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Thermal buckling and free vibration of Euler–Bernoulli FG nanobeams based on the higher-order nonlocal strain gradient theory

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A SIZE-DEPENDENT EULER–BERNOULLI BEAM MODEL is derived within the framework of the higher-order nonlocal strain gradient theory. Nonlocal equations of motion are derived by applying Hamilton's principle and solved with an analytical solution. The solution is obtained using the Navier solution procedure. In the case of simply supported boundary conditions, the analytical solutions of natural frequencies and critical buckling temperature for free vibration problems are obtained. The paper investigates the thermal effects on buckling and free vibrational characteristics of functionally graded size-dependent nanobeams subjected to various types of thermal loading. The influence of higher-order and lower-order nonlocal parameters and strain gradient scale on buckling and vibration are investigated for various thermal conditions. The obtained results are compared with previous research.

Key words: Euler–Bernoulli beam theory, thermal buckling, vibration, functionally graded materials, higher-order strain gradient theory.

1. Introduction

FUNCTIONALLY GRADED MATERIALS are a new class of composite materials, which have recently been used in many studies. The properties of these materials can vary along the beam length, when they are known as axially functionally graded materials, or along the beam thickness. Material properties can also vary in two directions, and such materials are known as bi–directional functionally graded materials (BDFGM). The scientific community has been motivated to conduct static and dynamic analysis of solid structural nanocomponents, such as nanobeams, nanotubes, nanoplates or nanoshells. Nanostructures are commonly used as components in electromechanical systems. The combination of those fields is of particular interest.

There are various methods for static and dynamic analysis of nanostructures, such as molecular dynamics simulations [1, 2] and non-classical continuum mechanics. Eringen's nonlocal elasticity theory [3, 4] is one of the non-classical continuum methods, which includes size-dependent effects, i.e. where stress at a reference point depends not only on the strain in this point but also on the strain in other points in the nearby region. The gradient elasticity theories [5, 6] are also examples of the non-classical continuum theories that can predict the stiffness enhancement effect. Based on the gradient elasticity theories, materials should be considered as atoms with a higher-order deformation mechanism at a small scale.

Lim *et al.* [7] presented the higher-order nonlocal strain gradient theory starting from the point of view that the length scale present in the nonlocal elasticity and the strain gradient theory describe two entirely different physical characteristics. The nonlocal elasticity theory does not include the nonlocality of a higher-order stress. On the other hand, the strain gradient theory only considers local higher-order strain gradients. The higher-order nonlocal strain theory is primarily based on the nonlocal effects of the strain gradient field, i.e. nonlocal effects in a global sense.

Thermal buckling of nanobeams has been the topic of many studies in the past. Ebrahimi *et al.* [8] investigated the influence of thermal effect on free vibration of functionally graded size-dependent nanobeams, using the differential transform method (DTM). Ebrahimi *et al.* [9] showed the effect of buckling due to the effect of a thermal load. An analytical solution was presented, where the equation was given using Hamilton's principle, based on the nonlocal third–order shear deformation theory, with the material varying through the thickness of the nanobeam. Also, the authors presented a different type of change in the critical buckling load with the changes in the nonlocal parameter and the ratio of the length and the thickness of the beam. Ebrahimi *et al.* [10] investigated thermal buckling and free vibration of a Timoshenko, simply supported, FG nanobeam. They used the classical nonlocal elasticity theory and obtained the solutions analytically. This paper also provides the results on the change in the critical buckling load, with changes in temperature, material graduation and beam thickness, as well as with a change in nonlocality parameters. The variation of dimensionless frequencies with the variation of temperature and material graduation was also considered. The consideration of the buckling characteristics of an FG microbeam in thermal environment is given in the paper [11]. The beam rested on a Pasternak type foundation, and apart from a thermal load, the beam was also loaded with an axial load on both its left and right ends.

