

MUSTAPHA KACI  (Oran)

SONIA RADJEF  (Oran)

## A new geometric approach for sensitivity analysis in linear programming

**Abstract** In this paper, we present a new geometric approach for sensitivity analysis in linear programming that is computationally practical for a decision-maker to study the behavior of the optimal solution of the linear programming problem under changes in program data. First, we fix the feasible domain (fix the linear constraints). Then, we geometrically formulate a linear programming problem. Next, we give a new equivalent geometric formulation of the sensitivity analysis problem using notions of affine geometry. We write the coefficient vector of the objective function in polar coordinates and we determine all the angles for which the solution remains unchanged. Finally, the approach is presented in detail and illustrated with a numerical example.

*2010 Mathematics Subject Classification:* Primary: 90C05; Secondary: 52B12; 90C31; 93B35.

*Key words and phrases:* linear programming, sensitivity analysis, geometric approach.

### 1. Introduction

One of the star models of operational research is linear programming. It is a field of mathematical programming the most studied. It concerns the optimization of a mathematical program where the objective function and the functions defining the constraints are linear, see [2, 4]. Geometrically, linear programming problems are of convex programming, the linear constraints form a convex polyhedron; so, the results of the convexity are exploited. The vertices of the convex polyhedron form the basic solutions, one of them can be the optimal solution.

One of the essential parts of linear programming is sensitivity analysis, also called post-optimality analysis, because it starts from the original optimal solution, see [1, 3].

Linear programming models concrete problems such as maximizing a company's profits, but changes in market data require updates for the initial problems, and the sensitivity analysis is used to illustrate the margins of the linear program parameters for which the solution of the initial problem remains stable. Another situation that requires a stability study of the optimal

solution is the committed errors in the data of the problem and it will allow us to avoid restarting the procedure of resolution several times.

In this work, we give a geometric approach to study the stability of the optimal solution when we change the coefficients of two decision variables in the objective function. This approach allows us to do the stability study with the variation of the two parameters at the same time. It uses the Euclidean norm as well as the linearity of the constraint functions to formulate a new problem equivalent to sensitivity analysis problem, then this new problem is solved with a simple calculation. We fix a corner of the feasible domain and we find all the linear forms that reach their optimum value at this point. Geometrically, this consists to determine hyperplanes in  $\mathbb{R}^3$ , therefore the variation of the coefficients of the objective function corresponds to the rotation of the plan.

First, we consider a linear maximization programming problem (1), then we give a geometric formulation of a linear programming problem. Then, we use the latter to determine an equivalent problem to sensitivity analysis problem which consists in determining an angle interval on which the vector of the objective function coefficients is allowed to rotate to determine a linear form that reached its optimal value in initial optimal solution. Then, we look to the objective function as a plan of  $\mathbb{R}^3$ , and the linear constraints as a convex subset of  $\mathbb{R}^2$ . The variation of the objective function coefficients corresponds to the rotation of the plan.

This paper is structured as follow: First, we give some preliminaries in section 2. Then, we give the mathematical formulation of a sensitivity analysis problem in section 3. Next, a geometric formulation of a linear programming problem is given in section 4. After that, we give a geometric approach to sensitivity analysis in linear programming in section 5. Then, we illustrate this approach with a numerical example in section 6. Finally, a conclusion is given in section 7.

## 2. Preliminary

NOTATIONS 1 Let  $A = (a_{ij})$  be a matrix of size  $m_1 \times m_2$ , where  $m_1, m_2$  are any integers. We denote by  $A^t$  the transposed matrix of  $A$ , defined by

$$A^t = (a_{ji}).$$

DEFINITION 2.1 (Isometry of  $\mathbb{R}^2$ ) We call an isometry of  $\mathbb{R}^2$ , a linear application  $u \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$  which checks one of the following properties:

- a)  $u$  preserves the norm:  $\forall x \in \mathbb{R}^2, \|u(x)\| = \|x\|$ ,
- b)  $u$  preserves the scalar product:  $\forall x, y \in \mathbb{R}^2, \langle u(x), u(y) \rangle = \langle x, y \rangle$ ,
- c)  $u$  transforms an orthonormal basis into an orthonormal basis.

