \leq M_2$. Let (D_n) be a sequence of open sets in Ω such that $|\Omega \backslash D_n| \to 0$ as $n \to \infty$, and let $\phi_n \equiv \phi_{D_n}$. Then for $\xi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla\phi_n|^{p-2}\nabla\phi_n \cdot \nabla\xi dx = \int_{D_n} h(\phi_n)\xi dx + \int_{\Omega\backslash D_n} g\xi dx. \tag{2.6}$$

Since $|\phi_n|_{C^{1,\nu}} \leq M_2$, there exists $\omega_p \in C^1(\bar{\Omega})$ and a subsequence of (ϕ_n), which we still denote by (ϕ_n), such that $\phi_n \to \omega_p$ in $C^1(\bar{\Omega})$.

Since

$$\int_{\Omega\setminus D_n} |g\xi| dx \leq M \int_{\Omega\setminus D_n} |\xi| dx \leq M \left(\int_{\Omega} |\xi|^p dx \right)^{\frac{1}{p}} |\Omega\setminus D_n|^{\frac{p-1}{p}},$$

it follows that $\int_{\Omega\setminus D_n} |g\xi| dx \to 0$ as $n \to \infty$. Hence by letting $n \to \infty$ in (2.6), we obtain

$$\int_{\Omega} |\nabla w_p|^{p-2} \nabla w_p \cdot \nabla \xi dx = \int_{\Omega} h(w_p) \xi dx$$

for all $\xi \in W_0^{1,p}(\Omega)$, i.e. ω_p is the solution of $-\Delta_p u = h(u)$ in $\Omega, u = 0$ on $\partial\Omega$. Thus $\phi_D \to \omega_p$ in $C^1(\bar{\Omega})$ as $|\Omega\setminus D| \to 0$, i.e. (i) holds.

(ii) Note that $\beta < p-1$ for $p < 2, p \sim 2$, which we assume. Since

$$-\Delta_p \omega_p = \omega_p^{\beta} \leq \|\omega_p\|_{\infty}^{\beta} \text{ in } \Omega,$$

it follows that

$$0 \leq -\Delta_p \left(\frac{\omega_p}{\|\omega_p\|_{\infty}^{\frac{\beta}{p-1}}} \right) \leq 1 \text{ in } \Omega. \tag{2.7}$$

By the comparison principle,

$$\frac{\omega_p}{\|\omega_p\|_{\infty}^{\frac{\beta}{p-1}}} \leq \psi \quad \text{in } \Omega, \tag{2.8}$$

where ψ is defined in (2.4). Let $R > 1$ be such that $\bar{\Omega} \subset B(0, R)$, where $B(0, R)$ denotes the open ball centered at 0 with radius R in $\mathbb{R}^n$. Let w satisfy

$$-\Delta_p w = 1 \text{ in } B(0, R), \quad w = 0 \text{ on } \partial B(0, R).$$

Then $\psi \leq w_p$ in Ω by Lemma 0 in [13]. Since

$$w(x) = \frac{N^{-\frac{1}{p-1}}(p-1)}{p} (R^{\frac{p}{p-1}} - |x|^{\frac{p}{p-1}}) \quad \text{for } x \in B(0, R),$$

it follows that

$$\psi \leq R^{\frac{p}{p-1}} \leq R^3 \text{ in } \Omega \text{ for } p > 3/2, \tag{2.9}$$

i.e ψ is uniformly bounded in Ω by a constant independent of p for $p > 3/2$.

Hence, (2.8) gives

$$\|\omega_p\|_{\infty} \leq R^{\frac{3(p-1)}{p-1-\beta}} \leq R^{\frac{4}{1-\beta}}$$

for $p < 2$ sufficiently close to 2, as $\frac{3(p-1)}{p-1-\beta} \downarrow \frac{3}{1-\beta}$ as $p \uparrow 2$. Thus ω_p is uniformly bounded by a constant independent of p for $p \sim 2$, $p < 2$.