The influence of material porosity of a functionally graded beam, the thermal effect and boundary conditions on the natural frequencies was investigated by Ebrahimi *et al.* [12]. The governing equations of motions were derived by applying Hamilton's principle and the solution was obtained by the differential transform method (DTM). The dynamic behavior of smart nanostructures was discussed by Ebrahimi and Barati [13]. They investigated the magneto–electro–elastic effect on vibrations of an FG nanobeam, with various types of boundary conditions. Material properties were changed across the thickness of the nanobeam. A refined shear deformation beam theory with the trigonometric shear strain function was used.

Li *et al.* [14] studied the free vibration of the FG Timoshenko and Euler–Bernoulli nanobeams based on the nonlocal strain gradient theory. Material properties of the FG nanobeam depended on the through-thickness power-law variation of the two materials. The authors investigated the effect of the power-law and small length-scaled effect on natural frequencies of a simply supported FG nanobeam. In this paper, comparisons of the natural frequencies of Timoshenko and Euler beams were performed. Khaniki *et al.* [15] considered the flexural vibration of an Euler–Bernoulli nanobeam model, with the nonlocal strain gradient theory. They investigated three different types of nonuniformity of the nanobeam with variations in width and thickness. The results were obtained by using the generalized differential quadrature method (GDQM). The influence of dynamic instability of a Timoshenko FG nanobeam, with thermal and magnetic loads, is given in the paper by Jalaei *et al.* [16]. Equations were derived by using Hamilton's principle, within the nonlocal strain gradient theory. Lu *et al.* [17] studied the free vibration of the sinusoidal shear deformation of a nanobeam model based on the nonlocal strain gradient theory. Navier's method was utilized to obtain analytical solutions for natural frequencies of simply supported nanobeams.

Barati *et al.* [18] studied wave propagation of a porous double-nanobeam system on an elastic substrate. They developed a general bi–Helmholtz nonlocal strain–gradient elasticity model, where the equations were derived using Hamilton's principle, while the results were obtained analytically.

Based on the nonlocal strain gradient theory and various higher-order shear deformation theories, Al-Shuajiri *et al.* [19] studied the buckling and free vibration of functionally graded sandwich micro-beams resting on an elastic foundation. The authors reported on the effects of the nonlocal parameter, length scale parameter, gradient index, different cross-section shapes, temperature change and stiffnesses of Winkler and shear layer springs on the dimensionless critical buckling load and dimensionless frequencies.

Within the framework of the nonlocal strain gradient theory, Čanađija *et al.* [20] investigated longitudinal and transversal displacement of a nanobeam model, with a slope for different values of nonlocal parameters, with clamped ends and in an inhomogeneous temperature field. The static problems of a nanobeam model, for different boundary conditions, were investigated in Apuzzo *et al.* [21]. They provided values of the maximum nondimensional deflection for different values of nanoscale parameters. Barretta *et al.* [22] presented a stress-driven nonlocal model for the structural analysis of a nanobeam model, with reference to thermoelastic behavior. They presented a graph view of the total axial deformation, for different values of the nonlocal parameter with uniform and non-uniform temperature loads, for a beam with clamped ends.

Pavlović *et al.* [23, 24] studied a stability and instability problem of a nanobeam subjected to a compressive axial load based on the higher-order nonlocal strain gradient theory. According to the direct Lyapunov method, the authors obtained the bounds of the almost sure asymptotic stability and instability, which were verified by numerical results using the Monte Carlo simulation method.

In this paper, size-dependent Euler–Bernoulli nanobeam models, which account for through-thickness power-law variation of two-constituent FG materials, are deduced within the framework of the higher-order nonlocal strain gradient theory (HONSGT). This theory is employed to study the effects of buckling and vibrational behavior of nanobeams in different thermal environments. Equations of motion are derived using Hamilton's principle. By employing an analytical solution procedure, the closed-form critical buckling temperature and frequency are obtained for simply supported boundary conditions. The obtained results are compared with the literature to confirm the validity of the solution. The influence of higher-order and lower-order nonlocal parameters and strain gradient scale on buckling and vibration are investigated. Finally, certain important conclusions are summarized.