The application  $u$  is also called orthogonal automorphism. The vector space of all the isometries (orthogonal automorphisms) of  $\mathbb{R}^2$  is denoted by  $\mathcal{O}(\mathbb{R}^2)$ .

PROPERTY 1 Let  $u_1, u_2$  be two isometries of  $\mathcal{O}(\mathbb{R}^2)$ , then

1.  $u_1$  and  $u_2$  are one to one maps.
2.  $u_1 \circ u_2$  and  $u_1^{-1}$  are isometries.
3. The translations and rotations of  $\mathbb{R}^2$  are isometries.

EXAMPLE 2.2

1. Let  $v$  be a vector in  $\mathbb{R}^2$ . A translation of  $\mathbb{R}^2$  in the direction of  $v$  is a one to one application noted by  $T_v$ , which associates to an element  $x$  of  $\mathbb{R}^2$  another element  $T_v(x) = x + v$  of  $\mathbb{R}^2$ .
2. Let  $\theta$  be an angle between 0 and  $2\pi$ . A rotation of  $\mathbb{R}^2$  of angle  $\theta$  is a one to one application, defined by

$$R_\theta : \quad \mathbb{R}^2 \quad \longrightarrow \quad \mathbb{R}^2 \\ (x_1, x_2) \quad \longrightarrow \quad R_\theta(x_1, x_2) = \text{Rot}(\theta)(x_1, x_2)^t,$$

where  $\text{Rot}(\theta)$  is the matrix associated to  $R_\theta$ , defined by

$$\text{Rot}(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Note that for any angle  $\theta$ , we have

$$\begin{aligned} \text{Rot}(\theta)^t \text{Rot}(\theta) &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{Id}_2. \end{aligned}$$

DEFINITION 2.3 Let  $F_1$  and  $F_2$  be two supplementary linear subspaces of  $\mathbb{R}^n$ . Then, we call a projection on  $F_1$  parallel to  $F_2$ , the map  $P_{F_1}$  which associates to  $x$  of  $\mathbb{R}^n$  the unique element  $P_{F_1}(x)$  of  $F_1$  such that  $x = P_{F_1}(x) + z$  where  $z \in F_2$ . Such an application is also called a projector.

PROPERTY 2 Consider two lines  $(d)$  and  $(d')$  that form an angle  $\theta'$ . If  $M$  and  $N$  are two points that belong to  $(d)$  and  $M' = P_{(d)}(M)$ ,  $N' = P_{(d')}(N)$ , their orthogonal projection respectively, see the figure (1), we obtain

$$M'N' = MN \cos(\theta').$$

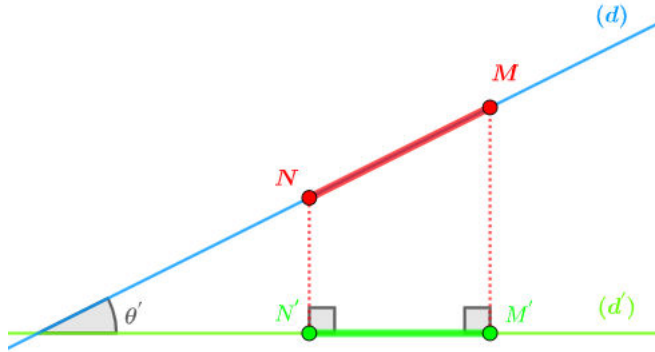


Figure 1: Projection of a segment on a line.

**3. Problem formulation**

Consider the following initial linear programming problem in the standard form:

$$\begin{cases} \max_x f(x) = c_0^t x \\ Ax \leq b \\ x \geq 0 \end{cases}, \tag{1}$$

with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \quad c_0 = \begin{pmatrix} c_1^0 \\ c_2^0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Where  $c_0$  is the vector of the objective function,  $c_1^0, c_2^0$  are constants,  $x$  is a  $2 \times 1$  vector of decision variables,  $A$  is a  $m \times 2$  matrix of constants,  $b$  is a  $m \times 1$  vector of constants and  $m$  is the number of linear constraints.

With the hypothesis that  $c_1^0, c_2^0$  are positive numbers, we will see in the following that it is an artificial hypothesis that we made only to simplify the continuation of this presentation, see the remark (4.3).