By (2.7)–(2.8) and Lieberman's regularity result [14, Theorem 1], there exist constants $\nu \in (0,1)$ and $C > 0$ independent of such p such that

$$\frac{|\omega_p|_{C^{1,\nu}}}{\|\omega_p\|_\infty^{\frac{\beta}{p-1}}} \le C,$$

which implies

$$|\omega_p|_{C^{1,\nu}} \le C\|\omega_p\|_\infty^{\frac{\beta}{p-1}} \le CR^{\frac{4\beta}{(1-\beta)(p-1)}} \le CR^{\frac{8\beta}{1-\beta}}$$

for $p > 3/2$, i.e. ω_p is bounded in $C^{1,\nu}(\bar{\Omega})$ by a constant independent of p for $p < 2$, $p \sim 2$. To show that $\omega_p \to \omega_2$ in $C^1(\bar{\Omega})$ as $p \to 2$, $p < 2$, let (p_n) be such that $p_n < 2, p_n \to 2$ as $n \to \infty$. Then for $\xi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla \omega_{p_n}|^{p_n-2}\nabla\omega_{p_n}\cdot\nabla\xi\, dx = \int_\Omega \omega_{p_n}^\beta \xi\, dx. \tag{2.10}$$

Since (ω_{p_n}) is bounded in $C^{1,\nu}(\bar{\Omega})$, it has a subsequence which we still denote by (ω_{p_n}) and a function $\phi \in C^1(\bar{\Omega})$ such that $\omega_{p_n} \to \phi$ in $C^1(\bar{\Omega})$ as $n \to \infty$.

Let $n \to \infty$ in (2.10), we obtain

$$\int_\Omega \nabla\phi\cdot\nabla\xi\, dx = \int_\Omega \phi^\beta\xi\, dx \text{ for all } \xi \in W_0^{1,p}(\Omega),$$

i.e. $\phi = \omega_2$ in Ω. Hence $\omega_p \to \omega_2$ in $C^1(\bar{\Omega})$ as $p \to 2$, $p < 2$, which completes the proof. □

Next, we establish a comparison principle.

Lemma 2.2. *Let h, g and D be as in Lemma 2.1. Let $u, v \in C^1(\bar{\Omega})$ satisfy $\inf_\Omega \frac{u}{d} > 0$ and*

$$\begin{gathered} -\Delta_p u \ge \begin{cases} h(u) & \text{in } D, \\ g(x) & \text{in } \Omega\backslash D \end{cases}, \quad u \ge 0 \text{ on } \partial\Omega \\ \left(\text{resp.} -\Delta_p u \le \begin{cases} h(u) & \text{in } D, \\ g(x) & \text{in } \Omega\backslash D \end{cases}, \quad u \le 0 \text{ on } \partial\Omega\right), \end{gathered} \tag{2.11}$$

$$-\Delta_p v = \begin{cases} h(v) & \text{in } D, \\ g(x) & \text{in } \Omega\backslash D \end{cases}, \quad v = 0 \text{ on } \partial\Omega.$$

Then $u \ge v$ in Ω (resp. $u \le v$ on $\partial\Omega$).

Proof. Since $\inf_\Omega \frac{u}{d} > 0$ and $v \in C^1(\bar{\Omega})$, $\inf_\Omega \frac{u}{v} > 0$. Let c be the largest number such that $u \ge cv$ in Ω and suppose $c < 1$. Then

$$-\Delta_p u \ge h(u) \ge h(cv) \ge c^q h(v) \text{ in } D,$$

which implies

$$-\Delta_p\left(\frac{u}{c^{\frac{q}{p-1}}}\right) \geq \begin{cases} h(v) & \text{in } D, \\ g(x) & \text{in } \Omega\backslash D. \end{cases}$$

By the weak comparison principle [21, Lemma A.2], $u \geq c^{\frac{q}{p-1}} v$ in Ω. This implies $c \geq c^{\frac{q}{p-1}}$ and so $c \geq 1$, a contradiction. Thus $u \geq v$ in Ω.