2. Mathematical model. Problem description

Consider a nanobeam of functionally graded material, where the graded properties are assumed to be in the through-thickness direction. The system of interest is a rectangular functionally graded nanobeam of length L, width b and thickness h (Fig. 1). The beam is subjected to an in-plane thermal loading, where, according to the rule of mixture, the effective material properties P_f are distributed as follows [25]

$$P_f(T, z) = P_c(T)V_c(z) + P_m(T)V_m(z), \tag{2.1}$$

where the volume fraction of the ceramic $V_c(z)$ and the volume fraction of the metal $V_m(z)$ constituents of the beam may be expressed using the power-law

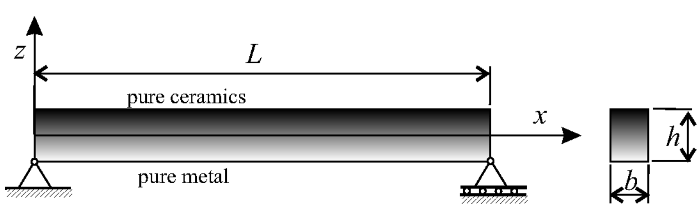

Fig. 1. Geometry and coordinates of the functionally graded beam.

distribution

$$V_c(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^p, \qquad V_m(z) = 1 - V_c(z), \qquad -\frac{h}{2} \le z \le \frac{h}{2}. \tag{2.2}$$

The temperature-dependent material properties (such as Young's modulus E, thermal expansion coefficient α, mass density ρ and thermal conductivity κ) can be written as follows [26]

$$P(T) = P_0(P_{-1}T^{-1} + 1 + P_1T + P_2T^2 + P_3T^3), \tag{2.3}$$

where P_0, P_{-1}, P_1, P_2 and P_3 are the coefficients that can be seen in the table of material properties for Si_3N_4 and SUS304 (Table 1).

Table 1. Temperature-dependent coefficient of Young's modulus E, thermal expansion coefficient α, mass density ρ and thermal conductivity κ for Si_3N_4 and SUS304.

Material	Properties	P_0	P_{-1}	P_1	P_2	P_3
Si_3N_4	E [Pa]	348.43e+9	0	−3.010e-4	2.160e-7	−8.946e-11
	α [K^{-1}]	5.8723e-6	0	9.095ee-4	0	0
	ρ [kg/m^3]	2370	0	0	0	0
	κ [W/mK]	13.723	0	−1.032e-3	5.466e-7	−7.876e-11
$SUS304$	E [Pa]	201.04e+9	0	3.079e-4	−6.534e-7	0
	α [K^{-1}]	12.330e-6	0	8.086e-4	0	0
	ρ [kg/m^3]	8166	0	0	0	0
	κ [W/mK]	15.379	0	−1.264e-3	2.092e-6	−7.223e-10

For the power-law distribution (2.2), the effective material properties are

$$P(z,T) = (P_c(T) - P_m(T))\left(\frac{z}{h} + \frac{1}{2}\right)^p + P_m(T). \tag{2.4}$$

The bottom surface ($z = -h/2$) of the FG beam is pure metal (SUS304) and the top surface ($z = h/2$) is pure ceramic (Si_3N_4).

3. The higher-order nonlocal strain gradient model for the FG nanobeam

Based on the higher-order nonlocal strain gradient theory [7], the nonlocal stress in a reference point x depends not only on the strain at that location but also on the strains in all other points in the nearby region. According to this theory, the internal strain energy density function can be expressed as

$$U_0 = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \int_V \alpha_0(|x - x'|, e_0 a) \varepsilon'_{kl} \, dV' + \frac{l^2}{2} C_{ijkl} \varepsilon_{ij,m} \int_V \alpha_1(|x - x'|, e_1 a) \varepsilon'_{kl,m} \, dV', \tag{3.1}$$

where C_{ijkl} is the elastic modulus tensor of classical elasticity, ε_{ij} and ε'_{ij} are the Cartesian components of the strain tensor in points x and x'; α_0 and α_1 are the kernel function related to the nonlocal effects with respect to the strain field and the first order strain gradient field; $e_0$and e_1 are the nonlocal material constants, $a$is the internal characteristic length and l is the strain gradient length scale parameter.

