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ANALYSIS OF A MULTIPHASE FREE BOUNDARY PROBLEM

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Abstract. In this paper, we investigate a free boundary problem relevant in several applications, such as tumor growth models. Our problem is expressed as an elliptic equation involving discontinuous nonlinearities in a specified domain with a moving boundary. We establish the existence and uniqueness of solutions and provide a qualitative analysis of the free boundaries generated by the nonlinear term (inner boundaries). Furthermore, we analyze the dynamics of the outer region boundary. The final result demonstrates that under certain conditions, our problem is solvable in the neighborhood of a radial solution.

Keywords: discontinuous nonlinearity, free boundary, perturbation, tumor growth.

Mathematics Subject Classification: 34R35, 35J25, 92B05, 35R35.

1. INTRODUCTION

In various contexts, free boundaries often take the form of curves or surfaces whose exact positions are initially undetermined, serving as interfaces between regions with differing properties. An example is found in tumor growth models, where the boundary of the tumor region (separating it from healthy tissue) is not given a priori and changes during the process. This lack of knowledge requires a major study of the free boundary.

Over the last decades, several mathematical models have been proposed to describe the growth of tumor spheroids. The starting point in the study of free boundary problems arising in tumor growth models is due to the pioneering papers of Greenspan [20, 21]. Later, with a rigorous analysis, Byrne and Chaplain [8, 9] extended similar results to the case of discontinuous nonlinearity and recently, Friedman and collaborators in a series of papers [12, 15, 16, 18] develop mathematical techniques to analyze the existence, uniqueness and bifurcation of solutions. For the historical development of mathematical cancer modeling, we refer the reader to [2].

Inspired by the previous relevant works, the authors investigate in [1], for $\varepsilon > 0$ the following problem:

$$\begin{cases} \Delta u = \lambda(\varepsilon + (1-\varepsilon)H(u-\mu)) & \text{in } \Omega(t), \\ u = \overline{u}_\infty & \text{on } \partial\Omega(t), \end{cases} \tag{1.1}$$

where μ is the critical value for which the tumor $\Omega(t)$ developed a region of slow growth in its center due to lack of nutrients. Note that the problem (1.1) appears in other contexts like Budyko climate models. We refer the reader to [4] and [5] for more details.

Spherical coordinates, which are well suited to represent the spherical configuration of tumors are useful in analyzing our problem. So, let $u := u(r,t)$ be the concentration nutrient in $\Omega(t)$ where $r = |x|$. The dynamic of the outer tumor radius denoted by $R(t)$ is governed by

$$\begin{aligned}\frac{d}{dt}\Big(\frac{4\pi R^3(t)}{3}\Big) &= \int\int\int_{\Omega(t)} S(u) r^2 \sin\theta \; d\theta \; d\phi \; dr \\ &\quad - \int\int\int_{\Omega(t)} N(u) r^2 \sin\theta \; d\theta \; d\phi \; dr,\end{aligned} \tag{1.2}$$

where $S(u)$ and $N(u)$ are the proliferation and the mortality rate functions.

The purpose of this work is to investigate the following problem:

$$\begin{cases} \Delta u = \lambda\Big(\varepsilon H(u-\mu_1) + (1-\varepsilon) H(u-\mu_2)\Big) & \text{in } \Omega(t), \\ u = \overline{u}_\infty & \text{on } \partial\Omega(t), \end{cases} \tag{1.3}$$

where $\Omega(t) \subset \mathbb{R}^3$ is the domain at time $t > 0$ with a moving boundary $\partial\Omega(t)$, $\varepsilon \in (0,1)$, $\overline{u}_\infty$, λ, μ_1, μ_2 are positive parameters such that $\mu_1 < \mu_2 < \overline{u}_\infty$ and H is the Heaviside function, i.e.

$$H(t) = \begin{cases} 1, & \text{for } t \geq 0, \\ 0, & \text{for } t < 0. \end{cases}$$

So, the present paper can be considered as a natural continuation of the previous paper by the authors [1] in which the same program of research was devoted to the one discontinuous nonlinearity. Here, we consider the sum of discontinuous nonlinearities, and as we shall see, our discontinuities generate free boundaries which are not given explicitly like [1] and this requires more attention. The challenge is to develop mathematical concepts that can handle the complexity of the problem.

In this work, we focus the study on the following multiphase free boundary problem

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2 \frac{\partial u}{\partial r}\big) = \lambda\Big(\varepsilon H(u-\mu_1) + (1-\varepsilon)H(u-\mu_2)\Big), & 0 < r < R(t),\ t > 0, \\ u(R(t),t) = u_\infty,\ \frac{\partial u}{\partial r}(0,t) = 0 & \text{for } t > 0, \\ R^2(t)\frac{dR(t)}{dt} = \int_0^{R(t)} S(u) r^2 dr - \int_0^{R(t)} N(u) r^2 dr, & \\ R(0) = R_0, & \end{cases} \tag{1.4}$$

where μ_1, μ_2 are the critical values for which tumor goes from one phase to another verifying $\mu_1 < \mu_2 < u_\infty$ and

$$S(u) = \lambda(\varepsilon H(u-\mu_1) + (1-\varepsilon)H(u-\mu_2)), \quad N(u) = \eta_1 H(u-\mu_1) + \eta_2 H(\mu_1 - u),$$

where η_1, η_2 are positive constants.

Let us introduce the notations:

$$\omega_{\mu_i}(t) = \{u(r,t) < \mu_i\}, \quad \omega_{\mu_i} = \{u(r) < \mu_i\}$$

and

$$\Gamma_{\mu_i}(t) = \partial\omega_{\mu_i}(t), \quad \Gamma_{\mu_i} = \partial\omega_{\mu_i},$$

where $\mu_i \in (0, \overline{u}_\infty)$ for $i = 1, 2$.

Hence, the discontinuous nonlinearities generate two free boundaries namely $\Gamma_{\mu_1}(t)$ and $\Gamma_{\mu_2}(t)$ in addition to the outer boundary $\partial\Omega(t)$ (moving boundary). This is the first time that the problem (1.4) is considered. One of the main difficulties in problem (1.4) is that the free boundaries are not explicitly given (contrary to the paper [1]).

First, for the stationary solution and regarding the position of u with respect to μ_1 and μ_2 dealing with the three phases, we give the existence of a stationary solution for problem (1.4). Moreover, we show the existence of their corresponding free boundaries given by

$$u(r_i) = \mu_i, \quad i = 1, 2.$$

We further obtain the long time behavior of all transient solutions with the form $\big(u(r,t), R(t)\big)$. Finally, we will assume that $\Omega(t)$ is a perturbation of the ball $B(0, R(t))$. More precisely, we assume what follows:

$$\begin{aligned}&\text{The boundary } \partial\Omega(t) \text{ can be parameterized as } R(t) + \beta(\theta) \text{ for } t > 0,\\ &\text{where } \beta \in C^2(S),\ \theta \in S \text{ and } S \text{ is the unit sphere.}\end{aligned} \tag{C}$$

Using this parametrization of our free boundaries which are the unknowns of our problem, we reduce the study to the solvability of a nonlinear integral equation and show the existence of functions namely b_i such that

$$u(r_i(t) + b_i(\theta), \theta) = \mu_i, \quad i = 1, 2,\ t > 0.$$

Moreover, this perturbation method gives the existence of exceptional free boundaries in the case of a stationary solution with the form

$$u(r_i(t) + h(\theta), \theta) = \mu_i, \quad i = 1, 2,$$

where $h \in C(S)$. This result is far from obvious.

