Opuscula Math. 41, no. 1 (2021), 145–155
https://doi.org/10.7494/OpMath.2021.41.1.145 Opuscula Mathematica

EXPONENTIAL STABILITY RESULTS FOR VARIABLE DELAY DIFFERENCE EQUATIONS

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Communicated by Josef Diblík

Abstract. Sufficient conditions that guarantee exponential decay to zero of the variable delay difference equation

$$x(n+1) = a(n)x(n) + b(n)x(n-g(n))$$

are obtained. These sufficient conditions are deduced via inequalities by employing Lyapunov functionals. In addition, a criterion for the instability of the zero solution is established. The results in the paper generalizes some results in the literature.

Keywords: exponential stability, Lyapunov functional, instability.

Mathematics Subject Classification: 34D20, 34D40, 34K20.

1. INTRODUCTION

The study of the qualitative properties of difference equations have gained the attention of many researchers in recent times, see [1, 2, 5, 6], and [7] and the references cited therein. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{Z}^+$ denote the set of positive integers. In this paper we consider the scalar linear difference equation with variable delay

$$x(n+1) = a(n)x(n) + b(n)x(n-g(n)), \tag{1.1}$$

where $a, b : \mathbb{Z}^+ \to \mathbb{R}$ and $0 < g(n) \leq h$, for some positive constant h and the function $n - g(n)$ is strictly increasing so that it has an inverse $r(n)$. We will obtain some inequalities regarding the solutions of (1.1) by employing Lyapunov functionals. These inequalities can be used to deduce exponential asymptotic stability of the zero solution. Also, by means of a Lyapunov functional an instability criterion of the zero solution of equation (1.1) will be provided.

In [7], Raffoul obtained sufficient conditions that guarantee exponential stability and instability of the zero solution of equation (1.1) when $g(n) = h$ for some constant delay h. In view of the fact that the delay in (1.1) is a variable, the results obtained in [7] does not hold for (1.1). Thus, our goal in this paper is to obtain exponential stability and instability results for (1.1) by using Lyapunov functionals.

Let $\psi : [-h, 0] \to (-\infty, \infty)$ be a given bounded initial function with

$$\|\psi\| = \max_{-h \le s \le 0} |\psi(s)|.$$

We also denote the norm of a function $\varphi : [-h, \infty) \to (-\infty, \infty)$ with

$$\|\varphi\| = \sup_{-h \le s \le \infty} |\varphi(s)|.$$

We say that $x(n) \equiv x(n, n_0, \psi)$ is a solution of (1.1) if $x(n)$ satisfies (1.1) for $n \ge n_0$ and $x_{n_0} = x(n_0 + s) = \psi(s)$, $s \in [-h, 0]$.

Throughout this paper Δ denotes the forward difference operator

$$\Delta x(n) = x(n+1) - x(n)$$

for any sequence $\{x(n), \ n = 0, 1, 2, \ldots\}$. Also, we define the operator E by $Ex(n) = x(n+1)$.

Lemma 1.1. *Assume that $r(n)$ is the inverse of $n - g(n)$. Then equation* (1.1) *is equivalent to*

$$\Delta x(n) = \Big(a(n) + b(r(n)) - 1\Big)x(n) - \Delta_n \sum_{s=n-g(n)}^{n-1} b(r(s))x(s). \tag{1.2}$$

Proof. Considering the second term on the right hand side of (1.2) we have

$$\begin{aligned}
\Delta_n \sum_{s=n-g(n)}^{n-1} b(r(s))x(s) &= \sum_{s=n+1-g(n+1)}^{n} b(r(s))x(s) - \sum_{s=n-g(n)}^{n-1} b(r(s))x(s) \\
&= b(r(n))x(n) + \sum_{s=n+1-g(n+1)}^{n-1} b(r(s))x(s) \\
&\quad - b(r(n-g(n)))x(n-g(n)) - \sum_{s=E(n-g(n))}^{n-1} b(r(s))x(s) \\
&= b(r(n))x(n) + \sum_{s=E(n-g(n))}^{n-1} b(r(s))x(s) \\
&\quad - b(n)x(n-g(n)) - \sum_{s=E(n-g(n))}^{n-1} b(r(s))x(s) \\
&= b(r(n))x(n) - b(n)x(n-g(n))
\end{aligned} \tag{1.3}$$