By using Eq. (3.1), the classical stress tensor σ, the higher-order stress tensor $\sigma^{(1)}$ and the total stress t can be written as follows

$$\begin{aligned} \boldsymbol{\sigma} &= \int_V \boldsymbol{\alpha}_0(|x - x'|, e_0 a) \mathrm{C} : \varepsilon' dV', \\ \boldsymbol{\sigma}^{(1)} &= l^2 \int_V \boldsymbol{\alpha}_1(|x - x'|, e_0 a) \mathrm{C} : \nabla \boldsymbol{\varepsilon}' dV', \quad \mathbf{t} = \boldsymbol{\sigma} - \nabla \boldsymbol{\sigma}^{(1)}. \end{aligned} \tag{3.2}$$

In Eq. (3.2) the symbol ":" is used to denote the double-dot product. For an elastic material in the one-dimensional case, the generalized nonlocal constitutive relations in a differential form based on the higher-order nonlocal strain gradient theory may be simplified as

$$\left(1 - \mu_1 \frac{\partial^2}{\partial x^2}\right)\left(1 - \mu_0 \frac{\partial^2}{\partial x^2}\right) t_{xx} = E\left[\left(1 - \mu_1 \frac{\partial^2}{\partial x^2}\right) - l^2 \left(1 - \mu_0 \frac{\partial^2}{\partial x^2}\right) \frac{\partial^2}{\partial x^2}\right] \varepsilon_{xx}, \tag{3.3}$$

where $\mu_0 = (e_0 a)^2$ and $\mu_1 = (e_1 a)^2$. The normal of the total stress tensor of the nonlocal strain gradient theory are defined as

$$t_{xx} = \sigma_{xx} - \frac{\partial \sigma_{xx}^{(1)}}{\partial x}, \tag{3.4}$$

where σ_{xx} is the classical normal stress component, $\sigma_{xx}^{(1)}$ is the higher-order normal stress component, ε_{xx} is the normal strain, E is Young's modulus.

It is worth mentioning that the integral formulation of the fully nonlocal elasticity theory leads to problems. ROMANO *et al.* [27] showed that the integral presentation of the nonlocal elasticity theory has a unique solution only if constitutive boundary conditions are satisfied. They also presented a paradox of the transformation from the integral to the differential form of the nonlocal model, for beam bending problems, where the solution of the bending moment within the differential form framework should be checked in the integral formulation. BARRETTA and SCIARRA [28] applied constitutive boundary conditions for a cantilever nanobeam, subjected to end-point loading. It should be noted that a differential formulation of the nonlocal elasticity theory was used in that paper.

4. Kinematic relations

The displacement components of any material point in the x, y and z direction can be written as

$$q_x = u(x,t) - z\frac{\partial w}{\partial x}, \tag{4.1}$$

$$q_z = w(x,t), \tag{4.2}$$

where u and w are the displacement components of the mid-plane in the x and z direction. Using Eqs. (4.1) and (4.2), the nonzero component of the beam is obtained as

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}, \tag{4.3}$$

The governing equations of motion are obtained based on Hamilton's principle, which is expressed as

$$\int\limits_{t_1}^{t_2} (\delta U + \delta V - \delta K)dt = 0, \tag{4.4}$$

in a time interval $t_1 < t < t_2$. δU is the virtual strain energy

$$\begin{aligned}\delta U &= \int\limits_v \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{xx}^{(1)}\frac{\partial(\delta\varepsilon_{xx})}{\partial x}\right)dv \\ &= \int\limits_L \left(N\delta\frac{\partial u}{\partial x} - M\delta\frac{\partial^2 w}{\partial x^2}\right)dx + \left[N^{(1)}\delta\frac{\partial u}{\partial x} - M^{(1)}\delta\frac{\partial^2 w}{\partial x^2}\right]_0^L.\end{aligned} \tag{4.5}$$