The structure of the rest of this paper is arranged as follows. In Section 2, we study the stationary solutions when $\Omega = B(0, R(t))$ with different phases. Section 3 is devoted to establishing the existence of a transient solution and studying its asymptotic behavior. In Section 4, using perturbation, we reformulate our free boundary problem and prove the existence of free boundaries through the use of local methods. Finally, some comments are given.

2. STATIONARY SOLUTIONS

This section deals with the existence of stationary solutions for the problem (1.4). Hence, we consider the following problem:

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \lambda\Big(\varepsilon H(u-\mu_1) + (1-\varepsilon)H(u-\mu_2)\Big), & 0 < r < R, \\ u(R) = u_\infty,\ \frac{\partial u}{\partial r}(0) = 0, \end{cases} \tag{2.1}$$

and the integral equation

$$\frac{1}{R^2}\left(\int_0^R S(u)r^2 dr - \int_0^R N(u)r^2 dr\right) = 0. \tag{2.2}$$

We recall that the nonnegative solution of (2.1) is strictly convex function such that

$$\min_{0<r<R} u(r) = u(0).$$

The sets Γ_{μ_1}, Γ_{μ_2} are the free boundaries corresponding to (2.1). So, there are three cases to be considered: $u(0) > \mu_2$, $\mu_1 < u(0) \le \mu_2$ and $u(0) \le \mu_1$.

We denote by u_λ^* the solution of (2.1) in the case $u(0) > \mu_2$ (without a free boundary), by u_{λ,μ_2} the solution in the case $\mu_1 < u(0) \le \mu_2$ (with one free boundary) and by u_{λ,μ_1,μ_2} the solution in the case $u(0) \le \mu_1$ (with two free boundaries).

Theorem 2.1. *For $\varepsilon \in (0,1)$, we have the following result:*

(i) *If $\lambda < \lambda_1 := \frac{6(u_\infty - \mu_2)}{R^2}$, then there exists a unique solution u_λ^* of (2.1). Moreover,*

$$u_\lambda^*(0) = \frac{-\lambda}{6}R^2 + u_\infty > \mu_2,$$

i.e. the line $(\lambda, \gamma^*(\lambda))$, where $u_\lambda^*(0) := \gamma^*(\lambda) := \frac{-\lambda}{6}R^2 + u_\infty$ defines a decreasing part of the bifurcation diagram (see Figure 1).

(ii) *There exists $\lambda^* > 0$ such that if $\lambda \in [\lambda_1, \lambda^*)$, then there exists a unique positive solution u_{λ,μ_2} of (2.1) giving rise to a free boundary given by*

$$r_\lambda = \frac{(\varepsilon - \frac{3}{2})}{3(\varepsilon-1)}R\Big[1 + 2\cos\Big(\frac{1}{3}\arccos\Big(1 + \frac{27(\varepsilon-1)^2\big(\frac{1}{2}R^2 - \frac{3(u_\infty-\mu_2)}{\lambda}\big)}{2(\varepsilon-\frac{3}{2})^3R^2}\Big) + \frac{4\pi}{3}\Big)\Big].$$

(iii) *If $\lambda \in [\lambda^*, +\infty)$, the problem (2.1) has a unique solution u_{λ,μ_1,μ_2} with two free boundaries namely r_{λ,μ_1} and r_{λ,μ_2} satisfying $0 < r_{\lambda,\mu_1} < r_{\lambda,\mu_2} < R$.*

Proof. First, to prove (i), we consider the case $u(0) > \mu_2$ (without a free boundary). We have the corresponding problem

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \lambda \quad \text{in } (0,R), \\ u(R) = u_\infty,\ u'(0) = 0. \end{cases}$$

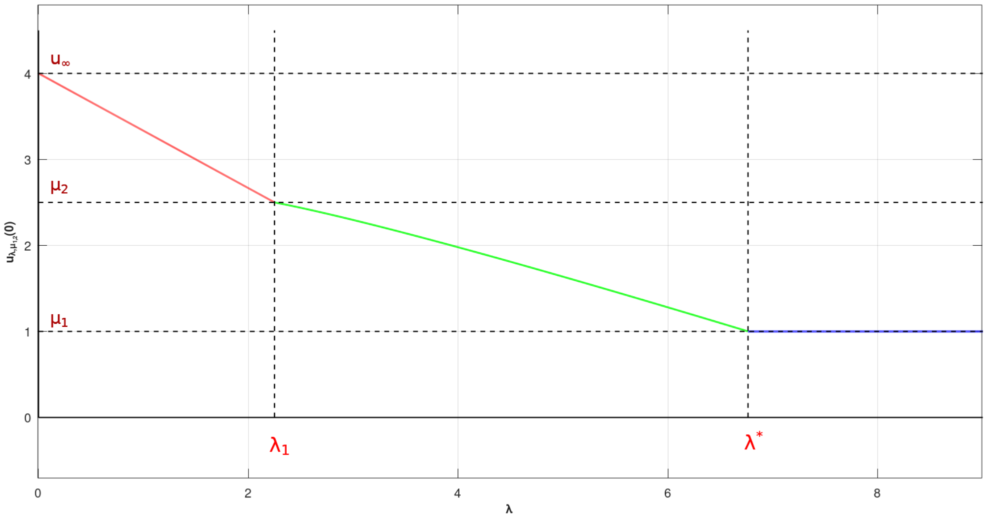

Fig. 1. A qualitative description of the bifurcation curve

So,

$$u_\lambda^*(r) = \frac{\lambda}{6}\big(r^2 - R^2\big) + u_\infty, \quad r \in [0, R].$$

Since

$$u_\lambda^*(0) = \frac{-\lambda}{6}R^2 + u_\infty > \mu_2 \Longleftrightarrow \lambda < \frac{6(u_\infty - \mu_2)}{R^2} := \lambda_1.$$

Secondly, to prove (ii), we assume that $\mu_1 < u(0) \leq \mu_2$. In this case, a solution take the value μ_2 at only one value of r, say r_λ, i.e. $u(r_\lambda) = \mu_2$. Hence, we have the following problem:

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial u}{\partial r}\big) = \lambda\Big(\varepsilon + (1-\varepsilon)H(u-\mu_2)\Big) & \text{in } (0, R), \\ u(R) = u_\infty,\ u'(0) = 0. \end{cases} \tag{2.3}$$

In [1], we have studied the solutions of problem (2.3) for all $\varepsilon > 0$. Here, we reduce the study for $\varepsilon \in (0, 1)$. More precisely, we have the following problem in the different regions $(0, r_\lambda)$ and (r_λ, R).

