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Classification of the norming sets of $\mathcal{L}_s({}^3 l_1^2)$

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Summary. Let $n \in \mathbb{N}, n \geqslant 2$. Let $(E, \|\cdot\|)$ be a Banach space. An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of $T \in \mathcal{L}({}^n E)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$, where $\mathcal{L}({}^n E)$ denotes the space of all continuous symmetric n-linear forms on E. For $T \in \mathcal{L}({}^n E)$, we define

$$\mathrm{Norm}(T) = \big\{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \text{ is a norming point of } T\big\}.$$

$\mathrm{Norm}(T)$ is called the *norming set* of T. In this paper, we classify $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}_s({}^3 l_1^2)$, where $\mathcal{L}_s({}^3 l_1^2)$ denotes the space of all continuous symmetric 3-linear forms on the plane with the l_1-norm.

Keywords
Norming points; symmetric 3-linear forms on on the plane with the l_1-norm

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1. Introduction

In 1961 Bishop and Phelps [2] showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon–Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon–Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jiménez-Sevilla and

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Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}, n \geqslant 2$. We write S_E for the unit sphere of a Banach space E. We denote by $\mathcal{L}(^n E)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup\{|T(x_1, \ldots, x_n)| : (x_1, \ldots, x_n) \in S_E \times \cdots \times S_E\}$. $\mathcal{L}_s(^n E)$ denote the closed subspace of all continuous symmetric n-linear forms on E. An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of T if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$.

For $T \in \mathcal{L}(^n E)$, we define

$$\operatorname{Norm}(T) = \left\{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \text{ is a norming point of } T\right\}.$$

$\operatorname{Norm}(T)$ is called the *norming set* of T. Notice that $(x_1, \ldots, x_n) \in \operatorname{Norm}(T)$ if and only if $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \operatorname{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \ldots, n)$. Indeed, if $(x_1, \ldots, x_n) \in \operatorname{Norm}(T)$, then

$$\left|T(\epsilon_1 x_1, \ldots, \epsilon_n x_n)\right| = \left|\epsilon_1 \ldots \epsilon_n T(x_1, \ldots, x_n)\right| = \left|T(x_1, \ldots, x_n)\right| = \|T\|,$$

which shows that $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \operatorname{Norm}(T)$. If $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \operatorname{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \ldots, n)$, then

$$(x_1, \ldots, x_n) = \left(\epsilon_1(\epsilon_1 x_1), \ldots, \epsilon_n(\epsilon_n x_n)\right) \in \operatorname{Norm}(T).$$

The following examples show that $\operatorname{Norm}(T)$ may be empty or an infinite set.

1.1. Examples.

(i) Let

$$T\left((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\right) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i y_i \in \mathcal{L}_s(^2 c_0).$$

We claim that $\operatorname{Norm}(T) = \varnothing$. Obviously, $\|T\| = 1$. Assume that $\operatorname{Norm}(T) \neq \varnothing$. Let $((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) \in \operatorname{Norm}(T)$. Then,

$$1 = \left|T\left((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\right)\right| \leqslant \sum_{i=1}^{\infty} \frac{1}{2^i} |x_i| |y_i| \leqslant \sum_{i=1}^{\infty} \frac{1}{2^i} = 1,$$

which shows that $|x_i| = |y_i| = 1$ for all $i \in \mathbb{N}$. Hence, $(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}} \notin c_0$. This is a contradiction. Therefore, $\operatorname{Norm}(T) = \varnothing$.

(ii) Let

$$T\left((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\right) = x_1 y_1 \in \mathcal{L}_s(^2 c_0).$$

Then,

$$\operatorname{Norm}(T) = \left\{\left((\pm 1, x_2, x_3, \ldots), (\pm 1, y_2, y_3, \ldots)\right) \in c_0 \times c_0 : |x_j| \leqslant 1, |y_j| \leqslant 1 \text{ for } j \geqslant 2\right\}.$$

A mapping $P: E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}({}^n E)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup\{|P(x)| : \|x\| = 1\}$.

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}({}^n E)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}({}^n E)$, we define

$$\text{Norm}(P) = \left\{x \in E : x \text{ is a norming point of } P\right\}.$$

Norm(P) is called the *norming set* of P. Notice that Norm(P) may be empty or an infinite set.