Here we consider the following stress resultant:

$$N = \int_A t_{xx} dA,\ M = \int_A z t_{xx} dA,\ N^{(1)} = \int_A \sigma_{xx}^{(1)} dA,\ M^{(1)} = \int_A z\sigma_{xx}^{(1)} dA, \tag{4.6}$$

variation of the work δV by thermal expansion

$$\begin{aligned} \delta V &= -\int_v E(T,z)\alpha(T,z)(T-T_0)\frac{\partial w}{\partial x}\frac{\partial}{\partial x}(\delta w) dv \\ &= -\int_0^L \left(N^T \frac{\partial w}{\partial x}\frac{\partial}{\partial x}(\delta w)\right) dx, \end{aligned} \tag{4.7}$$

where N^T is the thermal resultant

$$N^T = \int_{-h/2}^{h/2} E(T,z)\alpha(T,z)(T-T_0) b dz, \tag{4.8}$$

and $T_0 = 300$ K is the reference temperature.

δK is the virtual kinetic energy

$$\begin{aligned} \delta K &= \frac{1}{2}\int_v \rho(T,z)\delta\left(\left(\frac{\partial q_x}{\partial t}\right)^2 + \left(\frac{\partial q_z}{\partial t}\right)^2\right) dv \\ &= \int_0^L \left(I_0\left(\frac{\partial u}{\partial t}\delta\left(\frac{\partial u}{\partial t}\right) + \frac{\partial w}{\partial t}\delta\left(\frac{\partial w}{\partial t}\right)\right)\right. \\ &\quad \left. - I_1\left(\frac{\partial u}{\partial t}\delta\left(\frac{\partial^2 w}{\partial x \partial t}\right) + \frac{\partial^2 w}{\partial x\partial t}\delta\left(\frac{\partial u}{\partial t}\right)\right) + I_2 \frac{\partial^2 w}{\partial x\partial t}\delta\left(\frac{\partial^2 w}{\partial x\partial t}\right)\right) dx, \end{aligned} \tag{4.9}$$

where the mass moments of inertia are defined as follows

$$(I_0, I_1, I_2) = \int_A (1, z, z^2)\rho(z,T)\, dA. \tag{4.10}$$

By substituting Eqs. (4.5), (4.7) and (4.9) into Eq. (4.4), using integration by parts and setting the coefficients of δu and δw to zero, one obtains the following governing equations of motion based on the Euler–Bernoulli beam theory

$$\delta u: \ \frac{\partial N}{\partial x} = I_0 \frac{\partial^2 u}{\partial t^2} - I_1 \frac{\partial^3 w}{\partial x \partial t^2}, \tag{4.11}$$

$$\delta w: \ \frac{\partial^2 M}{\partial x^2} - N^T \frac{\partial^2 w}{\partial x^2} = I_0 \frac{\partial^2 w}{\partial t^2} + I_1 \frac{\partial^3 u}{\partial x \partial t^2} - I_2 \frac{\partial^4 w}{\partial x^2 \partial t^2}, \tag{4.12}$$

with the classical boundary conditions (at $x = 0$ or $x = L$)

$$N = 0 \quad \text{or} \quad u = 0, \tag{4.13}$$

$$\frac{\partial M}{\partial x} - I_1 \frac{\partial^2 u}{\partial t^2} + I_2 \frac{\partial^3 u}{\partial x \partial t^2} = 0 \quad \text{or} \quad w = 0, \tag{4.14}$$

$$M = 0 \quad \text{or} \quad \frac{\partial w}{\partial x} = 0, \tag{4.15}$$

and the non-classical boundary conditions (at $x = 0$ or $x = L$)

$$N^{(1)} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} = 0, \tag{4.16}$$

$$M^{(1)} = 0 \quad \text{or} \quad \frac{\partial^2 w}{\partial x^2} = 0. \tag{4.17}$$

