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# A NOTE ON BIPARTITE GRAPHS WHOSE $[1, k]$-DOMINATION NUMBER EQUAL TO THEIR NUMBER OF VERTICES

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**Abstract.** A subset $D$ of the vertex set $V$ of a graph $G$ is called an $[1, k]$-dominating set if every vertex from $V - D$ is adjacent to at least one vertex and at most $k$ vertices of $D$. A $[1, k]$-dominating set with the minimum number of vertices is called a $\gamma_{[1,k]}$-set and the number of its vertices is the $[1, k]$-domination number $\gamma_{[1,k]}(G)$ of $G$. In this short note we show that the decision problem whether $\gamma_{[1,k]}(G) = n$ is an $NP$-hard problem, even for bipartite graphs. Also, a simple construction of a bipartite graph $G$ of order $n$ satisfying $\gamma_{[1,k]}(G) = n$ is given for every integer $n \geq (k + 1)(2k + 3)$.

**Keywords:** domination, $[1, k]$-domination number, $[1, k]$-total domination number, bipartite graphs.

**Mathematics Subject Classification:** 05C69.

## 1. INTRODUCTION

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A subset $D$ of $V(G)$ is called a *dominating set*, if every vertex from $V(G) - D$ has at least one neighbor in $D$. The minimum cardinality of a dominating set is called the *domination number* of $G$ and is denoted by $\gamma(G)$. A dominating set $D$ of $G$ is called a $[1, k]$-*dominating set* if every vertex of $V - D$ is adjacent to at most $k$ vertices of $D$. The minimum cardinality of a $[1, k]$-dominating set is the $[1, k]$-*domination number* of $G$ and denoted by $\gamma_{[1,k]}(G)$. We call a $[1, k]$-dominating set of cardinality $\gamma_{[1,k]}(G)$ a $\gamma_{[1,k]}(G)$-*set*. Clearly $\gamma(G) \leq \gamma_{[1,k]}(G) \leq |V(G)|$, which are the trivial bounds for $\gamma_{[1,k]}(G)$.

The invariant $\gamma_{[1,k]}(G)$ was introduced by Chellali *et al.* in [3] in the more general setting of the $[j, k]$-domination number of a graph. They proved that computing $\gamma_{[1,2]}(G)$ is an NP-complete problem. Among other results, it was shown that the trivial bounds are strict for some graphs in the case of $k = 2$. They also posed several questions; one of them was to characterize graphs for which the trivial lower bound is strict for $k = 2$, that is $\gamma_{[1,2]}(G) = \gamma(G)$. Recently, see [4], it was shown that there is no polynomial recognition algorithm for graphs with $\gamma_{[1,k]}(G) = \gamma(G)$ unless $P = NP$.

Some other problems from [3] have been considered in [1,2,5,9]. For instance, in [9] authors find planar graphs and bipartite graphs of order $n$ with $\gamma_{[1,2]}(G) = n$. More precisely, for integer $n$ which is sufficiently large, they construct a bipartite graph $G$ of order $n$ with $\gamma_{[1,2]}(G) = n$. The construction is complicated and work only for large $n$.

In this note we present a simple construction of a bipartite graph $G$ of order $n$ with $\gamma_{[1,k]}(G) = n$ for any integers $k > 2$ and $n \geq (k+1)(2k+3)$. Hence, we generalize and simplify some results given in [9]. We also show that the decision problem $\gamma_{[1,k]}(G) = n$ is NP-hard for a given bipartite graph $G$ of order $n$, $n > k \geq 2$.

## 2. PRELIMINARIES

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. An *empty graph* on $n$ vertices $\overline{K_n}$ consists of $n$ isolated vertices with no edges. A tree which has exactly one vertex of degree greater than two is said to be *star-like*. The vertex of maximum degree of such a tree is called the *central vertex*. The graph $T - v$, where $T$ is a star-like tree and $v$ its central vertex, contains disjoint paths $P_{n_1}, \ldots, P_{n_k}$ and is denoted by $S(n_1, \ldots, n_k)$.