Next suppose the inequality $\leq$ in (2.11) holds. Let C be the smallest positive number such that $u \leq Cv$ in Ω and suppose $C > 1$. Then

$$-\Delta_p u \leq h(u) \leq h(Cv) \leq C^q h(v) \text{ in } D,$$

which implies

$$-\Delta_p\left(\frac{u}{C^{\frac{q}{p-1}}}\right) \leq \begin{cases} h(v) & \text{in } D, \\ g(x) & \text{in } \Omega\backslash D. \end{cases}$$

Hence $u \leq C^{\frac{q}{p-1}} v$ in Ω. This implies $C \leq C^{\frac{q}{p-1}}$ and so $C \leq 1$, a contradiction. Thus $u \leq v$ in Ω, which completes the proof. □

Lemma 2.3. *Let* (A_1)–(A_4) *hold,* $\beta < p-1$, *and* u_λ *be a positive solution of* (1.1). *Then*

$$\lim_{\lambda\to\infty} \frac{u_\lambda(x)}{\lambda^{\frac{1}{p-1-\beta}}\omega_p(x)} = 1 \tag{2.12}$$

uniformly for $x \in \Omega$, *where we recall that* $\omega_p \in C^1(\bar{\Omega})$ *is the unique solution of*

$$-\Delta_p u = u^\beta \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Proof. By Lemma 3.1 in [15],

$$u_\lambda \geq \mu\phi_1 \text{ in } \Omega$$

for $\lambda > \lambda_1/k$, where $k, \mu > 0$ are such that $f(z) > kz^{p-1}$ for $z \in (0, \mu]$.

Let K be a compact subset of Ω and $c = \min_K f(\mu\phi_1) > 0$. Then

$$-\Delta_p u_\lambda \geq \lambda c \chi_K \text{ in } \Omega,$$

where χ_K denotes the characteristic function on K. This implies

$$u_\lambda \geq (\lambda c)^{\frac{1}{p-1}} z \geq \lambda^{\frac{1}{p-1}} c_1 d \text{ in } \Omega, \tag{2.13}$$

where z is the positive solution of $-\Delta_p u = \chi_K$ in $\Omega, u = 0$ on $\partial\Omega$, and $c_1 = c^{\frac{1}{p-1}} \inf_\Omega \frac{z}{d} > 0$.

Let $\varepsilon \in (0,1)$. Then there exists a constant $A > 0$ such that

$$(1-\varepsilon)z^\beta \leq f(z) \leq (1+\varepsilon)z^\beta \text{ for } z > A \tag{2.14}$$

in view of (A_2). The left side inequality in (2.14) implies that

$$-\Delta_p u_\lambda \geq \lambda \begin{cases} (1-\varepsilon)u_\lambda^\beta, & u_\lambda > A, \\ 0, & u_\lambda < A. \end{cases}$$

Define $\tilde{u}_\lambda = \lambda^{-\frac{1}{p-1-\beta}} u_\lambda$. Then

$$-\Delta_p \tilde{u}_\lambda \geq \begin{cases} (1-\varepsilon)\tilde{u}_\lambda^\beta, & u_\lambda > A, \\ 0, & u_\lambda < A. \end{cases}$$

By Lemma 2.2 with $h(u) = (1-\varepsilon)u^\beta$, $g(x) = 0$, it follows that $\tilde{u}_\lambda \geq \check{u}_\lambda$ in Ω, where $\check{u}_\lambda$ satisfies

$$-\Delta_p \check{u}_\lambda = \begin{cases} (1-\varepsilon)\check{u}_\lambda^\beta, & u_\lambda > A, \\ 0, & u_\lambda < A. \end{cases}$$

Note that $\check{u}_\lambda = (1-\varepsilon)^{\frac{1}{p-1-\beta}} w_\lambda$, where w_λ satisfies

$$-\Delta_p w_\lambda = \begin{cases} w_\lambda^\beta, & u_\lambda > A, \\ 0, & u_\lambda < A. \end{cases}$$