Considering the above and integrating Eq. (3.3) over the beam's cross-section, the force-strain and moment-strain relation can be obtained as follows

$$\begin{aligned} &\left(1 - \mu_0 \frac{\partial^2}{\partial x^2}\right)\left(1 - \mu_1 \frac{\partial^2}{\partial x^2}\right) N \\ &\quad = A_{xx} \left[\left(1 - \mu_1 \frac{\partial^2}{\partial x^2}\right) - l^2 \left(1 - \mu_0 \frac{\partial^2}{\partial x^2}\right) \frac{\partial^2}{\partial x^2}\right] \frac{\partial u}{\partial x} \\ &\qquad - B_{xx} \left[\left(1 - \mu_1 \frac{\partial^2}{\partial x^2}\right) - l^2 \left(1 - \mu_0 \frac{\partial^2}{\partial x^2}\right) \frac{\partial^2}{\partial x^2}\right] \frac{\partial^2 w}{\partial x^2}, \end{aligned} \tag{4.18}$$

$$\begin{aligned} &\left(1 - \mu_0 \frac{\partial^2}{\partial x^2}\right)\left(1 - \mu_1 \frac{\partial^2}{\partial x^2}\right) M \\ &\quad = B_{xx} \left[\left(1 - \mu_1 \frac{\partial^2}{\partial x^2}\right) - l^2 \left(1 - \mu_0 \frac{\partial^2}{\partial x^2}\right) \frac{\partial^2}{\partial x^2}\right] \frac{\partial u}{\partial x} \\ &\qquad - C_{xx} \left[\left(1 - \mu_1 \frac{\partial^2}{\partial x^2}\right) - l^2 \left(1 - \mu_0 \frac{\partial^2}{\partial x^2}\right) \frac{\partial^2}{\partial x^2}\right] \frac{\partial^2 w}{\partial x^2}, \end{aligned} \tag{4.19}$$

in which the cross-sectional rigidities are

$$(A_{xx}, B_{xx}, C_{xx}) = \int_A (1, z, z^2) E(z, T)\, dA. \tag{4.20}$$

We introduce the following dimensionless parameters

$$\xi = \frac{x}{L},\ U(\xi, \tau) = \frac{u(x, t)}{L},\ W(\xi, \tau) = \frac{w(x, t)}{L},\ \tau = \frac{t}{L^2} \sqrt{\frac{E_c(T_c) I}{\rho_c(T_c) A}}, \tag{4.21}$$

where $E_c(T_c)$ and $\rho_c(T_c)$ are Young's modulus and the mass density of ceramic Si_3N_4 at the temperature T_c, $I = bh^3/12$ is the moment of inertia of the rectangular cross-section of the beam and $A = bh$. The explicit relation of the nonlocal normal force and bending moment can be derived by substituting the second and fourth derivatives of these values from Eqs. (4.11)–(4.12) into Eqs. (4.18)–(4.19) as follows

$$N = A_{xx}\left\{\mathcal{L}_{(3)}\left[\frac{\partial u}{\partial x} + k_B\frac{\partial^2 w}{\partial x^2}\right] + k_I\mathcal{L}_{(2)}\left[k_{I0}\frac{\partial^3 u}{\partial x\partial t^2} - k_{I1}\frac{\partial^4 w}{\partial x^2\partial t^2}\right]\right\}, \tag{4.22}$$

$$\begin{aligned} M = A_{xx}L\Big\{&\mathcal{L}_{(3)}\left[k_B\frac{\partial u}{\partial x} - k_C\frac{\partial^2 w}{\partial x^2}\right] \\ &+ \mathcal{L}_{(2)}\left[k_N\frac{\partial^2 w}{\partial x^2} + k_Ik_{I0}\frac{\partial^2 w}{\partial t^2} + k_Ik_{I1}\frac{\partial^3 u}{\partial x\partial t^2} - k_Ik_{I2}\frac{\partial^4 w}{\partial x^2\partial t^2}\right]\Big\}\end{aligned} \tag{4.23}$$