A subset $D$ of $V(G)$ is called a *total dominating set* if every $v \in V(G)$ is adjacent to a vertex from $D$. The minimum cardinality of a total dominating set in graph $G$ is denoted by $\gamma_t(G)$ and is called the *total domination number*. A total dominating set $D \subseteq V(G)$ is a *total* $[1,k]$-*dominating set*, if for every vertex $v \in V(G)$ is adjacent to at most $k$ vertices from $D$. While an $[1,k]$-dominating set exists for every graph $G$, there exist graphs which do not have any total $[1,k]$-dominating sets. By $\gamma_{t[1,k]}(G)$ we denote the minimum cardinality of a total $[1,k]$-dominating set (if it exist), it is $\infty$ if no total $[1,k]$-dominating set exists. An example of a graph $G_1 \cong S(2,2,2)$ with $\gamma_{t[1,2]}(G_1) = \infty$ is presented on Figure 2.

The *lexicographic product* of two graphs $G$ and $H$, denoted by $G \circ H$, is a graph with the vertex set $V(G \circ H) = V(G) \times V(H)$, where two vertices $(g,h)$ and $(g',h')$ are adjacent in $G \circ H$ if $gg' \in E(G)$ or $g = g'$ and $hh' \in E(H)$. It follows directly from the definition of the lexicographic product that $G \circ H$ is bipartite if and only if one factor is the empty graph $\overline{K_t}$ and the other is bipartite. Moreover, for a graph $G$ on at least two vertices, the graph $G \circ H$ is connected and bipartite if and only if $G$ is connected and bipartite and $H \cong \overline{K_t}$. See [7] for more informations about lexicographic and other products.

For any $h_0 \in V(H)$, we call the set

$$G^{h_0} = \{(g,h_0) \in V(G \circ H) : g \in V(G)\}$$

a *G-layer* of the graph $G \circ H$. Similarly, for $g_0 \in V(G)$, we call the set

$$H^{g_0} = \{(g_0,h) \in V(G \circ H) : h \in V(H)\}$$

an *H-layer* of the graph $G \circ H$.

Recently, see [8], $\gamma_{[1,k]}(G \circ H)$ was described as an optimization problem of some partitions of $V(G)$. For some special cases it is possible to present $\gamma_{[1,k]}(G \circ H)$ as an invariant of $G$. In particular, this is possible when $\gamma_{[1,k]}(H) > k$ and $H$ contains an isolated vertex.

**Theorem 2.1** ([8, Theorem 4.4]). *Let $G$ be a connected graph, $H$ a graph and $k \geq 2$ an integer. If $\gamma_{[1,k]}(H) > k$ and $H$ contains an isolated vertex, then*

$$\gamma_{[1,k]}(G \circ H) = \begin{cases} \gamma_{t[1,k]}(G), & \text{if } \gamma_{t[1,k]}(G) < \infty, \\ |V(G)| \cdot |V(H)|, & \text{otherwise.} \end{cases}$$

The following corollary is the direct consequence of Theorem 2.1 and will be useful later to construct bipartite graphs with $\gamma_{[1,k]}(G) = |V(G)|$.

**Corollary 2.2.** *Let $G$ be a connected graph and $H \cong \overline{K_{k+1}}$. Then*

$$\gamma_{[1,k]}(G \circ H) = |V(G \circ H)|$$

*if and only if $G$ has no total $[1,k]$-dominating set.*

## 3. COMPLEXITY

In this section we will show that it is $NP$-hard to check whether $\gamma_{[1,k]}(G) = |V(G)|$ for a bipartite graph $G$. For this aim, we first show that the related problem of checking whether $\gamma_{t[1,k]}(G) = |V(G)|$ is $NP$-hard for a bipartite graph $G$. The problem is called a BipTotal $[1,k]$-set problem. To prove this we use reduction from a kind of a set cover problem, called $[1,k]$-triple set cover problem, which is known to be $NP$-hard as shown in [6]. Then, using Theorem 2.1, we prove that for a bipartite graph $G$, checking whether $\gamma_{[1,k]}(G) = |V(G)|$ is an $NP$-hard problem.

| | |
|---|---|
| *Problem A:* | $[1,k]$-triple set cover |
| *Input:* | A finite set $X = \{x_1, \ldots, x_n\}$ and a collection $C = \{C_1, \ldots, C_t\}$ of 3-element subsets of $X$. |
| *Output:* | **Yes** if there exists a $C' \subseteq C$ such that every element of $X$ appears in at least one and at most $k$ elements of $C'$, **No** otherwise. |

| | |
|---|---|
| *Problem B:* | BipTotal $[1,k]$-set |
| *Input:* | A bipartite graph $G$. |
| *Output:* | **Yes** if there exists a $D \subseteq V(G)$ such that every element of $V(G)$ is adjacent to at least one and at most $k$ vertices of $D$, **No** otherwise. |

We are going to prove that the BipTotal $[1,k]$-set problem is $NP$-hard by giving a polynomial time reduction from the $[1,k]$-triple set cover problem.