Kim [7] has classified Norm(P) for every $P \in \mathcal{P}({}^2 l_\infty^2)$, where $l_\infty^2 = \mathbb{R}^2$ with the supremum norm.

If Norm$(T) \neq \emptyset$, $T \in \mathcal{L}({}^n E)$ is called a *norm attaining* n-linear form and if Norm$(P) \neq \emptyset$, $P \in \mathcal{P}({}^n E)$ is called a *norm attaining* n-homogeneoue polynomial (see [3]).

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

It seems to be natural and interesting to study about Norm(T) for $T \in \mathcal{L}({}^n E)$. For $m \in \mathbb{N}$, let $l_1^m := \mathbb{R}^m$ with the l_1-norm and $l_\infty^2 = \mathbb{R}^2$ with the supremum norm. Notice that if $E = l_1^m$ or l_∞^2 and $T \in \mathcal{L}({}^n E)$, Norm$(T) \neq \emptyset$ since S_E is compact. Kim ([6, 8–10]) classified Norm(T) for every $T \in \mathcal{L}_s({}^2 l_\infty^2), \mathcal{L}({}^2 l_\infty^2), \mathcal{L}({}^2 l_1^2), \mathcal{L}_s({}^2 l_1^3)$ or $\mathcal{L}_s({}^2 l_1^2)$. Kim [11] classified Norm(T) for every $T \in \mathcal{L}({}^2 \mathbb{R}^2_{h(w)})$, where $\mathbb{R}^2_{h(w)}$ denotes the plane with the hexagonal norm with weight $0 < w < 1$ $\|(x, y)\|_{h(w)} = \max\{|y|, |x| + (1 - w)|y|\}$. Kim [12] studied and investigated the norming set of a multilinear form on $\mathbb{R}^2$ with a certain norm.

In this paper, we classify Norm(T) for every $T \in \mathcal{L}_s({}^3 l_1^2)$.

2. Results

2.1. Theorem ([10]). *Let $n, m \geqslant 2$. Let $T \in \mathcal{L}({}^m l_1^n)$ with*

$$T\big((x_1^{(1)}, \ldots, x_n^{(1)}), \ldots, (x_1^{(m)}, \ldots, x_n^{(m)})\big) = \sum_{\substack{1 \leqslant i_k \leqslant n \\ 1 \leqslant k \leqslant m}} a_{i_1 \ldots i_m} x_{i_1}^{(1)} \ldots x_{i_m}^{(m)}$$

for some $a_{i_1 \ldots i_m} \in \mathbb{R}$. Then

$$\|T\| = \max\left\{|a_{i_1 \ldots i_m}| : 1 \leqslant i_k \leqslant n,\ 1 \leqslant k \leqslant m\right\}.$$

By simplicity we denote $T = (a_{i_1 \ldots i_m})_{1 \leqslant i_k \leqslant n, 1 \leqslant k \leqslant m}$. We call $a_{i_1 \ldots i_m}$'s the *coefficients* of T. Notice that if $\|T\| = 1$, then $|a_{i_1 \ldots i_m}| \leqslant 1$ for all $1 \leqslant i_k \leqslant n, 1 \leqslant k \leqslant m$.

2.2. Theorem ([10]). *Let $n, m \geqslant 2$. Let $T \in \mathcal{L}({}^m l_1^n)$ be the same as in Theorem 2.1. Suppose that $((t_1^{(1)}, \ldots, t_n^{(1)}), \ldots, (t_1^{(m)}, \ldots, t_n^{(m)})) \in$ Norm(T). If $|a_{i'_1 \ldots i'_m}| < \|T\|$ for $1 \leqslant i'_k \leqslant n, 1 \leqslant k \leqslant m$, then $t_{i'_1}^{(1)} \ldots t_{i'_m}^{(m)} = 0$.*

2.3. Theorem ([12]). *Let $n, m \geqslant 2$. Let $T = (a_{i_1\ldots i_m})_{1\leqslant i_k\leqslant n, 1\leqslant k\leqslant m} \in \mathcal{L}({}^m l_1^n)$ be the same as in Theorem 2.1 with $\|T\| = 1$. Let $\delta_{i_1\ldots i_m} = 1$ if $|a_{i_1\ldots i_m}| = 1$ and $\delta_{i_1\ldots i_m} = 0$ if $|a_{i_1\ldots i_m}| < 1$. We define*

$$T_\delta = \left(a_{i_1\ldots i_m}\delta_{i_1\ldots i_m}\right)_{1\leqslant i_k\leqslant n, 1\leqslant k\leqslant m} \in \mathcal{L}({}^m l_1^n).$$

Then, $\mathrm{Norm}(T) = \mathrm{Norm}(T_\delta)$.