where the linear differential operators are

$$\begin{gathered}\nabla = \frac{\partial}{\partial \xi}, \quad \mathcal{L}_{(0)} = 1 - k_{\mu 0}\nabla^2, \quad \mathcal{L}_{(1)} = 1 - k_{\mu 1}\nabla^2, \\ \mathcal{L}_{(2)} = k_{\mu 0} + k_{\mu 1} - k_{\mu 0}k_{\mu 1}\nabla^2, \quad \mathcal{L}_{(3)} = \mathcal{L}_{(1)} - k_l\mathcal{L}_{(0)}\nabla^2\end{gathered} \tag{4.24}$$

and the marks are

$$\begin{gathered} k_B = \frac{B_{xx}}{A_{xx}L}, \quad k_C = \frac{C_{xx}}{A_{xx}L^2}, \quad k_N = \frac{N^T}{A_{xx}}, \quad k_l = \frac{l^2}{L^2}, \quad k_{\mu 0} = \frac{\mu_0}{L^2}, \\ k_{\mu 1} = \frac{\mu_1}{L^2}, \quad k_I = \frac{I}{AL^2}, \quad k_{I0} = \frac{E_c(T_c)I_0}{\rho_c(T_c)A_{xx}}, \\ k_{I1} = \frac{E_c(T_c)I_1}{\rho_c(T_c)A_{xx}L}, \quad k_{I2} = \frac{E_c(T_c)I_2}{\rho_c(T_c)A_{xx}L^2}.\end{gathered} \tag{4.25}$$

Substituting the derivative for N and M from Eqs. (4.22)–(4.23) into Eqs. (4.11)–(4.12), the nonlocal governing equations of the Timoshenko FG nanobeam can be derived as follows

$$\mathcal{L}_{(3)}\left[\frac{\partial^2 U}{\partial \xi^2} - k_B\frac{\partial^3 W}{\partial \xi^3}\right] - k_I\mathcal{L}_{(4)}\left[k_{I0}\frac{\partial^2 U}{\partial \tau^2} - k_{I1}\frac{\partial^3 W}{\partial \xi\partial \tau^2}\right] = 0, \tag{4.26}$$

$$\begin{aligned}&\mathcal{L}_{(3)}\left[k_B\frac{\partial^3 U}{\partial \xi^3} - k_C\frac{\partial^4 W}{\partial \xi^4}\right] \\ &\quad - \mathcal{L}_{(4)}\left[k_N\frac{\partial^2 W}{\partial \xi^2} + k_I\left(k_{I0}\frac{\partial^2 W}{\partial \tau^2} + k_{I1}\frac{\partial^3 U}{\partial \xi\partial \tau^2} - k_{I2}\frac{\partial^4 W}{\partial \xi^2\partial \tau^2}\right)\right] = 0,\end{aligned} \tag{4.27}$$

where $\mathcal{L}_{(4)} = \mathcal{L}_{(0)}\mathcal{L}_{(1)} = 1 - \mathcal{L}_{(2)}\nabla^2 = 1 - (k_{\mu 1} + k_l)\nabla^2 + k_{\mu 0}k_l\nabla^4$.

5. Temperature rise

In the case of a uniform temperature rise (UTR), the temperature of the FG beam uniformly rises by ΔT. Since the temperature is constant in the z-direction, then

$$T(z) = T_0 + \Delta T = \text{const.} \tag{5.1}$$

In the case of a linear temperature rise (LTR), the temperature of the FG beam varies linearly along the thickness of the beam

$$T(z) = T_m + \Delta T\left(\frac{1}{2} + \frac{z}{h}\right), \tag{5.2}$$

where the temperature of the top and the bottom surface of the nanobeam is, respectively,

$$T_c = T\left(\frac{h}{2}\right), \qquad T_m = T\left(-\frac{h}{2}\right), \tag{5.3}$$

where $\Delta T = T_c - T_m$. In this paper, it is assumed that the temperature of the bottom surface is $T_m = T_0 + 5 = 305$ K.