Substituting (1.3) into (1.2) we obtain

$$\begin{aligned}\Delta x(n) &= \Big(a(n) + b(r(n)) - 1\Big)x(n) - b(r(n))x(n) + b(n)x(n - g(n)) \\ &= a(n)x(n) - x(n) + b(n)x(n - g(n)).\end{aligned}$$

This implies that

$$x(n+1) - x(n) = a(n)x(n) - x(n) + b(n)x(n - g(n))$$

Thus,

$$x(n+1) = a(n)x(n) + b(n)x(n - g(n)).$$

This completes the proof. □

Definition 1.2. The zero solution of (1.1) is said to be exponentially stable if any solution $x(n, n_0, \psi)$ of (1.1) satisfies

$$|x(n, n_0, \psi)| \le C(\|\psi\|, n_0)\zeta^{\gamma(n-n_0)}, \text{ for all } n \ge n_0,$$

where ζ is a constant with $0 < \zeta < 1, C : \mathbb{R}^+ \times \mathbb{Z}^+ \to \mathbb{R}^+$, and γ is a positive constant. The zero solution of (1.1) is said to be uniformly exponentially stable if C is independent of n_0.

It must be noted that if $u(n)$ is a sequence, then

$$\Delta u^2(n) = u(n+1)\Delta u(n) + u(n)\Delta u(n).$$

For more on the calculus of difference equations we refer to [3] and [4].

2. EXPONENTIAL STABILITY

In this section we obtain inequalities that can be used to deduce the exponential stability of (1.1). To simplify notation we let

$$Q(n) = a(n) + b(r(n)) - 1.$$

Lemma 2.1. *Assume that $r(n)$ is the inverse of $n - g(n)$ and for $\delta > 0$,*

$$-\frac{\delta}{\delta h + g(n)} \le Q(n) \le -\delta h b^2(r(n)) - Q^2(n), \tag{2.1}$$

holds. If

$$V(n) = \Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]^2 + \delta \sum_{s=-h}^{-1} \sum_{z=n+s}^{n-1} b^2(r(z))x^2(z), \tag{2.2}$$

then based on the solutions of (1.1) *we have*

$$\Delta V(n) \le Q(n)V(n). \tag{2.3}$$

Proof. Let $x(n, n_0, \psi)$ be a solution of (1.1) and let $V(n)$ be defined by (2.2). It must also be noted that in view of condition (2.1), $Q(n) < 0$ for all $n \ge 0$.

Then based on the solutions of (1.1) we have

$$
\begin{aligned}
\Delta V(n) &= \Big[x(n+1) + \sum_{s=n+1-g(n+1)}^{n} b(r(s))x(s)\Big]\Delta\Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big] \\
&\quad + \Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]\Delta\Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big] \\
&\quad + \delta \sum_{s=-h}^{-1} \Big[b^2(r(n))x^2(n) - b^2(r(n+s))x^2(n+s)\Big] \\
&= \Big[x(n+1) + \sum_{s=n+1-g(n+1)}^{n} b(r(s))x(s)\Big]Q(n)x(n) \\
&\quad + \Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]Q(n)x(n) \\
&\quad + \delta \sum_{s=-h}^{-1} \Big[b^2(r(n))x^2(n) - b^2(r(n+s))x^2(n+s)\Big] \\
&= \Big[\big(a(n) + b(r(n))\big)x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]Q(n)x(n) \\
&\quad + \Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]Q(n)x(n) \\
&\quad + \delta h b^2(r(n))x^2(n) - \delta \sum_{s=-h}^{-1} b^2(r(n+s))x^2(n+s) \\
&= \Big[\big(Q(n) + 1\big)x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]Q(n)x(n) \\
&\quad + \Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]Q(n)x(n) \\
&\quad + \delta h b^2(r(n))x^2(n) - \delta \sum_{u=n-h}^{n-1} \Big[b^2(r(u))x^2(u)\Big] \\
&= Q(n)x^2(n) + 2Q(n) \sum_{s=n-g(n)}^{n-1} b(r(s))x(s) \\
&\quad + \big(Q(n) + Q^2(n) + \delta h b^2(r(n))\big)x^2(n) \\
&\quad - \delta \sum_{s=-h}^{-1} b^2(r(n+s))x^2(n+s)
\end{aligned}
\tag{2.4}
$$

$$= Q(n)V(n) + \big(Q(n) + Q^2(n) + \delta h b^2(r(n))\big)x^2(n)$$
$$- \delta \sum_{s=-h}^{-1} b^2(r(n+s))x^2(n+s) - Q(n)\left(\sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\right)^2$$
$$- Q(n)\delta \sum_{s=-h}^{-1}\sum_{z=n+s}^{n-1} b^2(r(z))x^2(z).$$