By (2.13),

$$\{x : u_\lambda(x) < A\} \subset \left\{ x \in \Omega : d(x) < Ac_1 \lambda^{-\frac{1}{p-1}} \right\},$$

from which it follows that $|\{x : u_\lambda(x) < A\}| \to 0$ as $\lambda \to \infty$. Hence Lemma 2.1 gives $w_\lambda \to \omega_p$ in $C^1(\bar{\Omega})$, which implies $w_\lambda \geq (1-\varepsilon)\omega_p$ in Ω for λ large. Consequently,

$$u_\lambda = \lambda^{\frac{1}{p-1-\beta}} \tilde{u}_\lambda \geq \lambda^{\frac{1}{p-1-\beta}} \check{u}_\lambda \geq \lambda^{\frac{1}{p-1-\beta}} (1-\varepsilon)^{\frac{p-\beta}{p-1-\beta}} \omega_p \quad \text{in } \Omega. \tag{2.15}$$

for λ large. By choosing ε small, we obtain $u_\lambda \geq \omega_p/2$ in Ω for λ large, which we assume. Next, the right side inequality in (2.14) implies

$$-\Delta_p u_\lambda \leq \lambda \begin{cases} (1+\varepsilon)u_\lambda^\beta, & u_\lambda > A, \\ c_2, & u_\lambda < A, \end{cases}$$

where $c_2 = \sup_{z \in [0,A]} f(z)$. Hence

$$-\Delta_p \tilde{u}_\lambda \leq \begin{cases} (1+\varepsilon)\tilde{u}_\lambda^\beta, & u_\lambda > A, \\ c_2, & u_\lambda < A. \end{cases}$$

By Lemma 2.2, $\tilde{u}_\lambda \leq \hat{u}_\lambda$ in Ω, where $\hat{u}_\lambda$ satisfies

$$-\Delta_p \hat{u}_\lambda = \begin{cases} (1+\varepsilon)\hat{u}_\lambda^\beta, & u_\lambda > A, \\ c_2, & u_\lambda < A. \end{cases}$$

Note that $\hat{u}_\lambda = (1+\varepsilon)^{\frac{1}{p-1-\beta}} w_\lambda$.Since $w_\lambda \to \omega_p$ in $C^1(\bar{\Omega}), w_\lambda \leq (1+\varepsilon)\omega_p$ in Ω for λ large. Consequently,

$$u_\lambda = \lambda^{\frac{1}{p-1-\beta}} \tilde{u}_\lambda \leq \lambda^{\frac{1}{p-1-\beta}} \hat{u}_\lambda \leq \lambda^{\frac{1}{p-1-\beta}} (1+\varepsilon)^{\frac{p-\beta}{p-1-\beta}} \omega_p \quad \text{in } \Omega. \tag{2.16}$$

Combining (2.15) and (2.16), we deduce that

$$(1-\varepsilon)^{\frac{p-\beta}{p-1-\beta}} \leq \frac{u_\lambda}{\lambda^{\frac{1}{p-1-\beta}} \omega_p} \leq (1+\varepsilon)^{\frac{p-\beta}{p-1-\beta}} \text{ in } \Omega$$

for λ large, i.e. (2.12) holds, which completes the proof. □

Lemma 2.4. *Let* (A_1)–(A_4) *hold and* u_λ *be a positive solution of* (1.1) *with* $1 < p < 2$. *Then if* p *is sufficiently close to 2, there exists a constant* $M > 0$ *independent of* p *such that*

$$|u_\lambda|_{C^1} \le M\lambda^{\frac{1}{p-1-\beta}}$$

for λ *large.*

Proof. Let $\kappa > 1$ and $\beta_0 \in (\beta, 1)$. Then $\beta_0 < p - 1$ if p is sufficiently close to 2. Since $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$ in view of Lemma 2.3, it follows from (A_2) that

$$f(u) \le \kappa \|u\|_\infty^\beta$$

for λ large. Hence

$$-\Delta_p u \le \lambda\kappa\|u\|_\infty^\beta \text{ in } \Omega,$$

i.e.