NOTATIONS 2 Let us define:

- The feasible region:

$$S := \{x \in \mathbb{R}^2 : Ax \leq b, x \geq 0\}.$$

- The vector space  $\mathcal{L}$  of linear forms defined on  $\mathbb{R}^2$ , is defined by:

$$\mathcal{L}(\mathbb{R}^2) := \{g : \mathbb{R}^2 \rightarrow \mathbb{R} : g(x_1, x_2) = c_1 x_1 + c_2 x_2\}.$$

- $\|\cdot\|$  denotes the Euclidean norm on  $\mathbb{R}^2$ .

Let  $x^0 = (x_1^0, x_2^0)^t$  be an optimal solution of the problem (1). A sensitivity analysis problem is to find all linear formes  $g$  different from  $f$  verifying

$$\arg \max_{x \in S} f(x) = \arg \max_{x \in S} g(x), \quad g \in \mathcal{L}(\mathbb{R}^2), \quad (2)$$

where

$$\arg \max_{x \in S} f(x) = \{y \in S \mid f(y) \geq f(x), \quad x \in S\}.$$

#### 4. Geometric formulation of a linear programming problem

In this part, we will determine a problem equivalent to the linear programming problem (1). In  $\mathbb{R}^3$ , we find that the feasible region  $S$  is a polygon entirely contained in the vector hyperplane of equation  $z = 0$ , we can easily show that the graph of  $f$  is a vector hyperplane (linear subspace of dimension two).

Let's start by transforming the expression of the linear form  $f$  as follows:

$$\begin{aligned} f(x_1, x_2) = c_1^0 x_1 + c_2^0 x_2 &\Leftrightarrow c_1^0 x_1 + c_2^0 x_2 - f(x_1, x_2) = 0 \\ &\Leftrightarrow \langle (c_1^0, c_2^0, -1), (x_1, x_2, f(x_1, x_2)) \rangle = 0 \\ &\Leftrightarrow \langle n_f, (x_1, x_2, z) \rangle = 0, \end{aligned}$$

where  $x_1, x_2 \in \mathbb{R}$ ,  $z = f(x_1, x_2)$  and  $n_f = (c_1^0, c_2^0, -1)$ . So, the graph of the linear form  $f$ , defined by

$$\begin{aligned} Gf(f) &= \{(x_1, x_2, z) \in \mathbb{R}^3 \mid z = f(x_1, x_2)\} \\ &= \{(x_1, x_2, z) \in \mathbb{R}^3 \mid \langle n_f, (x_1, x_2, z) \rangle = 0\} \end{aligned}$$

is orthogonal to  $\text{vect}(n_f)$ , the linear subspace generated by the vector  $n_f$ , and therefore any vector of  $\mathbb{R}^3$  can be written as the sum of a vector of  $Gf(f)$  and another vector of  $\text{vect}(n_f)$ .

Now, we position ourselves on  $\mathbb{R}^2$  (the plane  $z = 0$ ) and we consider the line  $d_0$  defined by the intersection of  $Gf(f)$  with the plane  $z = 0$ :

$$(d_0) : c_1^0 x_1 + c_2^0 x_2 = 0, \quad (3)$$

$d_0$  is a linear subspace of  $\mathbb{R}^2$  of dimension 1 directed by the vector:

$$v_0 = (-c_2^0, c_1^0).$$

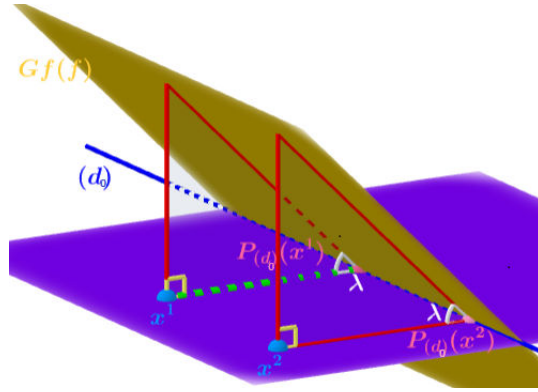


Figure 2: Graphic illustration of the lemma (4.2).