In the next few steps, we consider some of the terms in (2.4) in order to obtain a simplified version of (2.4). First, if we let $u = n+s$ then

$$\sum_{s=-h}^{-1} b^2(r(n+s))x^2(n+s) = \sum_{u=n-h}^{n-1} b^2(r(u))x^2(u) \tag{2.5}$$

Also, applying the Hölder inequality, we have

$$\left(\sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\right)^2 \le g(n) \sum_{s=n-g(n)}^{n-1} b^2(r(s))x^2(s)$$
$$\le g(n) \sum_{s=n-h}^{n-1} b^2(r(s))x^2(s).$$

Finally, we easily observe that

$$\sum_{s=-h}^{-1}\sum_{z=n+s}^{n-1} b^2(r(z))x^2(z) \le h \sum_{s=n-h}^{n-1} b^2(r(s))x^2(s). \tag{2.6}$$

Substituting (2.5), (2.6) and (2.6) in (2.4) we obtain

$$\Delta V(n) \le Q(n)V(n) + \big(Q(n) + Q^2(n) + \delta h b^2(r(n))\big)x^2(n)$$
$$+ \big[-(\delta h + g(n))Q(n) - \delta\big] \sum_{s=n-h}^{n-1} b^2(r(s))x^2(s)$$
$$\le Q(n)V(n).$$

□

Theorem 2.2. *Suppose the hypothesis of Lemma 2.1 hold. Then any solution $x(n) = x(n, n_0, \psi)$ of (1.1) satisfies the exponential inequality*

$$|x(n)| \le \sqrt{\frac{h+\delta}{\delta} V(n_0) \prod_{s=n_0}^{n-1} (b(r(s)) + a(s))} \tag{2.7}$$

for $n \ge n_0$.

Proof. Let $V(n)$ be defined by (2.2). That is,

$$V(n) = \Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]^2 + \delta \sum_{s=-h}^{-1}\sum_{z=n+s}^{n-1} b^2(r(z))x^2(z).$$

Now, changing the order of summation in the second term of $V(n)$ we obtain

$$\begin{aligned}\delta \sum_{s=-h}^{-1} \sum_{z=n+s}^{n-1} b^2(r(z))x^2(z) &= \delta \sum_{z=n-h}^{n-1} \sum_{s=-h}^{z-n} b^2(r(z))x^2(z) \\ &= \delta \sum_{z=n-h}^{n-1} b^2(r(z))x^2(z)(z-n+h+1) \\ &\geq \delta \sum_{z=n-h}^{n-1} b^2(r(z))x^2(z) \\ &\geq \delta \sum_{z=n-g(n)}^{n-1} b^2(r(z))x^2(z),\end{aligned}$$

where we have used the fact that if $n-h \leq z \leq n-1$ then $1 \leq z-n+h+1 \leq h$ and $n-h \leq n-g(n)$.

Also, we note that

$$\left(\sum_{z=n-g(n)}^{n-1} b(r(z))x(z) \right)^2 \leq h \sum_{z=n-g(n)}^{n-1} b^2(r(z))x^2(z).$$

Hence,

$$\delta \sum_{s=-h}^{-1} \sum_{z=n+s}^{n-1} b^2(r(z))x^2(z) \geq \frac{\delta}{h} \left(\sum_{z=n-g(n)}^{n-1} b(r(z))x(z) \right)^2.$$

Thus,

$$\begin{aligned}V(t) &\geq \left[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s) \right]^2 + \frac{\delta}{h} \left(\sum_{z=n-g(n)}^{n-1} b(r(z))x(z) \right)^2 \\ &= \frac{\delta}{h+\delta} x^2(t) + \left[\sqrt{\frac{h}{h+\delta}} x(t) + \sqrt{\frac{h+\delta}{h}} \sum_{z=n-g(n)}^{n-1} b(r(z))x(z) \right]^2 \geq \frac{\delta}{h+\delta} x^2(t).\end{aligned}$$