**Definition 3.1.** Let $X = \{x_1, \ldots, x_n\}$ and $C = \{C_1, \ldots, C_t\}$ be any given instance of Problem A. We construct a graph $G_{X,C}$ as follows:

$$V(G_{X,C}) = \bigcup_{i=1}^{t} (P_i \cup L_i) \cup X \cup \{c_1, \ldots, c_t\},$$

where for each integer $i$, $1 \leq i \leq t$, we have $P_i = \{p_{i,1}, \ldots, p_{i,k}\}$, $L_i = \{l_{i,1}, \ldots, l_{i,k}\}$, and

$$E(G_{X,C}) = \bigcup_{1 \leq j \leq t} \{c_j p_{j,1}, \ldots, c_j p_{j,k}, p_{j,1} l_{j,1}, \ldots, p_{j,k} l_{j,k}\} \cup \bigcup_{1 \leq i,j \leq t} \{x_i c_j : x_i \in C_j\}.$$

**Lemma 3.2.** *Let $X = \{x_1, \ldots, x_n\}$ be a finite set and $C = \{C_1, \ldots, C_t\}$ be a collection of 3-element subsets of $X$. Problem A for $(X, C)$ is a YES instance if and only if $G_{X,C}$ is a YES instance of Problem B.*

*Proof.* Suppose that $C'$ is a solution for the instance $(X, C)$ of Problem A. We construct $D$ as follows:

$$D = \bigcup_{1 \leq j \leq t} P_j \cup \bigcup_{C_j \in C'} \{c_j\} \cup \bigcup_{C_j \notin C'} L_j.$$

We can check easily that $D$ is a $[1, k]$-total set for $G_{X,C}$. Conversely, suppose that $G_{X,C}$ has a total $[1, k]$-set $D$. Clearly $D$ must contain all vertices of $P_j$ because every $p_{j,j'}$ is adjacent to at least one leaf $l_{j,j'}$. These vertices dominate every $c_j$ exactly $k$ times. Therefore, there is no vertex $x_i$ in $D$; in other words $D \cap X = \emptyset$. So, every $x_i$ must be dominated by a vertex of $\{c_1, \cdots, c_t\}$. It is easy to see that there is a solution $C' \subseteq C$ for $[1, k]$-triple set cover problem if and only if the corresponding vertices $C'$ of $V(G)$ dominate all vertices of $\{x_1, \ldots, x_n\}$ at least once and at most $k$ times. These vertices dominate all vertices $\{p_{j,1}, \ldots, p_{j,k}\}$ for $c_j \in C'$. To dominate all other vertices we add $\{l_{j,1}, \ldots, l_{j,k}\}$ to $D$ for $c_j \notin C'$. □

The following example help us to understand the definition and the lemma.

**Example 3.3.** Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ and $C = \{C_1, C_2, C_3, C_4, C_5\}$ such that $C_1 = \{x_1, x_2, x_4\}$, $C_2 = \{x_2, x_5, x_7\}$, $C_3 = \{x_4, x_5, x_6\}$, $C_4 = \{x_3, x_5, x_9\}$ and $C_5 = \{x_3, x_8, x_9\}$. For $k = 3$, the corresponding graph $G_{X,C}$ is shown in Figure 1. This is a YES-instance for the $[1, 3]$-set cover problem $(X, C)$, because $C' = \{C_1, C_2, C_3, C_5\}$ has the desired property. The vertices of total $[1, 3]$-set are black vertices shown in Figure 1.

**Theorem 3.4.** *The BipTotal $[1, k]$-set problem is $NP$-hard*

*Proof.* By Lemma 1 of [6] the $[1, k]$-triple set cover problem is $NP$-hard. Hence, using Lemma 3.2 the BipTotal $[1, k]$-set problem is also $NP$-hard. □

The following theorem which is the main result of this section is a direct consequence of Theorem 3.4 and Corollary 2.2.