The following shows that we can classify $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}({}^n l_1^m)$ with $\|T\| = 1$ if we have known $\mathrm{Norm}(S)$ for every $S = (b_{i_1\ldots i_m})_{1\leqslant i_k\leqslant n, 1\leqslant k\leqslant m} \in \mathcal{L}({}^n l_1^m)$ such that $\|S\| = 1 = |b_{i_1\ldots i_m}|$ for every $1 \leqslant i_k \leqslant n, 1 \leqslant k \leqslant m$.

2.4. Theorem ([12]). *Let $n, m \geqslant 2$. Let $T = (a_{i_1\ldots i_m})_{1\leqslant i_k\leqslant n, 1\leqslant k\leqslant m} \in \mathcal{L}({}^m l_1^n)$ with $\|T\| = 1$. Define $S = (b_{i_1\ldots i_m})_{1\leqslant i_k\leqslant n, 1\leqslant k\leqslant m} \in \mathcal{L}({}^n l_1^m)$ be such that $b_{i_1\ldots i_m} = a_{i_1\ldots i_m}$ if $|a_{i_1\ldots i_m}| = 1$ and $b_{i_1\ldots i_m} = 1$ if $|a_{i_1\ldots i_m}| < 1$. Then,*

$$\mathrm{Norm}(T) = \bigcap_{\substack{|a_{i'_1\ldots i'_m}|<1 \\ 1\leqslant i'_k\leqslant n, 1\leqslant k\leqslant m}} \Big\{\big((t_1^{(1)},\ldots,t_n^{(1)}),\ldots,(t_1^{(m)},\ldots,t_n^{(m)})\big) \in \mathrm{Norm}(S) : t_{i'_1}^{(1)} = 0, \ldots, \text{ or } t_{i'_m}^{(m)} = 0\Big\}.$$

Proof. For completeness we present a proof.

Let

$$\mathcal{F} = \Big\{\big((t_1^{(1)},\ldots,t_n^{(1)}),\ldots,(t_1^{(m)},\ldots,t_n^{(m)})\big) \in \mathrm{Norm}(S) : t_{i'_1}^{(1)}\ldots t_{i'_m}^{(m)} = 0 \text{ if } |a_{i'_1\ldots i'_m}| < 1 \text{ for some } 1 \leqslant i'_k \leqslant n,\ 1 \leqslant k \leqslant m\Big\}.$$

We will show that $\mathrm{Norm}(T) = \mathcal{F}$. Note that by Theorem 2.1, $\|S\| = 1$.

($\subseteq$). Let $((t_1^{(1)},\ldots,t_n^{(1)}),\ldots,(t_1^{(m)},\ldots,t_n^{(m)})) \in \mathrm{Norm}(T)$. Then

$$\begin{aligned}
1 = \|S\| &\geqslant \left|S\big((t_1^{(1)},\ldots,t_n^{(1)}),\ldots,(t_1^{(m)},\ldots,t_n^{(m)})\big)\right| \\
&= \Big|\sum_{|a_{i'_1\ldots i'_m}|<1} t_{i'_1}^{(1)}\ldots t_{i'_m}^{(m)} + \sum_{|a_{i_1\ldots i_m}|=1} a_{i_1\ldots i_m} t_{i_1}^{(1)}\ldots t_{i_m}^{(m)}\Big| \\
&= \Big|\sum_{|a_{i_1\ldots i_m}|=1} a_{i_1\ldots i_m} t_{i_1}^{(1)}\ldots t_{i_m}^{(m)}\Big| \quad \big(\text{by Theorem 2.2, } t_{i'_1}^{(1)}\ldots t_{i'_m}^{(m)} = 0\big) \\
&= \Big|\sum_{|a_{i'_1\ldots i'_m}|<1} a_{i'_1\ldots i'_m} t_{i'_1}^{(1)}\ldots t_{i'_m}^{(m)} + \sum_{|a_{i_1\ldots i_m}|=1} a_{i_1\ldots i_m} t_{i_1}^{(1)}\ldots t_{i_m}^{(m)}\Big| \\
&= \left|T\big((t_1^{(1)},\ldots,t_n^{(1)}),\ldots,(t_1^{(m)},\ldots,t_n^{(m)})\big)\right| = \|T\| = 1,
\end{aligned}$$

