Nonlinear Vibrations of a Slender Beam Interacting with a Periodic Viscoelastic Subsoil

Marcin ŚWIĄTEK
Department of Structural Mechanics, Łódź University of Technology,
al. Politechniki 6, 90-924 Łódź, Poland,
marcin.swiatek@edu.p.lodz.pl

Jarosław JĘDRYSIAK
Department of Structural Mechanics, Łódź University of Technology,
al. Politechniki 6, 90-924 Łódź, Poland,
jarek@p.lodz.pl

Łukasz DOMAGALSKI
Department of Structural Mechanics, Łódź University of Technology,
al. Politechniki 6, 90-924 Łódź, Poland,
lukasz.domagalski@p.lodz.pl

Abstract

The paper describes nonlinear vibrations of Euler-Bernoulli beams interacting with a periodic viscoelastic foundation. The original model equations with highly oscillating periodic coefficients are transformed using the tolerance modelling technique. Newly delivered equations have constant coefficients and describe macro-dynamics of the beam including the effect of the microstructure size. The main purpose of this paper is to propose an equivalent approximate model describing the nonlinear vibrations of a beam interacting with a periodic viscoelastic subsoil.

Keywords: periodic beams, Euler-Bernoulli beams

1. Introduction

The paper deals with geometrically nonlinear vibrations of beams based on a foundation with periodically varying properties. Dynamics of such elements is described by differential equations with non-continuous, highly oscillating periodic coefficients. For this reason, other various approximate models of such structures are proposed, e. g. those based on the theory of asymptotic homogenization, cf. [2, 3, 9].

In this paper differential equations with highly oscillating coefficients are replaced by equations with constant coefficients using the tolerance averaging technique (TA), (cf. the book edited by Cz. Woźniak, Michalak and Jędrysiak [13]). The aforementioned method has wide application and can be used for modelling problems, described by differential equations with highly oscillating coefficients such as dynamic behaviour of thin functionally graded [8] and periodic [10] plates. Unlike the exact models, the resultant equations have constant coefficients, some of which explicitly depend on the microstructure size.

The literature on the problems of linear vibrations of periodic beams is extensive. A wave propagation and linear vibrations in periodic Euler-Bernoulli beams were considered in [4] and [11]. Frequency band gaps were analysed by the differential quadrature method in [14]. The transfer matrix method was applied in [15] in analysis of flexural wave propagation in the beam on elastic foundation. In [6] a wide literature study on composite beam vibration can be found. In order to determine a homogenized model of a composite beam with small periodicity the two-scale asymptotic expansion method is used in [7].

Nonlinear vibrations of homogeneous nano-beams interacting with a homogeneous viscoelastic substrate were considered in [12]. Nonlinear vibration models of the Euler-Bernoulli beam derived through FEM discretization and the finite differences method were compared in paper [1].

The main aim of this contribution is to derive equations of the nonlinear tolerance model of dynamics of a slender beam interacting with a periodic viscoelastic foundation and show an application for the special problem.

2. Governing equations of the model

The object under consideration is a linearly elastic prismatic beam, bilaterally interacting with a periodic viscoelastic foundation. Let us introduce an orthogonal Cartesian coordinate system *Oxyz*. The *Ox* axis is collinear with the axis of the beam, the cross section of the beam is symmetric with respect to the plane of the load *Oxz*, the load acts in the direction of the *Oz* axis. It is assumed that a small repetitive element called a periodicity cell can be distinguished in the beams structure. Let $\Omega\equiv[0, L]$, where L is the beam length and $l \ll L$ denotes the length of the cell. Hence, there can be introduced a range occupied by the periodicity cell, called microstructure parameter $\Delta\equiv[-l/2, l/2]$. Our considerations are based on the Euler-Bernoulli theory of beams with von Kármán type nonlinearity. It is assumed that vibrations take place only in transverse direction, so the effect of axial inertia is neglected from further considerations. The following notation is introduced: $\partial^k = \partial^k/\partial x^k$ is the k – th derivative with respect to the x coordinate and overdot stands for the derivative with respect to time. Let $w = w(x,t)$ be the transverse deflection, $u_0 = u_0(x,t)$ longitudinal displacement, $EA = E(x)A(x)$ and $EJ = E(x)J(x)$ tensile and flexural stiffness, $k = k(x)$ and $c = c(x)$ – elasticity and damping coefficients of the foundation, $\mu=\mu(x)$ mass per unit length and $q = q(x,t)$ – transverse load. Thus, the strain and kinetic energy density per unit length of the beam are:

$$\mathcal{W} = \tfrac{1}{2} EA\left(\partial u_0 + \tfrac{1}{2}\partial w \partial w\right)^2 + \tfrac{1}{2} EJ\left(\partial^2 w\right)^2, \quad \mathcal{K} = \tfrac{1}{2}\mu \dot{w}\dot{w}. \tag{1}$$

The Kelvin-Voight model is applied for the subsoil and the dissipative force is assumed in the form:

$$p = p(x,t,) = c(x)\dot{w}(x,t). \tag{2}$$

The equations of motion can be obtained from the extended (Woźniak et al., 2010) principle of stationary action $\mathcal{A} = \mathcal{A}(u_0,w)$ formulated as:

$$\delta\mathcal{A} = \delta\int_0^1\int_0^L \mathcal{L}dxdt = \int_0^1\int_0^L \delta\mathcal{L}dxdt = \int_0^1\int_0^L\left[\left(\frac{\partial\mathcal{L}}{\partial u_0} - \partial\frac{\partial\mathcal{L}}{\partial(\partial u_0)}\right)\delta u_0 \right.$$
$$\left. + \left(\frac{\partial\mathcal{L}}{\partial w} - \partial\frac{\partial\mathcal{L}}{\partial(\partial w)} + \partial^2\frac{\partial\mathcal{L}}{\partial(\partial^2 w)} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{w}} + \frac{d}{dt}\partial\frac{\partial\mathcal{L}}{\partial(\partial\dot{w})}\right)\delta w\right]dxdt = 0, \tag{3}$$

where the Lagrangian is

$$\mathcal{L} = \mathcal{W} - \mathcal{K} + pw + \tfrac{1}{2}kww - qw =$$
$$= \tfrac{1}{2}EA\left(\partial u0 + \tfrac{1}{2}\partial w\partial w\right)^2 + \tfrac{1}{2}\partial^2 wEJ\partial^2 w - \tfrac{1}{2}\mu\dot{w}\dot{w} + pw + \tfrac{1}{2}kww - qw. \tag{4}$$

As a result the system of nonlinear coupled partial differential equations for the longitudinal displacements u_0 and the transverse deflection w is obtained:

$$\partial\left[EA\left(\partial u_0 + \tfrac{1}{2}\partial w\partial w\right)\right] = 0,$$
$$\partial^2\left(EJ\partial^2 w\right) + kw + c\dot{w} + \mu\ddot{w} - \partial\left[EA\left(\partial u_0 + \tfrac{1}{2}\partial w\partial w\right)\partial w\right] = q. \tag{5}$$

The coefficients EA, EJ, k, μ, c are highly oscillating non-continuous functions of the x-coordinate.

3. Basic assumptions of the tolerance averaging technique

The tolerance averaging technique (TA) is based on a set of the following concepts: tolerance relations, slowly-varying functions and fluctuation shape functions. For the purpose of this article, only the most important assumptions of this method will be presented.

The averaging operation over a region of periodicity cell for an arbitrary integrable function f is defined as

$$\langle f\rangle(x) = \frac{1}{l}\int_{\Delta(x)} f(y)dy, \quad x\in\Omega_\Delta, \quad y\in\Delta(x). \tag{6}$$

where a cell at $x\in\Omega_\Delta$ is denoted by $\Delta(x) = x+\Delta$, $\Omega_\Delta = \{x\in\Omega : \Delta(x)\subset\Omega\}$.