$$-\Delta_p \left(\frac{u}{(\lambda\kappa)^{\frac{1}{p-1}} \|u\|_\infty^{\frac{\beta}{p-1}}} \right) \le 1,$$

from which it follows that

$$\frac{u}{(\lambda\kappa)^{\frac{1}{p-1}} \|u\|_\infty^{\frac{\beta}{p-1}}} \le \psi \text{ in } \Omega,$$

where ψ is defined in (2.4). Recall that $\|\psi\|_\infty$ is bounded independent of p for $p > 3/2$ in view of (2.9). Hence by [14, Theorem 1],

$$\frac{|u|_{C^1}}{(\lambda\kappa)^{\frac{1}{p-1}} \|u\|_\infty^{\frac{\beta}{p-1}}} \le K,$$

where $K > 1$ is a constant independent of λ, p. This implies $|u|_{C^1}^{1-\frac{\beta}{p-1}} \le K(\lambda\kappa)^{\frac{1}{p-1}}$, i.e.

$$|u|_{C^1} \le K^{\frac{p-1}{p-1-\beta}} (\lambda\kappa)^{\frac{1}{p-1-\beta}} \le K^{\frac{\beta_0}{\beta_0-\beta}} \kappa^{\frac{1}{\beta_0-\beta}} \lambda^{\frac{1}{p-1-\beta}} \equiv M\lambda^{\frac{1}{p-1-\beta}},$$

which completes the proof. □

3. PROOF OF THEOREM 1.1.

Proof. The existence of a positive solution to (1.1) for λ large follows from the method of sup- and supersolutions. Indeed, it is easy to see that for λ large enough, $\varepsilon\phi_1$ is a subsolution of (1.1) for ε small while $M\phi$ is a supersolution of (1.1) for M large, where ϕ satisfies $-\Delta_p\phi = 1$ in $\Omega, \phi = 0$ on $\partial\Omega$.

Let u, v be positive solutions of (1.1) for λ large and let $w = u - v$.

By (A_3), there exists a constant $\delta \in (0, 1)$ such that

$$\limsup_{\xi\to\infty} \frac{\xi f'(\xi)}{f(\xi)} < \delta. \tag{3.1}$$

Let $\delta_0, \delta_1 \in (0,1)$ be such that $\delta\delta_0^{2(\beta-1)} < \delta_1$. By making p close enough to 2, we can assume that

$$\omega_p \geq \delta_0 \omega_2 \ \text{ in } \Omega \tag{3.2}$$

(in view of Lemma 2.1(ii)), and $\delta_1 < p-1, (2M)^{2-p}\delta\delta_0^{2(\beta-1)} < \delta_1$, where M is defined in Lemma 2.4.

By (3.1) and (A_2), there exists a constant $A > 0$ such that

$$f'(\xi) \leq \frac{\delta}{\xi^{1-\beta}}. \tag{3.3}$$

for $\xi > A$. Multiplying the equation

$$-\Delta_p u - (-\Delta_p v) = \lambda \left(f(u) - f(v) \right) \ \text{ in } \Omega$$

by w and integrating, we obtain

$$\begin{aligned} \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \right) \cdot (\nabla u - \nabla v) dx &= \lambda \int_\Omega \left(f(u) - f(v) \right) w dx \\ &= \lambda \int_\Omega w^2 f'(\xi) dx, \end{aligned} \tag{3.4}$$

where ξ is between $u(x)$ and $v(x)$. Using the inequality

$$(|x| + |y|)^{2-p}(|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y) \geq (p-1)|x-y|^2$$

for $1 < p \leq 2$ and $x, y \in \mathbb{R}^n$ (see [17, Lemma 30.1]) with $x = \nabla u$ and $y = \nabla v$ in (3.4), we obtain from Lemma 2.4 that

$$(p-1) \int_\Omega |\nabla w|^2 dx \leq \lambda^{\frac{1-\beta}{p-1-\beta}} (2M)^{2-p} \int_\Omega w^2 f'(\xi) dx. \tag{3.5}$$