DEFINITION 4.1 The graph of the line  $d_0$ , defined by:

$$D_0 = \{(x_1, x_2) \in \mathbb{R}^2 \mid c_1^0 x_1 + c_2^0 x_2 = 0\}.$$

The upper half-space of  $\mathbb{R}^2$  deprived of  $(d_0)$ , defined by:

$$D_0^+ = \{(x_1, x_2) \in \mathbb{R}^2 \mid c_1^0 x_1 + c_2^0 x_2 > 0\}.$$

The lower half-space of  $\mathbb{R}^2$  deprived of  $d_0$ , defined by:

$$D_0^- = \{(x_1, x_2) \in \mathbb{R}^2 \mid c_1^0 x_1 + c_2^0 x_2 < 0\}.$$

LEMMA 4.2 For all  $x^1, x^2 \in D_0^+$  (respectively  $y^1, y^2 \in D_0^-$ ), we have

$$\frac{f(x^1)}{\|x^1 - P_{(d_0)}(x^1)\|} = \frac{f(x^2)}{\|x^2 - P_{(d_0)}(x^2)\|}, \quad (4)$$

and

$$\frac{f(y^1)}{\|y^1 - P_{(d_0)}(y^1)\|} = \frac{f(y^2)}{\|y^2 - P_{(d_0)}(y^2)\|}. \quad (5)$$

PROOF The intersection of  $Gf(f)$  with the plane  $z = 0$  forms a constant angle  $0 \leq \lambda \leq \pi$ , see figure (2). Then,

$$\begin{cases} \tan(\lambda) &= \frac{f(x^1)}{\|x^1 - P_{(d_0)}(x^1)\|} = \frac{f(x^2)}{\|x^2 - P_{(d_0)}(x^2)\|}, & x^1, x^2 \in D_0^+ \\ \tan(\pi - \lambda) &= \frac{f(y^1)}{\|y^1 - P_{(d_0)}(y^1)\|} = \frac{f(y^2)}{\|y^2 - P_{(d_0)}(y^2)\|}, & y^1, y^2 \in D_0^- \end{cases}.$$

REMARK 4.3 If  $c_1^0, c_2^0$  are any real numbers, we can come back to the hypotheses of the problem (1) by making the following changes:

1. Apply a rotation  $R_\theta$  over the feasible region  $S$  and the vector  $c_0$  simultaneously to obtain the problem:

$$\max_{x \in S} (R_\theta(c_0))^t R_\theta(x), \tag{6}$$

which is equivalent to the initial problem. Indeed, we have

$$(\text{Rot}(\theta)c_0)^t \text{Rot}(\theta)x = c_0^t (\text{Rot}(\theta)^t \text{Rot}(\theta)) = c_0^t \text{Id}_2 x = c_0^t x = f(x).$$

Then, an optimal solution  $x^0$  of the problem (1) is also optimal to the problem (6), indeed

$$(\text{Rot}(\theta)c_0)^t \text{Rot}(\theta)x^0 = f(x^0) \geq f(x) = (\text{Rot}(\theta)c_0)^t \text{Rot}(\theta)x.$$

After that, we pose

$$(\text{Rot}(\theta^0)c_0) = (|c_1^0|, |c_2^0|) \geq 0,$$

where  $\theta^0$  is solution of the following linear system:

$$\begin{cases} \cos(\theta)c_1^0 - \sin(\theta)c_2^0 &= |c_1^0| \\ \sin(\theta)c_1^0 + \cos(\theta)c_2^0 &= |c_2^0| \end{cases}.$$

2. Apply a translation  $T_{v_0}$  over  $S$  in the direction of  $\text{Rot}(\theta^0) \cdot (c_1^0, c_2^0)^t$  because the objective function is constant in the latter. Indeed, for all  $y \in T_{v_0}(S)$ , we have

$$\begin{aligned} f(y) = f(x + v_0) &= f(x) + f(v_0) \\ &\leq f(x^0) + f(v_0) \\ &\leq f(x^0 + v_0) \\ &\leq f(y^0) \end{aligned}, \quad x \in S.$$

where

$$y^0 = x^0 + v_0, \quad \text{and} \quad v_0 = \alpha \text{Rot}(\theta^0) (c_1^0, c_2^0)^t,$$

$\alpha$  is a strictly positive number sufficiently large.