The micro-macro decomposition is based on the observation that the response of a periodic structure is periodic-like. Thus, the unknown transverse deflection and axial displacement can be decomposed into their slowly-varying and tolerance periodic parts (here and hereafter a summation convention is used):

$$w(x,t) = W(x,t) + h^A(x)V^A(x,t), \quad A = 1,\ldots,N,$$
$$u_0(x,t) = U(x,t) + g^K(x)T^K(x,t), \quad K = 1,\ldots,M. \tag{7}$$

The new unknowns: averaged transverse deflection, axial displacement and their fluctuation amplitudes are slowly varying functions of second and first kind respectively:

$$W(\cdot), V^A(\cdot) \in SV_d^2(\Omega,\Delta), \quad U(\cdot), T^K(\cdot) \in SV_d^1(\Omega,\Delta), \tag{8}$$

and the corresponding *l*-periodic highly oscillating fluctuation shape functions:

$$h^A(\cdot) \in FS_d^2(\Omega,\Delta), \quad g^K(\cdot) \in FS_d^1(\Omega,\Delta), \tag{9}$$

The highly oscillating fluctuation shape functions (*FSFs*) h^A and g^K are proposed *a priori* for each problem under consideration. The *FSFs* describe the unknown fields oscillations caused by the structure inhomogeneity and have to ensure the *l*-periodicity constraint and provide the following conditions:

$$\left\langle \mu h^A \right\rangle = 0, \quad \left\langle \mu h^A h^B \right\rangle = 0 \text{ for A} \neq \text{B}; \quad \partial^m h^A \in O\left(l^{2-m}\right), \quad A,B = 1,\ldots,N. \tag{10}$$

4. Tolerance model of the beam

After substitution of micro-macro decompositions (7) into Lagrangian (4), the averaging over an arbitrary periodicity cell is performed (6) applying the aforementioned approximations (10).
The averaged action functional has the following form:

$$\delta\mathcal{A}_h = \delta\int_0^1\int_0^L \left\langle \mathcal{L}_h \right\rangle dx dt = \int_0^1\int_0^L \delta\left\langle \mathcal{L}_h \right\rangle dx dt = 0. \tag{11}$$

Under assumed boundary conditions it leads to a system of Euler-Lagrange equations:

$$\begin{gathered}\frac{\partial\langle\mathcal{L}_h\rangle}{\partial U} - \partial\frac{\partial\langle\mathcal{L}_h\rangle}{\partial(\partial U)} = 0, \quad \frac{\partial\langle\mathcal{L}_h\rangle}{\partial T^K} = 0, \quad \frac{\partial\langle\mathcal{L}_h\rangle}{\partial V^A} - \frac{d}{dt}\frac{\partial\langle\mathcal{L}_h\rangle}{\partial\left(\dot{V}^A\right)} = 0, \\ \frac{\partial\langle\mathcal{L}_h\rangle}{\partial W} - \partial\frac{\partial\langle\mathcal{L}_h\rangle}{\partial(\partial W)} + \partial^2\frac{\partial\langle\mathcal{L}_h\rangle}{\partial\left(\partial^2 W\right)} - \frac{d}{dt}\frac{\partial\langle\mathcal{L}_h\rangle}{\partial\left(\dot{W}\right)} + \frac{d}{dt}\partial\frac{\partial\langle\mathcal{L}_h\rangle}{\partial\left(\partial\dot{W}\right)} = 0.\end{gathered} \tag{12}$$

After some manipulations we derive the following system of equations in dimensionless form:

$$\begin{gathered}Dw_{\xi\xi\xi\xi} + D^A v^A{}_{\xi\xi} + Kw + \lambda^2 K^A v^A - \overline{N} w_{\xi\xi} - \lambda\tilde{N}^A v^A{}_\xi + \\ + Cw_\tau + \lambda^2 C^A v^A{}_\tau + Mw_{\tau\tau} + \lambda^2 M^A v^A{}_{\tau\tau} - Q = 0, \\ D^{AB} v^B + \lambda^4 K^{AB} v^B + \lambda^4 C^{AB} v^B{}_\tau + \lambda^4 M^{AB} v^B{}_{\tau\tau} + \lambda\tilde{N}^A w_\xi + \\ + \lambda^2 \tilde{N}^{AB} v^B + D^A w_{\xi\xi} + \lambda^2 K^A w + \lambda^2 M^A w_{\tau\tau} + \lambda^2 C^A w_\tau - \lambda^2 Q^A = 0,\end{gathered} \tag{13}$$

where coefficients in (13) are introduced in [5]

In contrast to the exact formulation (5), obtained system of partial differential equations for the macrodisplacements $U(\cdot)$, $W(\cdot)$ and for the fluctuation amplitudes of the axial displacement $T^K(\cdot)$ and of the deflection $V^A(\cdot)$ has constant coefficients. Underlined coefficients depend on the microstructure parameter *l*.