so $((t_1^{(1)},\ldots,t_n^{(1)}),\ldots,(t_1^{(m)},\ldots,t_n^{(m)})) \in \mathrm{Norm}(S)$ satisfying $t_{i'_1}^{(1)}\ldots t_{i'_m}^{(m)} = 0$ if $|a_{i'_1\ldots i'_m}| < 1$. Thus, $((t_1^{(1)},\ldots,t_n^{(1)}),\ldots,(t_1^{(m)},\ldots,t_n^{(m)})) \in \mathcal{F}$.

($\supseteq$). Let $((t_1^{(1)},\ldots,t_n^{(1)}),\ldots,(t_1^{(m)},\ldots,t_n^{(m)}))\in\mathcal{F}$. It follows that

$$\begin{aligned}
1 = \|S\| &= \left|S\big((t_1^{(1)},\ldots,t_n^{(1)}),\ldots,(t_1^{(m)},\ldots,t_n^{(m)})\big)\right|\\
&= \Big|\sum_{|a_{i'_1\ldots i'_m}|<1} t_{i'_1}^{(1)}\ldots t_{i'_m}^{(m)} + \sum_{|a_{i_1\ldots i_m}|=1} a_{i_1\ldots i_m} t_{i_1}^{(1)}\ldots t_{i_m}^{(m)}\Big|\\
&= \Big|\sum_{|a_{i_1\ldots i_m}|=1} a_{i_1\ldots i_m} t_{i_1}^{(1)}\ldots t_{i_m}^{(m)}\Big| \quad \big(\text{since } t_{i'_1}^{(1)}\ldots t_{i'_m}^{(m)}=0\big)\\
&= \left|T_\delta\big((t_1^{(1)},\ldots,t_n^{(1)}),\ldots,(t_1^{(m)},\ldots,t_n^{(m)})\big)\right| \quad (\text{by Theorem 2.3})\\
&= \left|T\big((t_1^{(1)},\ldots,t_n^{(1)}),\ldots,(t_1^{(m)},\ldots,t_n^{(m)})\big)\right| \leqslant \|T\| = 1,
\end{aligned}$$

which implies that $((t_1^{(1)},\ldots,t_n^{(1)}),\ldots,(t_1^{(m)},\ldots,t_n^{(m)}))\in \mathrm{Norm}(T)$. □

2.5. Theorem ([12]). *Let $n\in\mathbb{N}$ and $T\in\mathcal{L}({}^n l_1^2)$ with $\|T\|=1$. Then,*

$$\mathrm{Norm}(T)=\bigcup_{k=1}^{n}\big(A_k^+\cup A_k^-\cup B_{k,1}\cup B_{k,2}\big),$$

where

$$\begin{aligned}
A_k^+ = \Big\{&\big(\pm X_1,\ldots,\pm X_{k-1},\pm(t,1-t),\pm X_{k+1},\ldots,\pm X_n\big)\in\big(S_{\mathcal{L}({}^n l_1^2)}\big)^n :\\
&T\big(X_1,\ldots,X_{k-1},(1,0),X_{k+1},\ldots,\pm X_n\big)\\
&\times T\big(X_1,\ldots,X_{k-1},(0,1),X_{k+1},\ldots,X_n\big)=1,\ 0\leqslant t\leqslant 1\Big\},\\
A_k^- = \Big\{&\big(\pm X_1,\ldots,\pm X_{k-1},\pm(t,-(1-t)),\pm X_{k+1},\ldots,\pm X_n\big)\in\big(S_{\mathcal{L}({}^n l_1^2)}\big)^n :\\
&T\big(X_1,\ldots,X_{k-1},(1,0),X_{k+1},\ldots,\pm X_n\big)\\
&\times T\big(X_1,\ldots,X_{k-1},(0,1),X_{k+1},\ldots,X_n\big)=-1,\ 0\leqslant t\leqslant 1\Big\},\\
B_{k,1} = \Big\{&\big(\pm X_1,\ldots,\pm X_{k-1},\pm(1,0),\pm X_{k+1},\ldots,\pm X_n\big)\in\big(S_{\mathcal{L}({}^n l_1^2)}\big)^n :\\
&1=\big|T\big(X_1,\ldots,X_{k-1},(1,0),X_{k+1},\ldots,\pm X_n\big)\big|\\
&>\big|T\big(X_1,\ldots,X_{k-1},(0,1),X_{k+1},\ldots,X_n\big)\big|\Big\},\\
B_{k,2} = \Big\{&\big(\pm X_1,\ldots,\pm X_{k-1},\pm(0,1),\pm X_{k+1},\ldots,\pm X_n\big)\in\big(S_{\mathcal{L}({}^n l_1^2)}\big)^n :\\
&1=\big|T\big(X_1,\ldots,X_{k-1},(0,1),X_{k+1},\ldots,\pm X_n\big)\big|\\
&>\big|T\big(X_1,\ldots,X_{k-1},(1,0),X_{k+1},\ldots,X_n\big)\big|\Big\}.
\end{aligned}$$