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial u}{\partial r}\big) = \lambda\varepsilon & \text{in } (0, r_\lambda), \\ \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial u}{\partial r}\big) = \lambda & \text{in } (r_\lambda, R), \\ u(r_\lambda) = \mu_2,\ u'(0) = 0,\ u(R) = u_\infty. \end{cases}$$

Thus, the following solution is obtained:

$$u_{\lambda,\mu_2}(r) = \begin{cases} \frac{\lambda\varepsilon}{6}\big(r^2 - r_\lambda^2\big) + \mu_2, & 0 \leq r \leq r_\lambda, \\ u_\infty + \frac{\lambda}{6}\big(r^2 - R^2\big) + \frac{(r-R)r_\lambda}{(r_\lambda - R)r}\big(\mu_2 - u_\infty - \frac{\lambda}{6}\big(r_\lambda^2 - R^2\big)\big), & r_\lambda \leq r \leq R. \end{cases}$$

Using the transmission condition

$$\frac{\partial u_{\lambda,\mu_2}}{\partial r}\big(r_\lambda^-\big) = \frac{\partial u_{\lambda,\mu_2}}{\partial r}\big(r_\lambda^+\big),$$

where $\frac{\partial u_{\lambda,\mu_2}}{\partial r}\big(r_\lambda^+\big)$, $\frac{\partial u_{\lambda,\mu_2}}{\partial r}\big(r_\lambda^-\big)$ are the right and the left derivatives at the value $r = r_\lambda$, respectively. Thus,

$$\lambda = \frac{3(u_\infty - \mu_2)}{(\varepsilon - 1)r_\lambda^2\frac{(R-r_\lambda)}{R} + \frac{1}{2}(R^2 - r_\lambda^2)} := g(r_\lambda).$$

Then, for $\lambda > g(0) := \lambda_1$ and $\varepsilon \in (0,1)$ we obtain the explicit form of r_λ:

$$r_\lambda = \frac{(\varepsilon - \frac{3}{2})}{3(\varepsilon - 1)}R\Big[1 + 2\cos\Big(\frac{1}{3}\arccos\Big(1 + \frac{27(\varepsilon-1)^2\big(\frac{1}{2}R^2 - \frac{3(u_\infty - \mu_2)}{\lambda}\big)}{2(\varepsilon - \frac{3}{2})^3R^2}\Big) + \frac{4\pi}{3}\Big)\Big].$$

A complete analysis can be found in [1].

Now, to prove (iii) we shall assume that $u(0) \le \mu_1$. As in the case (ii), we look for the free boundaries $\{(r_{\lambda,\mu_i},\theta) : \theta \in S\}$, $i = 1,2$, with $u(r_{\lambda,\mu_i}) = \mu_i$. So, we consider the corresponding problems verified by u in the different regions $(0, r_{\lambda,\mu_1})$, $(r_{\lambda,\mu_1}, r_{\lambda,\mu_2})$ and (r_{λ,μ_2}, R). On $(0, r_{\lambda,\mu_1})$, we have the following problem:

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial u}{\partial r}\big) = 0 \quad \text{in } (0, r_{\lambda,\mu_1}), \\ u(r_{\lambda,\mu_1}) = \mu_1,\ u'(0) = 0. \end{cases}$$

Then,

$$u_{\lambda,\mu_1,\mu_2}(r) = \mu_1, \quad 0 \le r \le r_{\lambda,\mu_1}.$$

On $(r_{\lambda,\mu_1}, r_{\lambda,\mu_2})$, we get the problem

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial u}{\partial r}\big) = \lambda\varepsilon \quad \text{in } (r_{\lambda,\mu_1}, r_{\lambda,\mu_2}), \\ u(r_{\lambda,\mu_2}) = \mu_2,\ \frac{\partial u}{\partial r}\big(r_{\lambda,\mu_1}^-\big) = \frac{\partial u}{\partial r}\big(r_{\lambda,\mu_1}^+\big) \end{cases}$$

and thus,

$$u_{\lambda,\mu_1,\mu_2}(r) = \frac{\lambda\varepsilon}{6}(r^2 - r_{\lambda,\mu_2}^2) + \frac{\lambda\varepsilon}{3}r_{\lambda,\mu_1}^3\Big(\frac{1}{r} - \frac{1}{r_{\lambda,\mu_2}}\Big) + \mu_2, \quad r_{\lambda,\mu_1} \le r \le r_{\lambda,\mu_2}.$$

Finally, we examine the problem

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial u}{\partial r}\big) = \lambda \quad \text{in } (r_{\lambda,\mu_2}, R), \\ u(R) = u_\infty, \quad \frac{\partial u}{\partial r}\big(r_{\lambda,\mu_2}^-\big) = \frac{\partial u}{\partial r}\big(r_{\lambda,\mu_2}^+\big). \end{cases}$$

So,

$$\begin{aligned} u_{\lambda,\mu_1,\mu_2}(r) &= \frac{\lambda}{6}(r^2 - R^2) \\ &\quad + \Big(\frac{\lambda(\varepsilon-1)}{3}\, r_{\lambda,\mu_2}^3 - \frac{\lambda\varepsilon}{3}\, r_{\lambda,\mu_1}^3\Big)\Big(\frac{1}{R} - \frac{1}{r}\Big) + u_\infty, \quad r_{\lambda,\mu_2} \le r \le R. \end{aligned}$$

Our solutions must satisfy the conditions

$$\begin{cases} u_{\lambda,\mu_1,\mu_2}(r^+_{\lambda,\mu_1}) = \mu_1, \\ u_{\lambda,\mu_1,\mu_2}(r^+_{\lambda,\mu_2}) = \mu_2. \end{cases}$$

More precisely, we obtain the system

$$\begin{cases} \frac{\lambda\varepsilon}{6}(r^2_{\lambda,\mu_1} - r^2_{\lambda,\mu_2}) + \frac{\lambda\varepsilon}{3}\ r^3_{\lambda,\mu_1}\big(\frac{1}{r_{\lambda,\mu_1}} - \frac{1}{r_{\lambda,\mu_2}}\big) = \mu_1 - \mu_2, \\ \frac{\lambda}{6}(r^2_{\lambda,\mu_2} - R^2) + \Big(\frac{\lambda(\varepsilon-1)}{3}\ r^3_{\lambda,\mu_2} - \frac{\lambda\varepsilon}{3}\ r^3_{\lambda,\mu_1}\Big)\Big(\frac{1}{R} - \frac{1}{r_{\lambda,\mu_2}}\Big) = \mu_2 - u_\infty, \end{cases} \tag{2.4}$$

where $0 \le r_{\lambda,\mu_1} < r_{\lambda,\mu_2} < R$. The resolution of the system (2.4) gives the existence of the free boundaries r_{λ,μ_1} and r_{λ,μ_2} to conclude with the study of solutions of (2.1). The first equation in the system (2.4) can be regarded as an algebraic equation, where the unknown variable is r_{λ,μ_2}. We have

$$r^3_{\lambda,\mu_2} - \Big(3r^2_{\lambda,\mu_1} + \frac{6(\mu_2-\mu_1)}{\lambda\varepsilon}\Big)\ r_{\lambda,\mu_2} + 2r^3_{\lambda,\mu_1} = 0. \tag{2.5}$$