By Lemma 2.3,

$$u, v \geq \delta_0 \lambda^{\frac{1}{p-1-\beta}} \omega_p \text{ in } \Omega \tag{3.6}$$

for λ large. This, together with (3.2) and (3.3), implies

$$\begin{aligned} \int_{\xi > A} w^2 f'(\xi) dx &\leq \delta \int_{\xi > A} \frac{w^2}{\xi^{1-\beta}} dx \leq \frac{\delta}{\delta_0^{1-\beta} \lambda^{\frac{1-\beta}{p-1-\beta}}} \int_{\xi > A} \frac{w^2}{\omega_p^{1-\beta}} dx \\ &\leq \delta\delta_0^{2(\beta-1)} \lambda^{\frac{\beta-1}{p-1-\beta}} \int_\Omega \frac{w^2}{\omega_2^{1-\beta}} dx \leq \delta\delta_0^{2(\beta-1)} \lambda^{\frac{\beta-1}{p-1-\beta}} \int_\Omega |\nabla w|^2 dx, \end{aligned} \tag{3.7}$$

where we have used the inequality $\int_\Omega w^2 \omega_2^{\beta-1} dx \leq \int_\Omega |\nabla w|^2 dx$ in [15, Lemma 3.5]. Thus

$$\lambda^{\frac{1-\beta}{p-1-\beta}} (2M)^{2-p} \int_{\xi > A} w^2 |f'(\xi)| dx \leq (2M)^{2-p} \delta\delta_0^{2(\beta-1)} \int_\Omega |\nabla w|^2 dx \leq \delta_1 \int_\Omega |\nabla w|^2 dx. \tag{3.8}$$

By (A_5), there exists a constant $C > 0$ such that

$$|f'(\xi)| \le \frac{C}{\xi^{1+\alpha}} \text{ for } \xi \in (0, A]. \tag{3.9}$$

By Hardy's inequality [2, p. 194], there exists a constant $m > 0$ such that

$$\int\limits_{\Omega} \left|\frac{z}{d}\right|^2 dx \le m \int\limits_{\Omega} |\nabla z|^2 dx,$$

for all $z \in H_0^1(\Omega)$, where $d(x)$ denotes the distance function.

This, together with (3.2), (3.6), and (3.9), implies

$$\begin{aligned}\int\limits_{\xi<A} w^2|f'(\xi)|dx &\le C \int\limits_{\xi<A} \frac{w^2}{\xi^{1+\alpha}}dx \le \frac{C}{\delta_0^{2(1+\alpha)}\lambda^{\frac{1+\alpha}{p-1-\beta}}} \int\limits_{\xi<A} \frac{w^2}{\omega_2^{1+\alpha}}dx \\ &\le \frac{C\lambda^{-\frac{1+\alpha}{p-1-\beta}}}{\delta_0^{2(1+\alpha)}c_0^{1+\alpha}} \int\limits_{\xi<A} \frac{w^2}{d^{1+\alpha}}dx \le C_0\lambda^{-\frac{1+\alpha}{p-1-\beta}} \int\limits_{\Omega} \left|\frac{w}{d}\right|^2 dx \\ &\le C_1\lambda^{-\frac{1+\alpha}{p-1-\beta}} \int\limits_{\Omega} |\nabla w|^2 dx,\end{aligned}$$

where

$$c_0 = \inf_{\Omega} \frac{\omega_2}{d} > 0, \quad C_0 = \frac{C\|d\|_\infty^{1-\alpha}}{\delta_0^{2(1+\alpha)}c_0^{1+\alpha}}, \quad \text{and} \quad C_1 = C_0 m.$$

Consequently,

$$\lambda^{\frac{1-\beta}{p-1-\beta}}(2M)^{2-p} \int\limits_{\xi<A} w^2|f'(\xi)|dx \le C_1(2M)^{2-p}\lambda^{-\frac{\alpha+\beta}{p-1-\beta}} \int\limits_{\Omega} |\nabla w|^2 dx. \tag{3.10}$$