Let $T\big((x_1,y_1),(x_2,y_2),(x_3,y_3)\big)=ax_1x_2x_3+by_1y_2y_3+c\big(x_2x_3y_1+x_1x_3y_2+x_1x_2y_3\big)+d(x_3y_1y_2+x_2y_1y_3+x_1y_2y_3)\in\mathcal{L}_s({}^3 l_1^2)$ be such that $\|T\|=1$, $a,b,c,d\in\mathbb{R}$. Note that we may assume that $a\geqslant 0, b\geqslant 0$. Indeed, if $a<0$, we take $-T$. Thus we may assume that $a\geqslant 0$. If $b<0$, we take $T_1\in\mathcal{L}_s({}^3 l_1^2)$ such that

$$T_1\big((x_1,y_1),(x_2,y_2),(x_3,y_3)\big):=T\big((x_1,-y_1),(x_2,-y_2),(x_3,-y_3)\big)$$

for $x_j, y_j\in\mathbb{R}$ $(j=1,2,3)$.

In order to find $\mathrm{Norm}(T)$ with $\|T\| = 1$, by Theorems 2.1 and 2.4 it suffices to assume that $1 = a = |b| = |c| = |d|$.

Let $\mathcal{W} \subseteq (S_{l_1^2})^3$. We denote

$$\mathrm{Sym}(\mathcal{W}) := \left\{(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}) : X_1, X_2, X_3 \in \mathcal{W}, \sigma \text{ is a permutation on } \{1,2,3\}\right\}.$$

We are in a position to classify $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}_s({}^3 l_1^2)$.

2.6. Theorem. *Let* $T((x_1,y_1),(x_2,y_2),(x_3,y_3)) = ax_1x_2x_3+by_1y_2y_3+c(x_2x_3y_1+x_1x_3y_2+x_1x_2y_3) + d(x_3y_1y_2 + x_2y_1y_3 + x_1y_2y_3) \in \mathcal{L}_s({}^3 l_1^2)$ *such that* $\|T\| = 1 = a = b = |c| = |d|$. *Then the following statements hold.*

(i) *If* $a = b = c = d = 1$, *then*

$$\mathrm{Norm}(T) = \left\{\left(\pm(t, 1-t), \pm(s, 1-s), \pm(u, 1-u)\right) : 0 \leqslant t, s, u \leqslant 1\right\}.$$

(ii) *If* $a = b = -c = d = 1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{&\left(\pm(t, -(1-t)), \pm(s, -(1-s)), \pm(1,0)\right),\\ &\left(\pm(t, 1-t), \pm(0,1), \pm(0,1)\right) : 0 \leqslant t, s \leqslant 1\Big\}\Big).\end{aligned}$$

(iii) *If* $a = b = c = -d = 1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{&\left(\pm(t, -(1-t)), \pm(s, -(1-s)), \pm(0,1)\right),\\ &\left(\pm(t, 1-t), \pm(1,0), \pm(1,0)\right) : 0 \leqslant t, s \leqslant 1\Big\}\Big).\end{aligned}$$

(iv) *If* $a = b = -c = -d = 1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{&\left(\left(\pm(t, -(1-t)), \pm(1,0), \pm(1,0)\right),\right.\\ &\left(\pm(t, -(1-t)), \pm(0,1), \pm(0,1)\right), \left(\pm(t, 1-t) \pm (1,0), \pm(0,1)\right) : 0 \leqslant t \leqslant 1\Big\}\Big).\end{aligned}$$

Proof. We use Theorem 2.5.