PROPOSITION 4.4 Let  $c_1^0, c_2^0 \geq 0$ , then the problem (1) is equivalent to:

$$\max_{x \in S} \|x - P_{(d_0)}(x)\|. \tag{7}$$

PROOF

- ◇ Let  $x^0$  be an optimal solution of the problem (1), then

$$f(x^0) \geq f(x), \quad \forall x \in S, \quad (8)$$

from lemma (4.2), and inequality (8), we get

$$\frac{f(x^0)}{\|x^0 - P_{(d_0)}(x^0)\|} = \frac{f(x)}{\|x - P_{(d_0)}(x)\|} \leq \frac{f(x^0)}{\|x - P_{(d_0)}(x)\|}, \quad \forall x \in S.$$

Then, for all  $x \in S$ , we get

$$\|x^0 - P_{(d_0)}(x^0)\| \geq \|x - P_{(d_0)}(x)\|, \quad (9)$$

which proves that  $x^0$  is an optimal solution of the problem (7).

- ◇ Conversely, let  $x^0$  be an optimal solution of the problem (7), then  $x^0$  verifies the inequality (9), then

$$\frac{1}{\|x - P_{(d_0)}(x)\|} \geq \frac{1}{\|x^0 - P_{(d_0)}(x^0)\|}, \quad \forall x \in S.$$

We multiply the two sides of inequality by  $f(x)$ , then we apply lemma (4.2) to the left side, and get

$$\frac{f(x^0)}{\|x^0 - P_{(d_0)}(x^0)\|} = \frac{f(x)}{\|x - P_{(d_0)}(x)\|} \geq \frac{f(x)}{\|x^0 - P_{(d_0)}(x^0)\|},$$

which proves that  $x^0$  is an optimal solution of the problem (1),

$$f(x^0) \geq f(x), \quad \text{for all } x \in S.$$

## 5. A geometric approach to sensitivity analysis in linear programming

The geometric analysis we present here consists of writing a linear programming problem in polar coordinate system and then determining a relationship that links the coefficients of the objective function  $f$ . We already know that to any linear form  $g$  in  $\mathcal{L}(\mathbb{R}^2)$ , we can associate a single vector (gradient of  $g$ ) in  $\mathbb{R}^2$  and conversely. Another important fact is that we can get every corner of the feasible region  $S$  by solving a finite number of linear systems with two equations and two variables. This allows us to reformulate the problem (2) as follows:

PROBLEM 5.1 Let  $x^1, x^0, x^2$  be the successive corners (extreme points) of  $S$ , see figure (3), where  $x^0$  is the optimal solution of the problem (7). Find all vector lines  $(d)$  that simultaneously satisfy to the following two inequalities:

$$\|x^0 - P_{(d)}(x^0)\| \geq \|x^1 - P_{(d)}(x^1)\|, \quad (10)$$

and

$$\|x^0 - P_{(d)}(x^0)\| \geq \|x^2 - P_{(d)}(x^2)\|. \quad (11)$$



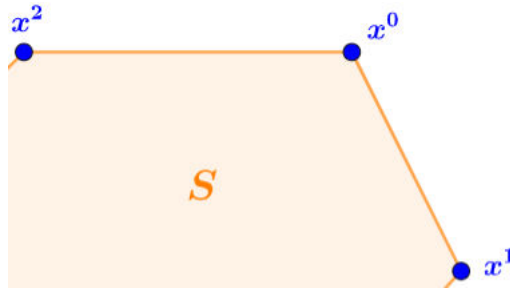


Figure 3: Extreme points close to the optimal solution  $x^0$ .

PROPOSITION 5.2 *The problem (2) and (5.1) are equivalents.*

PROOF This is a direct consequence of the proposition (4.4) and convexity of  $S$ . ■

REMARK 5.3 The fact that the problem (5.1) is built from another problem that admits a solution, then it still admit the vector line  $(d_0)$  as a trivial solution.