Taking $r_{\lambda,\mu_1} \coloneqq \eta$, this implies that

$$\lambda = \frac{6(\mu_2-\mu_1)\ r_{\lambda,\mu_2}}{\varepsilon\ \big(r^3_{\lambda,\mu_2} - 3\ \eta^2\ r_{\lambda,\mu_2} + 2\eta^3\big)}. \tag{2.6}$$

As in [1], we introduce the auxiliary function

$$K(r) = \frac{6(\mu_2-\mu_1)\ r}{\varepsilon\big(r^3 - 3\ \eta^2\ r + 2\eta^3\big)}, \quad r \in (\eta, R).$$

Hence,

$$K'(r) = \frac{12(\mu_2-\mu_1)(\eta^3 - r^3)}{\varepsilon\ \big(r^3_{\lambda,\mu_2} - 3\ \eta^2\ r_{\lambda,\mu_2} + 2\eta^3\big)^2} < 0, \quad r \in (\eta, R).$$

The function K is monotonely decreasing on (η, R). Then if

$$\lambda > K(R) = \frac{6(\mu_2-\mu_1)R}{\varepsilon\Big(R^3 - 3\eta^2 R + 2\eta^3\Big)} \coloneqq \lambda_2,$$

the equation (2.6) has one root $r_{\lambda,\mu_2} \in (\eta, R)$.

So, assuming that $\lambda > \lambda_2$ and using Cardano's method for the equation (2.5), we get

$$\begin{aligned} \Delta_2 &= \big(2\ r^3_{\lambda,\mu_1}\big)^2 - \frac{4}{27}\Big(3r^2_{\lambda,\mu_1} + \frac{6(\mu_2-\mu_1)}{\lambda\varepsilon}\Big)^3 \\ &= \frac{-8(\mu_2-\mu_1)}{\lambda\varepsilon}\Big[2\ r^2_{\lambda,\mu_1}\Big(r^2_{\lambda,\mu_1} + \frac{\mu_2-\mu_1}{\lambda\varepsilon}\Big) + \Big(r^2_{\lambda,\mu_1} + \frac{2(\mu_2-\mu_1)}{\lambda\varepsilon}\Big)^2\Big] < 0. \end{aligned}$$

Then, the free boundary r_{λ,μ_2} is given by

$$r_{\lambda,\mu_2} = 2\sqrt{r_{\lambda,\mu_1}^2 + \frac{2(\mu_2-\mu_1)}{\lambda\varepsilon}}\cos\left(\frac{1}{3}\arccos\left(-r_{\lambda,\mu_1}^3\frac{1}{\sqrt{\left(r_{\lambda,\mu_1}^2+\frac{2(\mu_2-\mu_1)}{\lambda\varepsilon}\right)^3}}\right)+\frac{4\pi}{3}\right).$$

Now, we look for the unknown variable r_{λ,μ_1} in the second equation of the system (2.4). We get

$$r_{\lambda,\mu_1} = \left(\Big(\frac{3(u_\infty-\mu_2)}{\lambda\varepsilon} + \frac{r_{\lambda,\mu_2}^2 - R^2}{2\varepsilon}\Big)\frac{R\, r_{\lambda,\mu_2}}{r_{\lambda,\mu_2}-R} + \Big(\frac{\varepsilon-1}{\varepsilon}\Big)r_{\lambda,\mu_2}^3\right)^{\frac{1}{3}}.$$

Hence, the free boundary r_{λ,μ_1} satisfies $0 < r_{\lambda,\mu_1} < r_{\lambda,\mu_2}$, if and only if the inequality $0 < h(r) < r^2$ has a solution on $(0,R)$, where

$$h(r) = \Big(\frac{3(u_\infty-\mu_2)}{\lambda\varepsilon} + \frac{r^2-R^2}{2\varepsilon}\Big)\frac{R}{r-R} + \Big(\frac{\varepsilon-1}{\varepsilon}\Big)r^2$$

and

$$h'(r) = \frac{3(u_\infty-\mu_2)}{\lambda\varepsilon}\frac{-R}{(r-R)^2} + \frac{R}{2\varepsilon} + 2\Big(\frac{\varepsilon-1}{\varepsilon}\Big)r.$$

An easy calculation shows that the function h' is decreasing on $(0,R)$. Then, we distinguish two cases (see Figures 2 and 3).

(1) If $h'(0) \le 0$, i.e. $\lambda \le \lambda_1$, the function h is decreasing and negative on $(0,R)$.
(2) If $h'(0) > 0$, i.e. $\lambda > \lambda_1$, the function h has a positive maximum in $r^* \in (0,R)$. Moreover, h is monotone increasing on $(0,r^*)$ and monotone decreasing on (r^*,R).

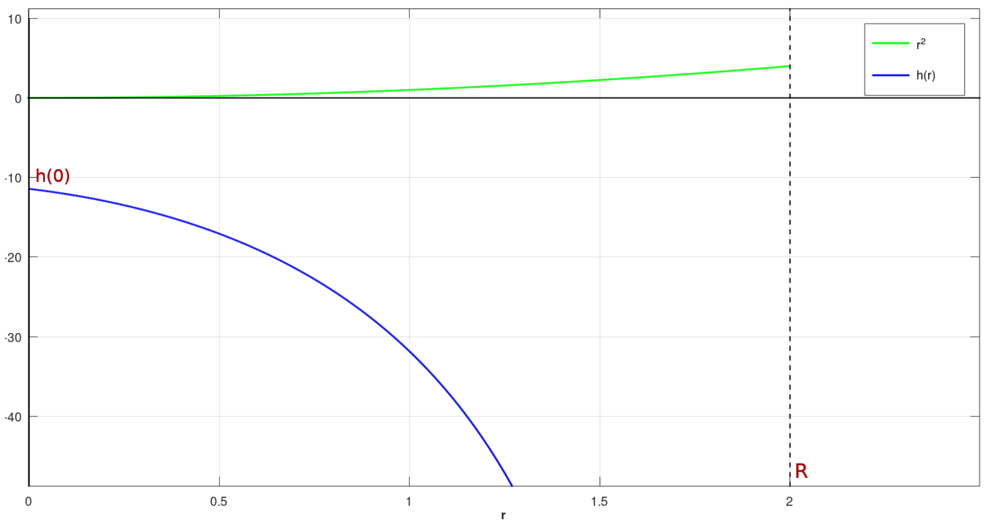

Fig. 2. The graph of the function h and the square function when $\lambda \le \lambda_1$