The proof of (i) is immediate. Let $(x_k, y_k) \in \{(1,0),(0,1)\}$ for $k = 1,2,3$.

(ii). $a = b = -c = d = 1$. Note that if $(0,1)$ appears once in the term $T((x_1,y_1),(x_2,y_2),(x_3,y_3))$, then

$$T\big((x_1,y_1),(x_2,y_2),(x_3,y_3)\big) = -1$$

and otherwise

$$T\big((x_1,y_1),(x_2,y_2),(x_3,y_3)\big) = 1.$$

By symmetry of T we may consider three subcases as following:

Subcase 1. $(x_j, y_j) = (1,0)$ for $j = 2,3$. Note that

$$T\big((1,0),(1,0),(1,0)\big) = 1 = -T\big((0,1),(1,0),(1,0)\big).$$

Thus,

$$T\big((t,-(1-t)),(1,0),(1,0)\big) = 1 \quad \text{for } 0 \leqslant t \leqslant 1.$$

Note that

$$T\big((t,-(1-t)),(0,1),(1,0)\big) = -1 \quad \text{for } 0 \leqslant t \leqslant 1.$$

Thus

$$T\big((t,-(1-t)),(s,-(1-s)),(1,0)\big) = 1 \quad \text{for } 0 \leqslant t,s \leqslant 1.$$

Note that for $0 < t, s < 1$,

$$1 > \left|T\big((t,-(1-t)),(s,-(1-s)),(0,1)\big)\right|$$

and

$$T\big((0,1),(0,1),(0,1)\big) = 1.$$

Thus, the set of the norming points appeared in Subcase 1 is

$$\mathrm{Sym}\Big(\big\{\big(\pm(t,-(1-t)),\pm(s,-(1-s)),\pm(1,0)\big),\big(\pm(0,1),\pm(0,1),\pm(0,1)\big): 0 \leqslant t,\ s \leqslant 1\big\}\Big).$$

Subcase 2. $(x_2, y_2) = (0,1)$ and $(x_3, y_3) = (1,0)$. Note that

$$T\big((1,0),(0,1),(1,0)\big) = -1 = -T\big((0,1),(0,1),(1,0)\big).$$

Thus, for $0 \leqslant t \leqslant 1$,

$$T\big((t,-(1-t)),(0,1),(1,0)\big) = -1 = -T\big((t,-(1-t)),(1,0),(1,0)\big).$$

Thus

$$T\big((t,-(1-t)),(s,-(1-s)),(1,0)\big) = 1 \quad \text{for } 0 \leqslant t,s \leqslant 1.$$

Note that for $0 < t, s < 1$,

$$1 > \left|T\big((t,-(1-t)),(s,-(1-s)),(0,1)\big)\right|$$

and

$$T\big((0,1),(0,1),(0,1)\big) = 1.$$

By the same argument as in Subcase 1, the set of the norming points appeared in Subcase 2 is

$$\mathrm{Sym}\Big(\big\{\big(\pm(t,-(1-t)),\pm(s,-(1-s)),\pm(1,0)\big),\big(\pm(0,1),\pm(0,1),\pm(0,1)\big): 0 \leqslant t,\ s \leqslant 1\big\}\Big).$$

Subcase 3. $(x_j, y_j) = (0,1)$ for $j = 2, 3$. Note that

$$T\big((1,0),(0,1),(0,1)\big) = 1 = T\big((0,1),(0,1),(0,1)\big).$$

Thus, $T((t,1-t),(0,1),(0,1)) = 1$ for $0 \leqslant t \leqslant 1$ and Thus $|T((t,1-t),(0,1),(1,0))| < 1$ for $0 < t < 1$. Note that

$$T\big((1,0),(0,1),(1,0)\big) = -1 = -T\big((0,1),(0,1),(1,0)\big).$$

Thus, the set of the norming points appeared in Subcase 3 is

$$\mathrm{Sym}\Big(\Big\{\big(\pm(t,1-t),\pm(0,1),\pm(0,1)\big),\big(\pm(1,0),\pm(1,0),\pm(0,1)\big):0 \leqslant t \leqslant 1\Big\}\Big).$$