But

$$V(n) \leq V(n_0) \prod_{s=n_0}^{n-1} \big(b(r(s)) + a(s) \big).$$

This implies that

$$\frac{\delta}{h+\delta} x^2(t) \leq V(n_0) \prod_{s=n_0}^{n-1} \big(b(r(s)) + a(s) \big).$$

Hence,

$$|x(n)| \leq \sqrt{\frac{h+\delta}{\delta} V(n_0) \prod_{s=n_0}^{n-1} (b(r(s)) + a(s)).}$$

This completes the proof. $\square$

Corollary 2.3. *Suppose that the hypotheses of Theorem 2.2 hold. Suppose also that there exists a number $\alpha < 1$ such that*

$$0 < b(r(n)) + a(n) \leq \alpha. \tag{2.8}$$

Then the zero solution of (1.1) *is exponentially stable.*

Proof. It follows from (2.7) that

$$\begin{aligned} |x(n)| &\leq \sqrt{\frac{h+\delta}{\delta} V(n_0) \prod_{s=n_0}^{n-1} (b(r(s)) + a(s))} \\ &\leq \sqrt{\frac{h+\delta}{\delta} V(n_0) \alpha^{n-n_0}} \end{aligned}$$

for $n \geq n_0$. Since $\alpha \in (0,1)$ the proof is complete. □

Before we end this section we compare our results with some results in the literature. For the sake of comparison we consider the difference equation with variable delay

$$x(n+1) = 1.3x(n) - 0.4x\Big(n - \frac{1}{n+1}\Big). \tag{2.9}$$

Then with $a = 1.3, b = -0.4, g(n) = \frac{1}{n+1}$ and $\delta = 0.5$ it can easily be verified that conditions (2.1) and (2.8) are satisfied for equation (2.9) and so we conclude that the zero solution of (2.8) is exponentially stable. However, in view of the fact that equation (2.9) contains a variable delay, the results in [7] does not apply to this equation. In [7] however, the author demonstrated that the results obtained in the paper improved the results of [1] and [2]. This implies that our results improve the results obtained in [1, 2, 7]. Moreover, in [5] and [6] the authors required that

$$\prod_{s=0}^{n-1} a(s) \to 0, \quad \text{as} \quad n \to \infty$$

for the asymptotic stability of the zero solution of (2.9). Clearly, this condition is not satisfied by (2.9) since $a(s) = 1.3$ and yet we have been able to conclude exponential stability for the zero solution of (2.9).

3. INSTABILITY CRITERIA

In this section we consider the problem of finding a criteria for instability of the zero solution of (1.1). A suitable Lyapunov functional will be used to obtain the instability criteria.

Theorem 3.1. *Suppose that $r(n)$ is the inverse of $n - g(n)$, and let $\rho > h$ be a constant. Assume that $Q(n) > 0$ such that*

$$Q^2(n) + Q(n) - \rho b^2(r(n)) \geq 0. \tag{3.1}$$

If

$$V(n) = \Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]^2 - \rho \sum_{s=n-h}^{n-1} b^2(r(s+h))x^2(s) \tag{3.2}$$

then, based on the solutions of (1.1) *we have*

$$\Delta V(n) \geq Q(n)V(n).$$

Proof. Let $x(n, n_0, \psi)$ be a solution of (1.1) and let $V(n)$ be defined by (3.2). Then based on the solutions of (1.1) we have

$$\begin{aligned}
\Delta V(n) &= \Big[x(n+1) + \sum_{s=n+1-g(n+1)}^{n} b(r(s))x(s)\Big]\Delta\Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big] \\
&\quad + \Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]\Delta\Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big] \\
&\quad + \rho\Big[b^2(r(n+h))x^2(n) - b^2(r(n))x^2(n-h)\Big] \\
&= \Big[x(n+1) + \sum_{s=n+1-g(n+1)}^{n} b(r(s))x(s)\Big]Q(n)x(n) \\
&\quad + \Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]Q(n)x(n) \\
&\quad + \rho\Big[b^2(r(n+h))x^2(n) - b^2(r(n))x^2(n-h)\Big] \\
&= \Big[\big(a(n) + b(r(n))\big)x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]Q(n)x(n) \\
&\quad + \Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]Q(n)x(n) \\
&\quad - \delta b^2(r(n+h))x^2(n) + \delta b^2(r(n))x^2(n-h) \\
&= \Big[\big(Q(n) + 1\big)x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]Q(n)x(n) \\
&\quad + \Big[x(n) + \sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]Q(n)x(n) \\
&\quad - \delta b^2(r(n+h))x^2(n) + \delta b^2(r(n))x^2(n-h)
\end{aligned}$$