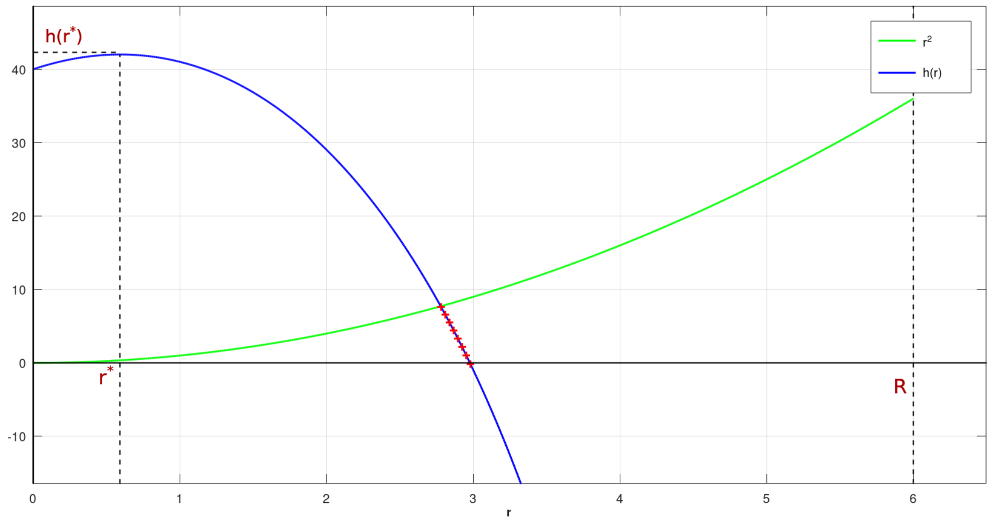

Fig. 3. The graph of the function h and the square function when $\lambda > \lambda_1$

So, for $\lambda > \lambda_1$, we get that the inequality $0 < h(r) < r^2$ has at least one solution on $(0, R)$.

Finally, we assure the existence of $\lambda^* > \max(\lambda_1, \lambda_2)$ such that when $\lambda \in [\lambda^*, +\infty)$, the free boundaries r_{λ,μ_1} and r_{λ,μ_2} satisfy $0 < r_{\lambda,\mu_1} < r_{\lambda,\mu_2} < R$. Thus, by continuity of the solution with respect to λ, there exists λ^* such that if $\lambda \in [\lambda_1, \lambda^*[$, then there exists a unique solution u_{λ,μ_2} such that $u_{\lambda,\mu_2}(r_\lambda) = \mu_2$ and when $\lambda \in [\lambda^*, +\infty[$, we have a unique solution with two free boundaries r_{λ,μ_1} and r_{λ,μ_2}. The proof of Theorem 2.1 ends. $\square$

Let the pair (u, R_s) denote a solution of the problem (2.1)–(2.2) such that $R_s > 0$. Then R_s is obtained as a zero of the equation $I(R) = 0$, where the continuous function $I : [0, \infty) \to \mathbb{R}$ is defined by

$$I(R) = \frac{1}{R^3}\left(\lambda \int_0^R f(u) r^2 dr - \int_0^R \big(\eta_1 H(u - \mu_1) + \eta_2 H(\mu_1 - u)\big) r^2 dr\right). \tag{2.7}$$

According to Theorem 2.1, we see that we have three phases, $u(0) > \mu_2$, $\mu_1 < u(0) \le \mu_2$ and $u(0) \le \mu_1$.

The tumor remains in phase 1 (without free boundary) until $u^*_\lambda(0) = \mu_2$ giving

$$R^* = \sqrt{\frac{6(u_\infty - \mu_2)}{\lambda}}.$$

The tumor now in phase 2 until $u_{\lambda,\mu_2}(0) = \mu_1$ giving the radius R^{**} and the tumor moves from phase 2 into phase 3. So, when $R < R^*$, the tumor region is in phase 1 corresponds to the absence of free boundary (normal growth case).

When $R^* \le R < R^{**}$, the tumor region is in phase 2 with one free boundary (slow growth case), $R \ge R^{**}$ the tumor is in phase 3 corresponds to the necrotic core (with two free boundaries). Then we have the following result:

Theorem 2.2. *Assume that*

$$\lambda\varepsilon \ge \eta_1 \tag{2.8}$$

is satisfied. Then there exists a unique positive value R_s such that $I(R_s) = 0$.

The proof of Theorem 2.2 is based on the following lemma:

Lemma 2.3. *The operator I is continuously differentiable on $(0, +\infty)$.*

Proof. Consider the functions

$$\begin{cases} F_1(x_1, x_2, y) = \frac{\lambda\varepsilon}{6}(x_1^2 - x_2^2) + \frac{\lambda\varepsilon}{3}\, x_1^3\big(\frac{1}{x_1} - \frac{1}{x_2}\big) - \mu_1 + \mu_2, \\ F_2(x_1, x_2, y) = \frac{\lambda}{6}(x_2^2 - y^2) + \Big(\frac{\lambda(\varepsilon-1)}{3}\, x_2^3 - \frac{\lambda\varepsilon}{3}\, x_1^3\Big)\Big(\frac{1}{y} - \frac{1}{x_2}\Big) - \mu_2 + u_\infty. \end{cases} \tag{2.9}$$

The Jacobian matrix of F_1, F_2 with respect to x_1, x_2 is given by

$$J = \begin{pmatrix} \lambda\varepsilon\big(x_1 - \frac{x_1^2}{x_2}\big) & \frac{\lambda\varepsilon}{3}\big(-x_2 + \frac{x_1^3}{x_2^2}\big) \\ -\lambda\varepsilon x_1^2\big(\frac{1}{y} - \frac{1}{x_2}\big) & \frac{\lambda(3-2\varepsilon)}{3} x_2 + \lambda(\varepsilon - 1)\frac{x_2^2}{y} - \frac{\lambda\varepsilon}{3}\frac{x_1^3}{x_2^2} \end{pmatrix}.$$

By (2.4), we have

$$F_1(r_{\lambda,\mu_1}, r_{\lambda,\mu_2}, R) = F_2(r_{\lambda,\mu_1}, r_{\lambda,\mu_2}, R) = 0.$$

Using the fact that $r_{\lambda,\mu_1} < r_{\lambda,\mu_2} < R$, we obtain that $\det\ J > 0$ in the neighborhood of $(r_{\lambda,\mu_1}, r_{\lambda,\mu_2}, R)$. Hence, by the implicit function theorem, in the neighborhood of $(r_{\lambda,\mu_1}, r_{\lambda,\mu_2}, R)$ there are two differentiable functions $x_1(y)$ and $x_2(y)$ satisfy the system (2.4). So, our free boundaries $R \to r_{\lambda,\mu_1}(R), r_{\lambda,\mu_2}(R)$ are differentiable on R.