Collecting the norming sets of T in Subcases 1–3, we have

$$\begin{aligned}\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{&\big(\pm(t,-(1-t)),\pm(s,-(1-s)),\pm(1,0)\big),\\ &\big(\pm(t,1-t),\pm(0,1),\pm(0,1)\big):0 \leqslant t,\ s \leqslant 1\Big\}\Big).\end{aligned}$$

(iii). $a = b = c = -d = 1$. Note that if $(0,1)$ appears twice in the term $T\big((x_1,y_1),(x_2,y_2),(x_3,y_3)\big)$, then

$$T\big((x_1,y_1),(x_2,y_2),(x_3,y_3)\big) = -1$$

and otherwise

$$T\big((x_1,y_1),(x_2,y_2),(x_3,y_3)\big) = 1.$$

By symmetry of T we may consider three subcases as following.

Subcase 1. $(x_j, y_j) = (1,0)$ for $j = 2, 3$. Note that

$$T\big((1,0),(1,0),(1,0)\big) = 1 = T\big((0,1),(1,0),(1,0)\big).$$

Thus,

$$T\big((t,1-t),(1,0),(1,0)\big) = 1 \quad \text{for } 0 \leqslant t \leqslant 1.$$

Note that for $0 < t < 1$,

$$1 > \big|T\big((t,1-t),(0,1),(1,0)\big)\big| = \big|T\big((t,1-t),(1,0),(0,1)\big)\big|$$

and $T\big((1,0),(0,1),(0,1)\big) = -1$. Thus, the set of the norming points appeared in Subcase 1 is

$$\mathrm{Sym}\Big(\Big\{\big(\pm(t,1-t),\pm(1,0),\pm(1,0)\big),\big(\pm(1,0),\pm(0,1),\pm(0,1)\big):0 \leqslant t \leqslant 1\Big\}\Big).$$

Subcase 2. $(x_2, y_2) = (0,1)$ and $(x_3, y_3) = (1,0)$. Note that

$$T\big((1,0),(0,1),(1,0)\big) = 1 = -T\big((0,1),(0,1),(1,0)\big).$$

Thus, for $0 \leqslant t \leqslant 1$, $T((t,-(1-t)),(0,1),(1,0)) = 1$ and $1 > |T((t,-(1-t)),(1,0),(1,0))|$ for $0 < t < 1$, $T((1,0),(1,0),(1,0)) = 1$. Thus, the set of the norming points appeared in Subcase 2 is

$$\operatorname{Sym}\Big(\big\{\big(\pm(t,-(1-t)),\pm(0,1),\pm(1,0)\big),\big(\pm(1,0),\pm(1,0),\pm(1,0)\big): 0 \leqslant t \leqslant 1\big\}\Big).$$

Subcase 3. $(x_j, y_j) = (0,1)$ for $j = 2,3$. Note that

$$T\big((1,0),(0,1),(0,1)\big) = -1 = -T\big((0,1),(0,1),(0,1)\big).$$

Thus, $0 \leqslant t \leqslant 1$,

$$-T\big((t,-(1-t)),(0,1),(0,1)\big) = 1 = T\big((t,-(1-t)),(1,0),(0,1)\big).$$

Thus for $0 \leqslant t, s \leqslant 1$,

$$T\big((t,-(1-t)),(s,-(1-s)),(0,1)\big) = 1.$$

Note that for $0 < t, s < 1$,

$$\big|T\big((t,-(1-t)),(s,-(1-s)),(1,0)\big)\big| < 1$$

and $T\big((1,0),(1,0),(1,0)\big) = 1$. Thus, the set of the norming points appeared in Subcase 3 is

$$\operatorname{Sym}\Big(\big\{\big(\pm(t,-(1-t)),\pm(s,-(1-s)),\pm(0,1)\big),\big(\pm(1,0),\pm(1,0),\pm(1,0)\big): 0 \leqslant t, s \leqslant 1\big\}\Big).$$