5. Solution methods

As an example there is considered a hinged-hinged beam with immovable ends, which fragment is shown in Fig. 1. The beam's cross section, Young's modulus and mass density are constant. It is assumed that cross section of the beam is rectangular. The beam is based on a viscoelastic foundation which stiffness and damping coefficient are varying periodically along the beams axis. The periodicity cell has symmetrical shape, see Fig. 2.

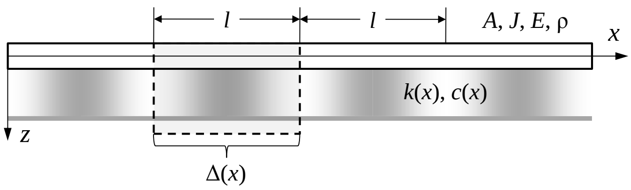

Figure 1. Fragment of considered Euler-Bernoulli beam

The fluctuation shape functions play a crucial role in the analysis. These functions represent the oscillations of displacements in the periodicity cell. The common practice is to use approximate functions, defined by trigonometric sine and cosine functions. Transverse and longitudinal approximate *l*-periodic trigonometric functions are introduced for the symmetric periodicity cell:

$$h^A(y) = l^2\left(\cos\left(\frac{2A\pi y}{l}\right) + c_h^A\right), \quad A = 1,\dots,N,$$
$$g^K(y) = l\left(\sin\left(\frac{2A\pi y}{l}\right) + c_g^K\right), \quad K = 1,\dots,M. \tag{14}$$

6. Calculational results

6.1. *Problem statement*

As an example there is considered a hinged-hinged beam with immovable ends. It is assumed that the considered beam is made of a linearly elastic homogeneous material with Young's modulus E = 205 GPa and density ρ = 7850 kg/m^3. The beam under consideration has length L = 1 m. The following ratios between its geometrical properties b/h = 1, h/l = 1/5 and $\lambda = l/L$ = 1/10 are introduced, where b, h and l are beam's cross section width, beam's cross section height and the length of the periodicity cell, respectively.

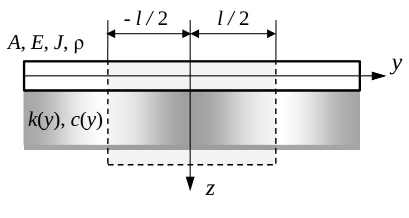

Figure 2. Considered periodicity cell

The properties of the substratum are assumed to change in a periodic manner. The stiffness of the foundation k_1 = 2000 kPa and the damping coefficient c_1 = 1. The l – periodic functions of the substratum are assumed in the form: $k(y) = k_1 + k_2(1 + \cos(2\pi y/l))$, $c(y) = c_1 + c_2(1 + \cos(2\pi y/l))$ for $y \in \Delta(x)$. For the purposes of this work there are introduced the following dimensionless ratios: $\alpha = k_2/k_1$ and $\beta = c_2/c_1$.

Firstly the linear eigenfrequencies of the beam are investigated. After that the beam is forced with vibrations frequencies close to the linear eigenfrequencies of the beam.

6.2. *Linear eigenfrequencies*

In order to calculate the linear eigenfrequencies of the beam we restrict to consider only the first term of Fourier series (m = 1) and one *FSF* (N = 1) so that the model has $m(1+N)$ = 2 degrees of freedom. As a result we derive the averaged dimensionless coefficients of equations (13):

$$\begin{Bmatrix} D \\ D^1 \\ D^{11} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 8\pi^4 \end{Bmatrix}, \quad \begin{Bmatrix} M \\ M^1 \\ M^{11} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{Bmatrix}, \quad \begin{Bmatrix} K \\ K^1 \\ K^{11} \end{Bmatrix} = \begin{Bmatrix} \frac{L^4(k_1+k_2)}{EJ} \\ \frac{L^4 k_2}{2EJ} \\ \frac{L^4(k_1+k_2)}{2EJ} \end{Bmatrix}. \tag{15}$$