PROPOSITION 5.4 *Let  $\theta_1, \theta_2 \in [0, \pi]$ ,  $\phi \in [0, \frac{\pi}{2}[$ , and  $r, r_1, r_2 \geq 0$ , and consider the following vectors  $x^{10}$  and  $x^{02}$  written in polar coordinate system:*

$$\begin{aligned} x^{10} &:= x^1 - x^0 = r_1(\cos(\theta_1), \sin(\theta_1)) \\ x^{02} &:= x^0 - x^2 = r_2(\cos(\theta_2), \sin(\theta_2)) \\ c &= (c_1, c_2) = r(\cos(\phi), \sin(\phi)). \end{aligned}$$

*Then, the solutions of the problem (5.1) are the line vectors defined by:*

$$(d) : r \cos(\phi)x_1 + r \sin(\phi)x_2 = 0 \quad \text{with } \theta_1 < \phi + \frac{\pi}{2} < \theta_2. \quad (12)$$

PROOF Let  $x^0$  be a solution of the problem (7), then from propriety (2), we get

$$\begin{cases} \|P_{(d)}(x^1) - P_{(d)}(x^0)\| = \|x^1 - x^0\| \cos(\delta_1) \\ \|P_{(d)}(x^2) - P_{(d)}(x^0)\| = \|x^2 - x^0\| \cos(\delta_2) \end{cases},$$

where  $\delta_1$  is the angle formed by the vector  $x^{10}$  with  $v_0$ , and  $\delta_2$  is the angle formed by the vector  $x^{02}$  with  $v_0$  (the order of vectors is important). On the other hand, see figure (4), we have

$$\begin{cases} |\delta_1| = |\phi + \frac{\pi}{2} - \theta_1| < \frac{\pi}{2} \\ |\delta_2| = |\phi + \frac{\pi}{2} - \theta_2| < \frac{\pi}{2} \end{cases}.$$

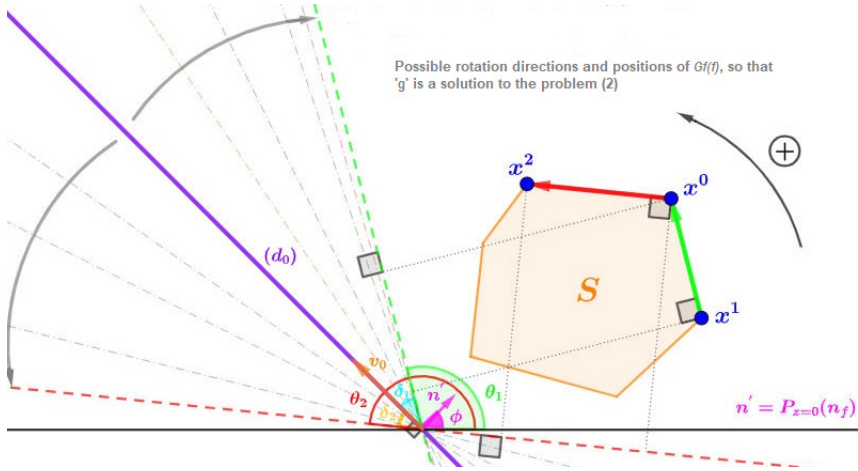


Figure 4: Graphic illustration of the proposition (5.4).

Therefore, we get

$$\begin{cases} \|P_{(d)}(x^1) - P_{(d)}(x^0)\| < \|x^1 - x^0\| \\ \|P_{(d)}(x^2) - P_{(d)}(x^0)\| < \|x^2 - x^0\| \end{cases}.$$

Since, for all  $i = 1, 2$  the quadrilateral  $x^0x^iP_{(d)}(x^i)P_{(d)}(x^0)$  is irregular, the two segments  $P_{(d)}(x^0)x^0$  and  $P_{(d)}(x^i)x^i$  are parallels, then

$$\begin{cases} \|P_{(d)}(x^1) - x^1\| < \|P_{(d)}(x^0) - x^0\| \\ \|P_{(d)}(x^2) - x^2\| < \|P_{(d)}(x^0) - x^0\| \end{cases}.$$