Collecting the norming sets of T in Subcases 1–3, we have

$$\begin{aligned}\operatorname{Norm}(T) = \operatorname{Sym}\Big(\big\{&\big(\pm(t,-(1-t)),\pm(s,-(1-s)),\pm(0,1)\big),\\ &\big(\pm(t,1-t),\pm(1,0),\pm(1,0)\big) : 0 \leqslant t, s \leqslant 1\big\}\Big).\end{aligned}$$

(iv). $a = b = -c = -d = 1$. Note that if $(0,1)$ appears once or twice in the term $T((x_1, y_1),$ $(x_2, y_2), (x_3, y_3))$, then

$$T\big((x_1,y_1),(x_2,y_2),(x_3,y_3)\big) = -1$$

and otherwise

$$T\big((x_1,y_1),(x_2,y_2),(x_3,y_3)\big) = 1.$$

By symmetry of T we may consider three subcases as following:

Subcase 1. $(x_j, y_j) = (1,0)$ for $j = 2,3$. Note that

$$T\big((1,0),(1,0),(1,0)\big) = 1 = -T\big((0,1),(1,0),(1,0)\big).$$

Thus,

$$T\big((t,-(1-t)),(1,0),(1,0)\big)=1 \quad \text{for } 0\leqslant t\leqslant 1.$$

Note that for $0<t<1$,

$$1>\big|T\big((t,-(1-t)),(0,1),(1,0)\big)\big|=\big|T\big((t,-(1-t)),(1,0),(0,1)\big)\big|$$

and $T\big((1,0),(0,1),(0,1)\big)=-1$. Thus, the set of the norming points appeared in Subcase 1 is

$$\mathrm{Sym}\big(\big\{\big(\pm(t,-(1-t)),\pm(1,0),\pm(1,0)\big),\big(\pm(1,0),\pm(0,1),\pm(0,1)\big):0\leqslant t\leqslant 1\big\}\big).$$

Subcase 2. $(x_2,y_2)=(0,1)$ and $(x_3,y_3)=(1,0)$. Note that

$$T\big((1,0),(0,1),(1,0)\big)=-1=T\big((0,1),(0,1),(1,0)\big).$$

Thus,

$$T\big((t,1-t),(0,1),(1,0)\big)=-1 \quad \text{for } 0\leqslant t\leqslant 1.$$

Note that for $0<t<1$,

$$1>\big|T\big((t,1-t),(1,0),(1,0)\big)\big|=\big|T\big((t,1-t),(0,1),(0,1)\big)\big|$$

and $T((1,0),(1,0),(1,0))=1=T((1,0),(0,1),(0,1))$. Thus, the set of the norming points appeared in Subcase 2 is

$$\mathrm{Sym}\big(\big\{\big(\pm(t,1-t),\pm(0,1),\pm(1,0)\big),\big(\pm(1,0),\pm(1,0),\pm(1,0)\big),\\ \big(\pm(1,0),\pm(0,1),\pm(0,1)\big):0\leqslant t\leqslant 1\big\}\big).$$

Subcase 3. $(x_j,y_j)=(0,1)$ for $j=2,3$. Note that

$$-T\big((1,0),(0,1),(0,1)\big)=1=T\big((0,1),(0,1),(0,1)\big).$$

Thus,

$$T\big((t,-(1-t)),(0,1),(0,1)\big)=-1 \quad \text{for } 0\leqslant t\leqslant 1.$$

Note that for $0<t<1$,

$$1>\big|T\big((t,-(1-t)),(1,0),(0,1)\big)\big|$$

and $T\big((1,0),(1,0),(0,1)\big)=1$. Thus, the set of the norming points appeared in Subcase 3 is

$$\mathrm{Sym}\big(\big\{\big(\pm(t,-(1-t)),\pm(0,1),\pm(0,1)\big),\big(\pm(1,0),\pm(1,0)\pm(0,1)\big):0\leqslant t\leqslant 1\big\}\big).$$

Collecting the norming sets of T in Subcases 1–3, we have

$$\text{Norm}(T) = \text{Sym}\Big(\big\{\big(\pm(t,-(1-t)),\pm(1,0),\pm(1,0)\big),\big(\pm(t,-(1-t)),\pm(0,1),\pm(0,1)\big),\\ \big(\pm(t,1-t)\pm(1,0),\pm(0,1)\big) : 0 \leqslant t \leqslant 1\big\}\Big).$$

We complete the proof. □

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