On the other hand, we have

$$\begin{aligned} I(R) &= \frac{1}{R^3}\Big(\lambda \int_0^R \big(\varepsilon H(u - \mu_1) + (1-\varepsilon)H(u - \mu_2)\big) r^2 dr \\ &\quad - \int_0^R \big(\eta_1 H(u - \mu_1) + \eta_2 H(\mu_1 - u)\big) r^2 dr\Big) \\ &= \frac{1}{R^3}\left(\int_{r_{\lambda,\mu_1}}^{r_{\lambda,\mu_2}} \lambda\varepsilon r^2 dr + \int_{r_{\lambda,\mu_2}}^{R} \lambda r^2 dr - \int_{r_{\lambda,\mu_1}}^{R} \eta_1 r^2 dr - \int_0^{r_{\lambda,\mu_1}} \eta_2 r^2 dr\right) \\ &= \frac{\lambda\varepsilon(r_{\lambda,\mu_2}^3 - r_{\lambda,\mu_1}^3) + \lambda(R^3 - r_{\lambda,\mu_2}^3) - \eta_1(R^3 - r_{\lambda,\mu_1}^3) - \eta_2 r_{\lambda,\mu_1}^3}{R^3}. \end{aligned}$$

Then the operator I is differentiable on $(0, +\infty)$. □

Lemma 2.4. *Assuming that* (2.8) *holds, then the function I is decreasing and satisfies*

$$-\frac{\eta_2}{3} \leq I(R) \leq \frac{\lambda - \eta_1}{3} \tag{2.10}$$

Proof. Setting $s = \frac{r}{R}$, we have

$$\begin{aligned} I(R) &= \lambda \int_0^1 f(u(R\,s))s^2 ds - \int_0^1 \big(\eta_1 H(u(R\,s) - \mu_1) + \eta_2 H(\mu_1 - u(R\,s))\big)s^2 ds \\ &= \int_0^1 g(u(R\,s))s^2 ds, \end{aligned}$$

where

$$\begin{aligned} g(u) &= (\lambda\varepsilon - \eta_1)H(u - \mu_1) + \lambda(1-\varepsilon)H(u - \mu_2) - \eta_2 H(\mu_1 - u) \\ &= \begin{cases} -\eta_2, & u < \mu_1, \\ \lambda\varepsilon - \eta_1, & \mu_1 \leq u < \mu_2, \\ \lambda - \eta_1, & u > \mu_2. \end{cases} \end{aligned}$$

Let $U(r, R) = u(r)$, then U is the solution of the following problem:

$$\begin{cases} \Delta_r U = \lambda f(U), \quad 0 < r < R, \\ U_r(0, R) = 0, \quad U(R, R) = u_\infty, \end{cases}$$

where

$$\Delta_r U = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2 \frac{\partial U}{\partial r}\Big).$$

Thus, $U_R \coloneqq \frac{\partial U}{\partial R}$ satisfies

$$\begin{cases} \Delta_r U_R = \frac{\partial \lambda}{\partial R} f(U), \quad 0 < r < R, \\ \frac{\partial}{\partial r} U_R(0, R) = 0,\ U_R(R, R) = 0. \end{cases}$$

Using the fact that $R \mapsto \lambda(R)$ is decreasing and by maximum principle, we obtain: $R \mapsto U(r, R)$ is decreasing. The assumption (2.8) implies that the function g is increasing. Hence, we deduce that the function I is decreasing on $(0, +\infty)$ and satisfies (2.10). This proves Lemma 2.4. □

Proof of Theorem 2.2. For $R < R^*$, we have

$$I(R) = \frac{\lambda - \eta_1}{3} > 0.$$

In [1], a simple calculation shows the existence of $\overline{R}$ such that

$$L(\overline{R}) = \frac{1}{\overline{R}^2}\left(\lambda \int_0^{\overline{R}} \tilde{f}(u) r^2 dr - \int_0^{\overline{R}} \eta_1 r^2 dr\right) = 0,$$

where

$$\tilde{f}(u(r)) = \begin{cases} \varepsilon, & \text{if } r < r_\lambda, \\ 1, & \text{if } r \geq r_\lambda \end{cases}$$

and r_λ is given in Theorem 2.1. Hence, $L(\overline{R}) = 0$ is equivalent to

$$(\lambda - \eta_1)\overline{R}^3 = -\lambda(\varepsilon - 1)r_\lambda^3 \tag{2.11}$$

So, we distinguish two cases:

(1) if $\overline{R} \in (0, R^{**})$, then $I(R) = L(R)/R$ and $I(\overline{R}) = 0$,
(2) if $\overline{R} \in [R^{**}, +\infty)$, then we have

$$I(\overline{R}) = \frac{1}{\overline{R}^3}\Big(\frac{\lambda(\varepsilon-1)}{3}r^3_{\lambda,\mu_2} + \frac{\eta_1 - \lambda\varepsilon - \eta_2}{3}r^3_{\lambda,\mu_1} + \frac{\lambda - \eta_1}{3}\overline{R}^3\Big)$$

using (2.11) and the assumption (2.8) we obtain

$$I(\overline{R}) = \frac{1}{\overline{R}^3}\Big(\frac{\lambda(\varepsilon-1)}{3}\big(r^3_{\lambda,\mu_2} - r^3_\lambda\big) + \frac{\eta_1 - \lambda\varepsilon - \eta_2}{3}r^3_{\lambda,\mu_1}\Big) < 0.$$

So, we conclude with the existence of R_s such that $I(R_s) = 0$ and uniqueness follows from the monotonicity of I proving in Lemma 2.4. □

3. TRANSIENT SOLUTION

In this section, we shall prove the global existence of a solution and the asymptotic behavior of solutions to the free boundary problem (1.4).

Theorem 3.1. *Assuming that* (2.8) *is satisfied, then for any* $R_0 > 0$ *and for* $0 < \varepsilon < 1$, *the problem* (1.4) *has a unique global solution* $(u(r,t), R(t))$ *for* $t > 0$. *Moreover,*

$$\lim_{t\to+\infty} R(t) = R_s, \quad \lim_{t\to+\infty} u(r,t) = u_{\lambda,\mu_1,\mu_2},$$

where $(u_{\lambda,\mu_1,\mu_2}, R_s)$ *is the stationary solution of the problem* (2.1)–(2.2).

To prove theorem 3.1, we use the following lemma:

Lemma 3.2 ([25])**.** *Let consider the problem*

$$\begin{cases} x'(t) = x(t)f(x(t)), & t > 0, \\ x(0) = x_0 > 0. \end{cases}$$

Assuming that f *is continuously and differentiable decreasing function on* $(0, +\infty)$. *If there exists a unique positive constant* x_s *such that* $f(x_s) = 0$, *then*

$$\lim_{t\to+\infty} x(t) = x_s.$$

Proof of Theorem 3.1. We know from Section 2 that problem (1.4) admits a unique solution $u(r, R(t))$. So, for $t > 0$, we determine $R(t)$ by solving the following problem:

$$\begin{cases} R'(t) = R(t)I(R(t)), & t > 0, \\ R(0) = R_0, \end{cases} \tag{3.1}$$

where I is given by (2.7). Using Lemmas 2.3 and 2.4, then the problem (3.1) has a unique global solution satisfying

$$R_0 e^{\left(\frac{-\eta_2}{3}\right)t} \leq R(t) \leq R_0 e^{\left(\frac{\lambda-\eta_1}{3}\right)t}, \quad t > 0.$$