We assume solutions in the following form:

$$\begin{aligned} w(\xi,\tau) &= w_1(\tau)\sin\pi\xi, \\ v^1(\xi,\tau) &= v_1^1(\tau)\sin\pi\xi. \end{aligned} \tag{16}$$

As a result the following algebraic system of equations is derived:

$$\begin{bmatrix} \frac{1}{2}\left(\pi^4 + \frac{L^4(k_1+k_2)}{EJ} - \omega^2\right) & \frac{L^2 k_2}{EJ} \\ \frac{L^2 k_2}{EJ} & 4\left(\pi^4 + \frac{k_1+k_2}{EJ} - \frac{\omega^2}{L^4}\right) \end{bmatrix} \begin{Bmatrix} w_1 \\ v_{1,1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{17}$$

The formula for eigenfrequencies ω_- (lower) and ω_+ (higher) is:

$$\begin{aligned} \omega_{\mp} = \Bigg[& \frac{\pi^4}{2}\left(L^4+1\right) + \frac{L^4(k_1+k_2)}{EJ} \mp \\ & \mp \frac{1}{2EJ}\sqrt{E^2J^2L^8\pi^8 - 2E^2J^2L^4\pi^8 + E^2J^2\pi^8 + 2L^8k_2^2}\Bigg]^{\frac{1}{2}} \end{aligned} \tag{18}$$

The solutions derived from the TA method are compared with the Ritz method in Table 1.

Table 1. Linear eigenfrequencies of the beam

Case	α	β	linear vibration frequency ω (dimensionless)			
			ω_-		ω_+	
			Ritz	TA	Ritz	TA
1÷8	25	0÷64	116.433	116.435	15585473.59	15585473.60
9	5	2	101.799	101.801	15585458.96	15585458.96
10	10		105.457	105.460	15585462.61	15585462.62
11	20		112.775	112.777	15585469.93	15585469.94
12	40		127.409	127.411	15585484.57	15585484.57
13	80		156.677	156.679	15585513.84	15585513.84
14	160		215.214	215.213	15585572.37	15585572.38
15	320		332.285	332.286	15585689.44	15585689.45
16	640		566.426	566.428	15585923.59	15585923.60

6.3. *Nonlinear vibrations*

After finding the free vibration frequencies, the beam is forced with a harmonic load frequency Ω_F. The deflection $w(L/2)$ in the middle of the beam span as a function of the number of excitation periods T for the 1. case is presented in Fig. 3 and Fig. 4, for $\Omega_F = \omega_-$ and $\Omega_F = \omega_+$, respectively.

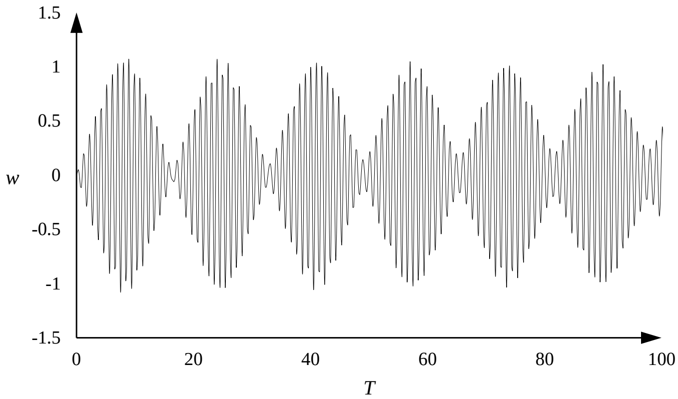

Figure 3. Deflection of the beam for a harmonic load frequency $\Omega_F = \omega_-$ as a function of the number of excitation periods T for case 1