**6. Numerical example** Consider the following linear programming problem:

$$\begin{cases} \max_{x_1, x_2} f(x_1, x_2) = 2x_1 + 3x_2 \\ A(x_1, x_2)^t \leq b \\ x_1 \geq 0; x_2 \geq 0 \end{cases}, \quad (13)$$

where

$$A = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{2}{5} & \frac{1}{5} \\ 0 & \frac{4}{5} \end{pmatrix}, \quad b = \begin{pmatrix} 40 \\ 40 \\ 40 \end{pmatrix}, \quad c_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We start by solving the problem (13) using the simplex method, we obtain an optimal solution equal to  $x^0 = (80, 40)$ . Then, we solve the following problem:

$$\arg \max_{x \in S} f(x) = \arg \max_{x \in S} g(x). \quad (14)$$

Consider the successive corners  $x^1, x^0, x^2$  of the feasible region, such that:

$$x^1 = (100, 0) \quad \text{and} \quad x^2 = (60, 50) .$$

Then we have

$$x^{10} = (-20, 40) = 20\sqrt{5} \left( -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right),$$

$$x^{02} = (-20, 10) = 10\sqrt{5} \left( -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right),$$

and therefore we get

$$\theta_1 = \arccos \left( -\frac{1}{\sqrt{5}} \right) \approx 116.5650^\circ$$

and

$$\theta_2 = \arccos \left( -\frac{2}{\sqrt{5}} \right) \approx 153.4349^\circ.$$

Finally, for all  $r > 0$  and  $\phi \in ]26.5650^\circ, 63.4349^\circ[$ , the linear form:

$$g(\cdot) = r \langle (\cos(\phi), \sin(\phi)), \cdot \rangle$$

is a solution of the problem (14).

Finally, to do the sensitivity analysis, we write the coefficients of the objective function  $f$  in polar coordinate system, as follows:

$$c_0 = r_f (\cos(\phi_f), \sin(\phi_f)),$$

where  $\phi_f = \arccos \left( \frac{2}{\sqrt{13}} \right) \approx 56.31^\circ$  and  $r_f = \sqrt{13}$ , and consider the solution  $g$  which stabilizes  $x^0$  as an optimal solution. After that, we calculate  $g(x^0)$ , we get

$$\begin{aligned} g(x^0) = g(80, 40) &\approx 40\sqrt{5}r \langle (\cos(\phi), \sin(\phi)), (\cos(26.5650^\circ), \sin(26.5650^\circ)) \rangle \\ &\approx 40\sqrt{5}r \cos(\phi - 26.5650^\circ) \\ &\approx 40\sqrt{5}r \cos(\phi_f + \nu - 26.5650^\circ) \\ &\approx 40\sqrt{5}r \cos(56.31^\circ + \nu - 26.5650^\circ) \\ &\approx 40\sqrt{5}r \cos(\nu + 29.745^\circ), \end{aligned}$$

where  $\nu \in ]-3.18^\circ, 33.8699^\circ[$  corresponds to the angle of rotation of  $\text{grad}(f)$ , to arrive at  $\text{grad}(g)$ , in other words the angle between  $\text{grad}(f)$  and  $\text{grad}(g)$ . So, if  $\nu \in ]-3.18^\circ, 0^\circ[$ , the optimal value of  $f$  increases, which means the optimal value of  $g$  at  $x^0$  is bigger than the optimal value of  $f$  at  $x^0$ . And, if  $\nu \in ]0^\circ, 33.8699^\circ[$ , the optimal value of  $f$  decreases.

**7. Summary.** In this work, we give a new formulation of the sensitivity analysis problem by the problem (5.1), which allowed us to study the stability of an optimal solution of a linear programming problem in dimension 2 of the variable space. This is by reducing it to the search for a single interval by a simple calculation that does not require the use of a machine. The other advantage is that we determine all the objective functions which reach their optimum value at a common point. On the other hand, the classical approach consists to do the study of stability under variation of just one coefficient, and the variation of the two coefficients leads us to two different intervals related to each other which makes the analysis difficult, see [1]. Finally, we illustrated our approach with a numerical example.