**Theorem 3.5.** *For bipartite graphs it is $NP$-hard to decide whether we have $\gamma_{[1,k]}(G) = |V(G)|$.*

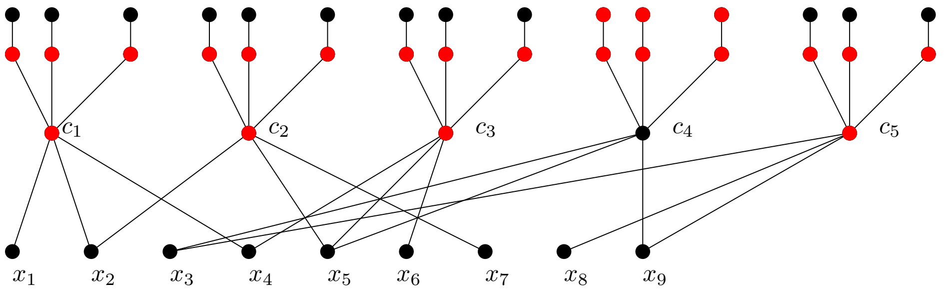


**Fig. 1.** $G_{X,C}$ from Example 3.3

## 4. CONSTRUCTION

Here, for any integers $k \geq 2$ and $n \geq (k+1)(2k+3)$, we construct a bipartite graph $G$ of order $n$ with $\gamma_{[1,k]}(G) = n$. As already mentioned, in [9] a bipartite graph $G$ of order $n$ was constructed for sufficiently large integer $n$ which satisfies $\gamma_{[1,2]}(G) = n$.

First, we give our construction in the case of $k = 2$, then we extend the result to the general case.

**Example 4.1.** If $G_1 \cong S(2,2,2)$, see Figure 2, and $H \cong \overline{K_3}$, then $G = G_1 \circ H$, see Figure 3, is bipartite and $\gamma_{[1,2]}(G) = |V(G)|$ by Theorem 2.1.

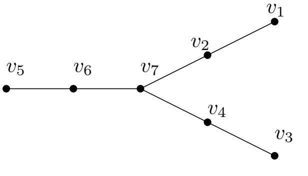


**Fig. 2.** Bipartite graph $G_1 \cong S(2,2,2)$ with $\gamma_{t[1,2]}(G_1) = \infty$

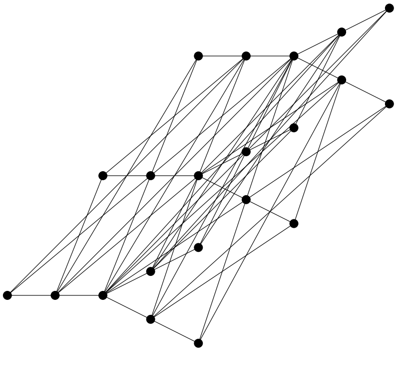

**Fig. 3.** Bipartite graph $G$ with $\gamma_{[1,2]}(G) = |V(G)|$

**Theorem 4.2.** *For any integer $n \geq 21$, there exists a bipartite graph $\Gamma$ with $n$ vertices such that $\gamma_{[1,2]}(\Gamma) = n$.*

*Proof.* Let $G_1 = S(2,2,2)$ and $G$ be graphs shown on Figures 2 and 3, respectively. By Example 4.1 $G$ is a bipartite graph with 21 vertices for which $\gamma_{[1,2]}(G) = 21$. Let $v_1, \ldots, v_7 \in V(G_1)$ be the vertices of $G_1$ as shown on Figure 2. For any integer $t \geq 1$, using the graph $G$ of Figure 3, we construct a new bipartite graph $\Gamma$ of order $n = 21 + t$ as follows:

$$\Gamma = (V(\Gamma), E(\Gamma)),$$

where

$$V(\Gamma) = V(G) \cup \{a_1, \ldots, a_t\} \quad \text{and} \quad E(\Gamma) = \bigcup_{h \in V(H)} \{a_1(v_2,h), \ldots, a_t(v_2,h)\} \cup E(G)$$

(see Figure 4).