Combining (3.5), (3.8) and (3.10), we obtain

$$(p-1)\int\limits_{\Omega} |\nabla w|^2 dx \le (\delta_1 + C_1\left((2M)^{2-p}\lambda^{-\frac{\alpha+\beta}{p-1-\beta}}\right) \int\limits_{\Omega} |\nabla w|^2 dx,$$

which implies $\int_\Omega |\nabla w|^2 dx = 0$, i.e. $w = 0$ on Ω, provided that λ is large enough so that

$$\delta_1 + C_1\left((2M)^{2-p}\lambda^{-\frac{\alpha+\beta}{p-1-\beta}}\right) < p-1.$$

This completes the proof of Theorem 1.1. □

REFERENCES

[1] A. Alsaedi, V.D. Radulescu, B. Ahmad, *Bifurcation analysis for degenerate problems with mixed regime and absorption*, Bull. Math. Sci. **11** (2021), Paper no. 2050017.

[2] H. Brezis, *Analyse fonctionnelle, théorie et applications*, 2nd ed., Masson, Paris, 1983 [in French].

[3] H. Brezis, L. Oswald, *Remark on sublinear elliptic equations*, Nonlinear Anal. **10** (1986), no. 1, 55–64.

[4] S. Chen, C.A. Santos, M. Yang, J. Zhou, *Bifurcation analysis for a modified quasilinear equation with negative exponent*, Adv. Nonlinear Anal. **11** (2022), no. 1, 684–701.

[5] K.D. Chu, D.D. Hai, R. Shivaji, *Uniqueness for a class of singular quasilinear Dirichlet problem*, Appl. Math. Lett. **106** (2020), 106306.

[6] P.T. Cong, D.D. Hai, R. Shivaji, *A uniqueness result for a class of singular p-Laplacian Dirichlet problem with non-monotone forcing term*, Proc. Amer. Math. Soc. **150** (2021), 633–637.

[7] E.N. Dancer, *Uniqueness for elliptic equations when a parameter is large*, Nonlinear Anal. **8** (1984), 835–836.

[8] E.N. Dancer, *On the number of positive solutions of semilinear elliptic systems*, Proc. London Math. Soc. **53** (1986), 429–452.

[9] J.I. Díaz, J.E. Saa, *Existence et unicité de solutions positives pur certaines équations elliptiques quasilinéaires*, C.R. Acad. Sci. Paris **305** (1987), 521–524.

[10] P. Drábek, J. Hernandez, *Existence and uniqueness of positive solutions for some quasilinear elliptic problems*, Nonlinear Anal. **44** (2001), 189–204.

[11] Z. Guo, J.R.L. Webb, *Uniqueness of positive solutions for quasilinear elliptic equations when a parameter is large*, Proc. Roy. Soc. Edinburgh **124** (1994), 189–198.

[12] D.D. Hai, *Uniqueness of positive solutions for a class of quasilinear problems*, Nonlinear Anal. **69** (2008), 2720–2732.

[13] B. Kawohl, *On a family of torsional creep problems*, J. Reine Angew. Math. **410** (1990), 1–22.

[14] G.M. Lieberman, *Boundary regularity for solutions of degenerate quasilinear elliptic equations*, Nonlinear Anal. **12** (1988), 1203–1219.

[15] S.S. Lin, *On the number of positive solutions for nonlinear elliptic equations when a parameter is large*, Nonlinear Anal. **16** (1991), 283–297.

[16] S.S. Lin, *Some uniqueness results for positone problems when a parameter is large*, Chinese J. Math. **13** (1985), 67–81.

[17] T. Oden, *Qualitative Methods in Nonlinear Mechanics*, Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1986.

[18] N.S. Papageorgiou, *Double phase problems: a survey of some recent results*, Opuscula Math. **42** (2022), no. 2, 257–278.

[19] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Nonlinear Analysis – Theory and Methods*, Springer Monographs in Mathematics, Cham, 2019.

[20] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Positive solutions for nonlinear Neumann problems with singular terms and convection*, J. Math. Pures Appl. **136** (2020), 1–21.

[21] S. Sakaguchi, *Concavity properties of solutions of some degenerate quasilinear elliptic Dirichlet problems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **14** (1987), 403–421.

[22] J.L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191–202.

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Received: February 22, 2023.
Revised: August 23, 2023.
Accepted: August 27, 2023.