## References

- [1] H. G. Daellenbach and J. A. George. *Introduction to operations research techniques*. Allyn and Bacon, Inc., Boston, MA, U.S.A., 1978. [Zbl 0455.90021](#). Cited on pp. 145 and 156.
- [2] G. B. Dantzig. *Linear programming and extensions*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, corrected edition, 1998. [MR 1658673](#). Cited on p. 145.
- [3] A. Shahin, P. Hanafizadeh, and M. Hladík. Sensitivity analysis of linear programming in the presence of correlation among right-hand side parameters or objective function coefficients. *CEJOR Cent. Eur. J. Oper. Res.*, 24(3):563–593, 2016. [MR 3522043](#). Cited on p. 145.
- [4] J. Stoer and C. Witzgall. *Convexity and optimization in finite dimensions. I*. Die Grundlehren der mathematischen Wissenschaften, Band 163. Springer-Verlag, New York-Berlin, 1970. [MR 0286498](#). Cited on p. 145.

## Geometryczna analiza wrażliwości w programowaniu liniowym.

[Mustapha Kaci and Sonia Radjef](#)

**Streszczenie** W niniejszym artykule przedstawiamy nowe podejście geometryczne do analizy wrażliwości w programowaniu liniowym, które jest praktyczne obliczeniowo. Pozwala analizować zachowania optymalnego rozwiązania problemu programowania liniowego przy zmianach danych zadania. Najpierw ustalamy dopuszczalną dziedzinę (naprawiamy ograniczenia liniowe). Następnie geometrycznie formułujemy problem programowania liniowego. Następnie podajemy nowe równoważne sformułowanie geometryczne problemu analizy wrażliwości przy użyciu pojęć geometrii afinicznej. Piszemy wektor współczynników funkcji celu we współrzędnych biegunowych i wyznaczamy wszystkie kąty, dla których rozwiązanie pozostaje niezmienione. Proponowane podejście zostało szczegółowo przedstawione i zilustrowane przykładem liczbowym.

2010 *Klasyfikacja tematyczna AMS (2010)*: 90C05; (52B12; 90C31; 93B35).


*Słowa kluczowe*: Programowanie liniowe, analiza wrażliwości, podejście geometryczne.




*Mustapha Kaci* prepares his doctorat (Mathematics) in University of Sciences and Technology of Oran Mohamed Boudiaf USTOMB under the supervision of Prof. Radjef Sonia. He has completed his Licence in Mathematics from the same university and Master from University of Science and Technology Houari Boumediene USTHB.



*Sonia Radjef* completed his doctorat (Mathematics) in university of Bejaia under the supervision of Prof. Bibi Mohand Ouamer. She is a teacher of mathematics at USTO MB university and her research areas are operations research, optimization and multi-objective programming. References to his research papers are listed in the European Mathematical Society, FIZ Karlsruhe, and the Heidelberg Academy of Sciences bibliography database known as zbMath under [ZBL ai:radjef.sonia](#), in MathSciNet under [ID:936857](#), and Scopus under [AU-ID:40462260900](#)

MUSTAPHA KACI   
 UNIVERSITY OF SCIENCES AND TECHNOLOGY OF ORAN MOHAMED BOUDIAF USTO-MB  
 DEPARTMENT OF MATHEMATICS  
 FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, SIGNAL IMAGE PAROLE (SIMPA) LABORATORY  
 EL MNAOUAR, BP 1505, BIR EL DJIR 31000, ORAN, ALGERIA  
*E-mail:* [mustapha.kaci@univ-usto.dz](mailto:mustapha.kaci@univ-usto.dz) and [kaci.mustapha.95@gmail.com](mailto:kaci.mustapha.95@gmail.com)

SONIA RADJEF   
 UNIVERSITY OF SCIENCES AND TECHNOLOGY OF ORAN MOHAMED BOUDIAF USTO-MB  
 DEPARTMENT OF MATHEMATICS  
 FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, SIGNAL IMAGE PAROLE (SIMPA) LABORATORY  
 EL MNAOUAR, BP 1505, BIR EL DJIR 31000, ORAN, ALGERIA  
*E-mail:* [soniaradjef@univ-usto.dz](mailto:soniaradjef@univ-usto.dz) and [soniaradjef@yahoo.fr](mailto:soniaradjef@yahoo.fr)

*Communicated by:* Jan Goncerzewicz

(Received: 22nd of November 2021; revised: 22nd of January 2022)

---