$$
\begin{aligned}
&= Q(n)x^2(n) + 2Q(n)\sum_{s=n-g(n)}^{n-1} b(r(s))x(s) \\
&\quad + \big(Q(n)+Q^2(n)-\delta b^2(r(n))\big)x^2(n) + \delta b^2(r(n))x^2(n-h) \\
&= Q(n)V(n) + \big(Q(n)+Q^2(n)-\delta b^2(r(n+h))\big)x^2(n) \\
&\quad + \delta b^2(r(n))x^2(n-h) - Q(n)\Big(\sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big)^2 \\
&\quad + Q(n)\rho\sum_{s=n-h}^{n-1} b^2(r(s+h))x^2(s) \\
&\ge Q(n)V(n),
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
\Bigg(\sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Bigg)^2 &\le g(n)\sum_{s=n-g(n)}^{n-1} b^2(r(s))x^2(s) \\
&\le h\sum_{s=n-h}^{n-1} b^2(r(s))x^2(s) \le \rho\sum_{s=n-h}^{n-1} b^2(r(s))x^2(s).
\end{aligned}
$$

This completes the proof. □

Theorem 3.2. *Suppose the hypothesis of Theorem 3.1 hold. Then the zero solution of* (1.1) *is unstable, provided that*

$$\prod_{s=0}^{\infty}(b(r(s))+a(s)) = \infty.$$

Proof. We have from Theorem 3.1 that

$$\Delta V(n) \ge Q(n)V(n),$$

which implies that

$$V(n) \ge V(n_0)\prod_{s=n_0}^{\infty}(b(r(s))+a(s)). \tag{3.3}$$

Using the definition of $V(n)$ in (3.2) we have that

$$
\begin{aligned}
V(n) = x^2(n) + 2x(n)\sum_{s=n-g(n)}^{n-1} b(r(s))x(s) + \Bigg[\sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Bigg]^2 \\
- \rho\sum_{s=n-h}^{n-1} b^2(r(s+h))x^2(s).
\end{aligned} \tag{3.4}
$$

Now let $\beta = \rho - h$, then from

$$\left(\frac{\sqrt{h}}{\sqrt{\beta}}a - \frac{\sqrt{\beta}}{\sqrt{h}}b\right)^2 \geq 0,$$

we have

$$2ab \leq \frac{h}{\beta}a^2 + \frac{\beta}{h}b^2.$$

It follows from this inequality that

$$\begin{aligned} 2x(n)\sum_{s=n-g(n)}^{n-1} b(r(s))x(s) &\leq 2|x(n)|\Big|\sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big| \\ &\leq \frac{h}{\beta}x^2(n) + \frac{\beta}{h}\Big[\sum_{s=n-g(n)}^{n-1} b(r(s))x(s)\Big]^2 \qquad (3.5) \\ &\leq \frac{h}{\beta}x^2(n) + \frac{\beta}{h}h\sum_{s=n-g(n)}^{n-1} b^2(r(s))x^2(s). \end{aligned}$$

Substituting (3.5) into (3.4) we obtain

$$\begin{aligned} V(n) &\leq x^2(n) + \frac{h}{\beta}x^2(n) + (\beta + h - \rho)\sum_{s=n-g(n)}^{n-1} b^2(r(s))x^2(s) \\ &= \frac{\beta+h}{\beta}x^2(n) \leq \frac{\rho}{\rho-h}x^2(n). \end{aligned}$$

Using the last inequality and (3.3) we obtain

$$|x(n)|^2 \geq \frac{\rho-h}{\rho}V(n) = \frac{\rho-h}{\rho}V(n_0)\prod_{s=n_0}^{\infty}[b(r(s)) + a(s)].$$

This completes the proof. □

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Received: October 19, 2020.
Revised: November 13, 2020.
Accepted: December 14, 2020.