Now, because $I(R)$ is decreasing function and from theorem 2.2, there exists R_s such that the equation $I(R_s) = 0$. Then, by Lemma 3.2, we get

$$\lim_{t\to+\infty} R(t) = R_s.$$

So, $\lim\limits_{t\to+\infty} u(r,t) = u_{\lambda,\mu_1,\mu_2}$. This concludes the proof of Theorem 3.1. $\square$

4. PERTURBATION RESULT

This section is devoted to study the problem (1.3) when $\Omega(t)$ verifies (C). Let r_1, r_2 denote the values r_{λ,μ_1} and r_{λ,μ_2} of theorem 2.1. We look for the free boundaries on the form $r_1 + b_1(\theta)$ and $r_2 + b_2(\theta)$, where b_1, b_2 are the perturbations caused by β. Hence, we consider the following problem:

$$\begin{cases} \Delta u = \lambda\Big(\varepsilon H(u-\mu_1) + (1-\varepsilon)H(u-\mu_2)\Big) & \text{in } \Omega_\beta, \\ u = \overline{u}_\infty & \text{on } \partial\Omega_\beta, \end{cases} \tag{4.1}$$

where $\Omega_\beta = B(0, R+\beta(\theta)) \subset \mathbb{R}^3$ and $\overline{u}_\infty$ is close to u_∞.

As in [1], we define the set of admissible surfaces in Ω_β by

$$S_\beta = \{f \in C(S) \,:\, (f(\theta),\theta) \in \Omega_\beta \text{ for } \theta \in S\}.$$

For $i = 1, 2$, we consider the functions $\psi_i \in S_\beta$ and the sets

$$\omega_{\psi_i} = \{(r,\theta) \in \Omega_\beta \,:\, r < \psi_i(\theta)\}, \quad i = 1, 2.$$

In the following proposition, we formulate nonlinear equations for ψ_1, ψ_2 and prove that by solving them, we can solve the problem (4.1).

Proposition 4.1. *For $\varepsilon \in (0,1)$, there exists $\lambda^* > 0$ such that if $\lambda \geq \lambda^*$, the following problem*

$$\begin{cases} \Delta u = \lambda\Big(\varepsilon\chi_{\Omega_\beta\setminus\omega_{\psi_1}} + (1-\varepsilon)\chi_{\Omega_\beta\setminus\omega_{\psi_2}}\Big) & \text{in } \Omega_\beta, \\ u = \overline{u}_\infty & \text{on } \partial\Omega_\beta \end{cases} \tag{4.2}$$

has a unique solution $u_\beta \in C^{1,\alpha}(\overline{\Omega_\beta})$ with $\alpha = 1 - \frac{3}{p}$, $p > 3$. Moreover, if $u_\beta(\psi_i(\theta),\theta) = \mu_i$ for $i = 1, 2$ and $\overline{u}_\infty > \mu_2 > \mu_1$, then u_β is a solution of (4.1).

Proof. We have $\chi_{\Omega_\beta\setminus\omega_{\psi_i}} \in L^p(\Omega_\beta)$, $p>1$, $i=1,2$. From [19], there exists a unique solution u_β of (4.2) in $W^{2,p}(\Omega_\beta)$. For $p>3$, $W^{2,p}(\Omega_\beta)\subset C^{1,\alpha}(\overline{\Omega_\beta},\mathbb{R})$ with $\alpha = 1-\frac{3}{p}$.

If we prove the existence of functions ψ_i such that $u_\beta(\psi_i(\theta),\theta)=\mu_i$ for $i=1,2$, then u_β will be a solution of the following problems:

$$\begin{cases}\Delta u_\beta = 0 & \text{in } \omega_{\psi_1},\\ u_\beta=\mu_1 & \text{on } \partial\omega_{\psi_1},\end{cases}\qquad \begin{cases}\Delta u_\beta = \lambda\varepsilon & \text{in } \omega_{\psi_2}\setminus\omega_{\psi_1},\\ u_\beta=\mu_1 & \text{on } \partial\omega_{\psi_1},\\ u_\beta=\mu_2 & \text{on } \partial\omega_{\psi_2},\end{cases}\qquad \begin{cases}\Delta u_\beta = \lambda & \text{in } \Omega_\beta\setminus\omega_{\psi_2},\\ u_\beta=\mu_2 & \text{on } \partial\omega_{\psi_2},\\ u_\beta=\overline{u}_\infty & \text{on } \partial\Omega_\beta.\end{cases}$$

Using the fact that $\overline{u}_\infty > \mu_2 > \mu_1$ and the maximum principle, we obtain that u_β satisfies

$$\begin{cases}\Delta u_\beta = \lambda\Big(\varepsilon H(u_\beta-\mu_1)+(1-\varepsilon)H(u_\beta-\mu_2)\Big) & \text{in } \Omega_\beta,\\ u_\beta = \overline{u}_\infty & \text{on } \partial\Omega_\beta.\end{cases}\qquad\square$$

As in [1], the variation of the domain Ω_β suggests the use of an appropriate transformation that maps the changing domain Ω_β to a constant domain:

$$\begin{aligned}&T_\beta\colon \Omega_\beta \to \Omega_0 = B(0,R),\\ &(r,\theta)\mapsto(\overline{r},\theta)=\Big(r+r\frac{\beta}{R},\theta\Big),\end{aligned}$$

where (r,θ) is the coordinates in Ω_β and $(\overline{r},\theta)$ the coordinates in Ω_0.

For a small β, the transformation T_β is a diffeomorphism of class C^2 of the domain Ω_β into Ω_0 and transforms S_β into S_0, hence

$$T_\beta(\psi_i(\theta),\theta) = (f_i(\theta),\theta),$$

where $f_i\in S_0$. So, the mapping T_β transforms the problem (4.2) and the equation $u_\beta(\psi_i(\theta),\theta)=\mu_i$ for $i=1,2$ to the problem

$$\begin{cases}\Delta\overline{u}+\delta_\beta\overline{u} = \lambda\Big(\varepsilon\ \chi_{\Omega_0\setminus\omega_{f_1}}+(1-\varepsilon)\ \chi_{\Omega_0\setminus\omega_{f_2}}\Big) & \text{in } \Omega_0,\\ \overline{u}=\overline{u}_\infty & \text{on } \partial\Omega_0\end{cases}\tag{4.3}$$

and the equation

$$\overline{u}(f_i(\theta),\theta)=\mu_i,\quad i=1,2.\tag{4.4}$$

where

$$\begin{aligned}\delta_\beta = {}&\frac{\beta}{R}\Big(2+\frac{\beta}{R}\Big)\frac{\partial^2}{\partial\overline{r}^2}\\ &+\frac{2\beta}{\overline{r}R}\frac{\partial}{\partial\overline{r}}\frac{1}{\overline{r}^2}\Big[a_{ij}(\theta)\Big[\frac{r}{R}\frac{\partial\beta}{\partial\theta_j}\Big(\frac{\partial^2}{\partial\overline{r}\partial\theta_i}+\frac{1}{R(1+\frac{\beta}{R})}\frac{\partial\beta}{\partial\theta_i}\frac{\partial}{\partial\overline{r}}+\frac{r}{R}\frac{\partial\beta}{\partial\theta_i}\frac{\partial^2}{\partial\overline{r}^2}\Big)\\ &\quad+\frac{r}{R}\frac{\partial^2\beta}{\partial\theta_j\partial\theta_i}\frac{\partial}{\partial\overline{r}}+\frac{r}{R}\frac{\partial\beta}{\partial\theta_i}\frac{\partial^2}{\partial\theta_j\partial\overline{r}}\Big]+b_i(\theta)\Big[\frac{r}{R}\frac{\partial\beta}{\partial\theta_i}\frac{\partial}{\partial\overline{r}}\Big]\Big]\end{aligned}$$

for $a_{i,j}, b_i\in C^2(S)$, $i,j=1,2$, and $f_i\in S_0$.