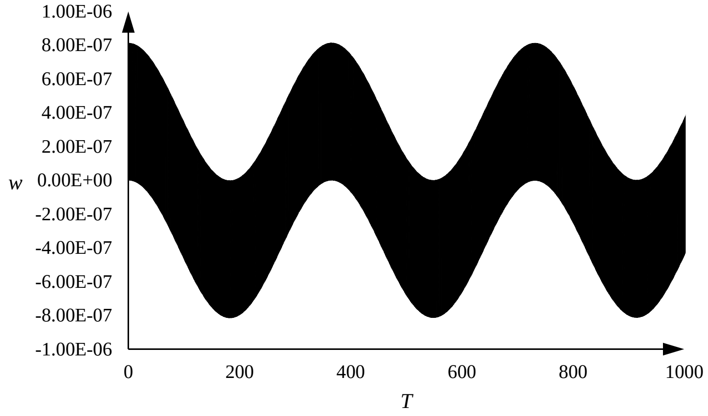

Figure 4. Deflection of the beam for a harmonic load frequency $\Omega_F = \omega_+$ as a function of the number of excitation periods *T* for case 1

7. Conclusions

In this paper the geometrically nonlinear vibrations of a beam interacting with a periodic viscoelastic substratum has been presented. The model equations are obtained by implementing the tolerance averaging technique. It can be observed that the proposed tolerance model makes it possible to investigate nonlinear dynamic problems of structures with periodically varying properties.

This contribution is supported by the National Science Centre of Poland under grant No. 2014/15/B/STS/03155.

References

1. J. Awrejcewicz, A. V. Krysko, J. Mrozowski, O. A. Saltykova, M. V. Zhigalov, *Analysis of regular and chaotic dynamics of the Euler-Bernoulli beams using finite difference and finite element methods*, Acta Mechanica Sinica, **27** (2011) 36 – 43.
2. N. S. Bakhvalov, G. P. Panasenko, *Averaging of processes in periodic media*, Nauka, Moskwa 1984 (in Russian).
3. A. Bensoussan, J. L. Lions, G. Papanicolaou, *Asymptotic analysis for periodic structures*, NorthHolland, Amsterdam 1978.
4. T. Chen, *Investigations on flexural wave propagation of a periodic beam using multi-reflection method*, Archive of Applied Mechanics 83, **2** (2013) 315 – 329.
5. Ł. Domagalski, J. Jędrysiak, *Nonlinear vibrations of periodic beams*, Journal of Theoretical and Applied Mechanics, **54** (2016) 1095 – 1108.
6. M. Hajianmaleki, M. S. Quatu, *Vibrations of straight and curved composite beams: A review*, Composite Structures, **100** (2013) 218 – 232.
7. W. M. He, W. Q. Chen, H. Qiao, *Frequency estimate and adjustment of composite beams with small periodicity*, Composites: Part B, **45** (2013) 742 – 747.

8. J. Jędrysiak, *Modelling of dynamic behaviour of microstructured thin functionally graded plates*, Thin-Walled Structures, **7** (2013) 71 – 102.
9. V. V. Jikov, S. M. Kozlov, O. A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer Verlag, Berlin-Heidelberg-New York 1994.
10. K. Mazur-Śniady, C. Woźniak, E. Wierzbicki, *On the modelling of dynamic problems for plates with a periodic structure*, Archive of Applied Mechanics, **74** (2004) 179 – 90.
11. K. Mazur-Śniady, *Macro-dynamics of micro-periodic elastic beams*, Journal of Theoretical and Applied Mechanics, **31** (1993) 781 – 793.
12. Y. Z. Wang, F. M. Li, *Nonlinear primary resonance of nano beam with axial initial load by nonlocal continuum theory*, International Journal of Non-Linear Mechanics, **61** (2014) 74 – 79.
13. C. Woźniak et al (eds.), *Mathematical modelling and analysis in continuum mechanics of microstructured media*, Silesian University of Technology Press, Gliwice 2010.
14. H. J. Xiang, Z. F. Shi, *Analysis of flexural vibration band gaps in periodic beams using differential quadrature method*, Computers and Structures, **87** (2009) 1559 – 1566.
15. D. L. Yu, J. H. Wen, H. J. Shen, Y. Xiao, X. S. Wen, *Propagation of flexural wave in periodic beam on elastic foundations*, Physics Letters A, **376** (2012) 626 – 630.