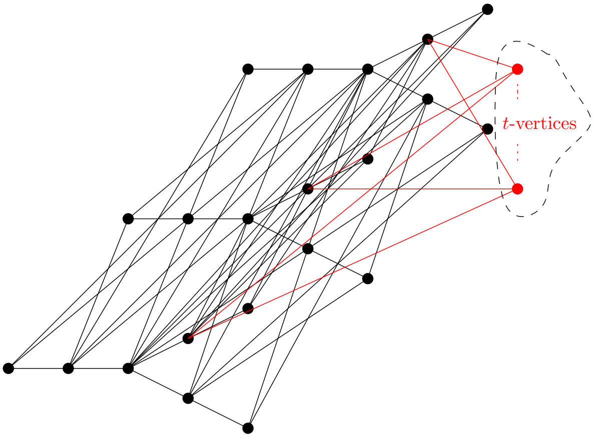


**Fig. 4.** Bipartite graph $\Gamma$ with $\gamma_{[1,2]}(\Gamma) = |V(\Gamma)|$

Let $S$ be a $[1,2]$-set for $\Gamma$. First, we claim that there exists a vertex $h \in H$ with $(v_2,h) \in S$. To dominate the three vertices of the $H$-layer $H^{v_1}$, either there exists a vertex $h \in H$ with $(v_2,h) \in S$ or $H^{v_1} \subseteq S$. If there exists a vertex $h \in H$ with $(v_2,h) \in S$, then there is nothing to prove. If $H^{v_1} \subseteq S$, then every vertex of $H^{v_2}$ is dominated at least three times, hence $H^{v_2} \subseteq S$. Therefore the claim is true and $H^{v_2} \cap S \neq \emptyset$. By the same reasoning we have $H^{v_4} \cap S \neq \emptyset$ and $H^{v_6} \cap S \neq \emptyset$. Hence, by the definition of lexicographic product of graphs, every vertex of $H^{v_7}$ is dominated at least three times. Therefore, we have

$$H^{v_7} \subseteq S. \tag{4.1}$$

Now, by (4.1) every vertex in $H^{v_2} \cup H^{v_4} \cup H^{v_6}$ is dominated at least three times and so we have

$$H^{v_2} \cup H^{v_4} \cup H^{v_6} \subseteq S. \tag{4.2}$$

And, then by (4.2) we conclude that

$$H^{v_1} \cup H^{v_3} \cup H^{v_5} \cup \{a_1, \ldots, a_t\} \subseteq S.$$

Therefore, $S = V(\Gamma)$, as desired. □

We end with a generalization of the above result from $k = 2$ to $k \geq 2$.

**Theorem 4.3.** *For integers $k \geq 2$ and $n \geq (k+1)(2k+3)$, there exists a bipartite graph $\Gamma$ with $n$ vertices such that $\gamma_{[1,k]}(\Gamma) = n$.*

*Proof.* Let $G_1 = S(2, 2, \ldots, 2)$ be a star-like tree with $2k+3$ vertices and let $H \cong \overline{K_{k+1}}$. Clearly $G = G_1 \circ H$ is a bipartite graph. Let $v_1 \in V(G_1)$ be a vertex of degree one and $v_2 \in V(G_1)$ be its only neighbor. For any integer $t \geq 1$, using the graph $G$, we construct a new bipartite graph $\Gamma$ of order $n = (k+1)(2k+3) + t$ as follows:

$$\Gamma = (V(\Gamma), E(\Gamma)),$$

where

$$V(\Gamma) = V(G) \cup \{a_1, \ldots, a_t\} \quad \text{and} \quad E(\Gamma) = \bigcup_{h \in V(H)} \{a_1(v_2, h), \ldots, a_t(v_2, h)\} \cup E(G).$$

Let $S$ be a $\gamma_{[1,k]}(\Gamma)$-set. By the same reasoning as in the proof of Theorem 4.2 one can show that $H^{v_i} \cap S \neq \emptyset$ for every vertex $v_i \in V(G_1)$ with $\deg_{G_1}(v_i) = 2$. Since there are $k+1$ such vertices in $G_1$, all vertices of $H^v$ must be in $S$ for a central vertex $v$ of $G_1$. This clearly leads to $\gamma_{[1,k]}(\Gamma) = |V(\Gamma)|$ because $|H^v| = k+1$. □

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