Thus, to solve the problem (4.1) it is sufficient to prove the existence of f_1, f_2 satisfying (4.4). We can proceed as follows.

First, we give the existence of $f_2(\theta)$ for $\theta \in S$. We denote by $\overline{u}_2$ the restriction of the solution $\overline{u}$ in the set $\Omega_0 \setminus \omega_{f_2}$. So, we have

$$\begin{cases} \Delta \overline{u}_2 + \delta_\beta \overline{u}_2 = \lambda \Big(\varepsilon + (1-\varepsilon)\ \chi_{\Omega_0 \setminus \omega_{f_2}} \Big) & \text{in } \Omega_0, \\ \overline{u}_2 = \overline{u}_\infty & \text{on } \partial\Omega_0. \end{cases} \tag{4.5}$$

The same technique used in [1] can be applied and the implicit function theorem gives the existence of $f_2(\theta)$, $\theta \in S$ such that

$$\overline{u}_2(f_2(\theta), \theta) = \mu_2.$$

Hence, from the construction and uniqueness of the solution we conclude that

$$\overline{u}(f_2(\theta), \theta) = \mu_2.$$

Now, consider the following problem

$$\begin{cases} \Delta \overline{u}_1 + \delta_\beta \overline{u}_1 = \lambda \varepsilon\ \chi_{\Omega_0 \setminus \omega_{f_1}} & \text{in } \Omega_0, \\ \overline{u}_1 = \overline{u}_\infty & \text{on } \partial\Omega_0. \end{cases}$$

We define the operator $J : \mathbb{R}^+ \times S_0 \times \mathbb{R}^+ \times D \to C(S, \mathbb{R})$ by

$$J(\overline{u}_\infty, f_1, \mu_1, \beta) = \overline{u}_1(f_1(\theta), \theta) - \mu_1,$$

where D is the neighborhood of zero in $C(S)$.

So, we obtain

$$\begin{aligned} J(\overline{u}_\infty, f_1, \mu_1, \beta) = \overline{u}_\infty \int_S P(f_1(\theta), \theta, \theta') d\theta' + \lambda \varepsilon \int_S \int_{f_1(\theta')}^{R} G(f_1(\theta), \theta, r', \theta') r'^2 dr' d\theta' \\ - \int_{\Omega_0} \delta_\beta \overline{u}_1(r', \theta') G(f_1(\theta), \theta, r', \theta') dr' d\theta' - \mu_1, \end{aligned}$$

where P is the Poisson kernel and G is the Green function for the Laplacian in Ω_0.

By the same argument as in [1] and using the implicit function theorem in the neighborhood of $(u_\infty, r_1, \mu_1, 0)$, we have the existence of a function f_1 depending on $\overline{u}_\infty$, μ_1, where β satisfies

$$J(\overline{u}_\infty, f_1(\overline{u}_\infty, \mu_1, \beta), \mu_1, \beta) = 0.$$

So, we conclude that

$$\overline{u}(f_1(\theta), \theta) = \mu_1.$$

Moreover, in the region ω_{f_1}, we have the following result:

Lemma 4.2. *Let γ be a Jordan curve in $\{x \in \Omega_0 : \overline{u}(x) = \mu_1\}$, where ω is the interior of γ. Then*

$$\omega \subset \{x \in \Omega_0 \,:\, \overline{u}(x) = \mu_1\}.$$

Proof. Suppose that there exists $x_0 \in \omega$ such that $\overline{u}(x_0) < \mu_1$. Let

$$\omega_1 = \{x \in \Omega_0 \,:\, \overline{u}(x) < \mu_1\}$$

which is an open set (by continuity of $\overline{u}$). So, $\overline{u}$ verifies

$$\begin{cases} \Delta u = 0 & \text{in } \omega_1, \\ u = \mu_1 & \text{on } \partial\omega_1. \end{cases}$$

By the maximum principle we get $\overline{u} \geq \mu_1$ which is a contradiction.

So, let u be a solution of the problem (2.1), then the set $\{x \in \Omega \,:\, u(x) = \mu_1\}$ is a ball of radius $r_1 \in (0, R)$.

Moreover, by the above perturbation method, we conclude that w_{f_1} is a perturbation of a ball of radius $r_1 + f_1(\theta)$, $\theta \in S$. □

5. FINAL COMMENTS

In this paper, we have studied the existence of stationary solutions for problem (1.3) when the domain is a ball. After, reducing the free boundary problem (1.3) into an initial problem of ordinary differential equation, we study the asymptotic behavior of solutions. Finally, using a rigorous perturbative approach, we derive a characterization of the free boundary under perturbation of the boundary condition and a smooth boundary of the domain. However, some open questions related to our problem still remain. Here, we give some interesting problems:

(1) A more general form of the problem (2.1) can be regarded as

$$\begin{cases} \Delta u = \lambda \sum_{i=1}^{n} \phi_i \; H(u - \mu_i) & \text{in } \Omega(t), \\ u = \overline{u}_\infty & \text{on } \partial\Omega(t), \end{cases} \tag{5.1}$$

where ϕ_i are positive constants for $i = 1, \ldots, n$. To the best of our knowledge, no investigation has been devoted to such problem in the literature. Recently, in [3] the second author studies a similar problem of (5.1) and shows that under suitable conditions there exists a solution with a convex level set. We refer to [3] for other results. Moreover, the problem (5.1) can be reformulated as an equivalent free boundary problem. For $t > 0$, we have

$$\begin{cases} \Delta u = \lambda \sum_{i=1}^{n} \phi_i \; \chi_{\Omega(t)\setminus\omega_{\mu_i}(t)} & \text{in } \Omega(t), \\ u = \overline{u}_\infty & \text{on } \partial\Omega(t), \end{cases} \tag{5.2}$$

where χ_ω denotes the characteristic function of the set ω. There is a mathematical challenge in the study of problem (5.2) and some technical details will be presented elsewhere.

(2) Certainly, for other models, we conjecture that the results of our work is still true for n free boundaries (the nonlinearity f can be written as $f(u) = \sum_{i=1}^{n} \phi_i H(u - \mu_i)$).
(3) The obtained bifurcation diagram can be very important to study the question of the stability of solution of problem (1.1). We recall that the stability results obtained by the Crandall–Rabinowitz theorem [10] can not be applied in our work. This question requires more attention.

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