In the case of heat conduction across the thickness, the temperature of the FG nanobeam varies nonlinearly (NLTR) along the thickness of the beam. The one-dimensional steady state heat conduction problem can be formulated by a differential equation [29]

$$\frac{d}{dz}\left(\kappa(z,T)\frac{dT(z)}{dz}\right) = 0, \tag{5.4}$$

where the known temperature boundary conditions on the bottom and the top surface are given as in (5.3). In order to present an analytical solution for Eq. (5.4), it is common to assume that thermal conductivity $\kappa = \kappa(z)$ is independent of temperature. Taking this into account, the solution of Eq. (5.4) can be obtained in a power series as follows

$$T(z) = T_m + \frac{\Delta T}{\lambda}\sum_{i=0}^{n}\frac{1}{ki+1}\left(\frac{1}{2}+\frac{z}{h}\right)^{ki+1}\left(\frac{\kappa_m - \kappa_c}{\kappa_m}\right)^{i}, \tag{5.5}$$

where

$$\lambda = \sum_{i=0}^{n}\frac{1}{ki+1}\left(\frac{\kappa_m - \kappa_c}{\kappa_m}\right)^{i}. \tag{5.6}$$

6. Solution procedures

This section presents the analytical solutions for the vibration problem described by Eqs. (4.26) and (4.27). The Navier solution approach is used to determine the analytical solutions of vibration frequencies and critical buckling temperature for simply supported boundary conditions.

The displacement functions can be assumed to be periodic in time in the form

$$U(\xi,\tau)=\sum_{n=1}^{\infty}U_n\cos(n\pi\xi)e^{i\omega_n\tau}, \tag{6.1}$$

$$W(\xi,\tau)=\sum_{n=1}^{\infty}W_n\sin(n\pi\xi)e^{i\omega_n\tau}, \tag{6.2}$$

where $i=\sqrt{-1}$, U_j, W_j $(j=1,2,\ldots,n)$ are the unknown Fourier coefficients to be determined for each n value and ω_n is the frequency of vibration. It can be checked that the series solution (6.1) and (6.2) satisfies the classical boundary conditions (4.13)–(4.15) and non-classical boundary conditions (4.16) and (4.17).

Eliminating U from Eqs. (4.26)–(4.27), the governing differential equation becomes

$$\left(\mathcal{L}_{(5)}\frac{\partial^4}{\partial\tau^4}+\mathcal{L}_{(6)}\frac{\partial^2}{\partial\tau^2}+\mathcal{L}_{(7)}\right)W=0, \tag{6.3}$$

where the linear differential operators are

$$\mathcal{L}_{(5)}=k_I^2(\beta_1\nabla^2-k_{I0}^2)\mathcal{L}_{(4)}^2, \tag{6.4}$$

$$\mathcal{L}_{(6)}=k_I\mathcal{L}_{(4)}(k_{I0}\mathcal{L}_{(3)}\nabla^2-k_Nk_{I0}\mathcal{L}_{(4)}\nabla^2-\beta_2\mathcal{L}_{(3)}\nabla^4), \tag{6.5}$$

$$\mathcal{L}_{(7)}=\mathcal{L}_{(3)}(\beta_3\mathcal{L}_{(3)}\nabla^6+k_N\mathcal{L}_{(4)}\nabla^4), \tag{6.6}$$

and

$$\beta_1=k_{I0}k_{I2}-k_{I1}^2,\quad \beta_2=k_{I2}-2k_Bk_{I1}+k_Ck_{I0},\quad \beta_3=k_C-k_B^2. \tag{6.7}$$

Substituting Eq. (6.2) into Eq. (6.3), we get the following characteristic equation

$$A_\omega\omega_n^4+B_\omega\omega_n^2+C_\omega=0, \tag{6.8}$$

where

$$\begin{aligned}
&A_\omega=-\alpha_4^2k_I^2[\beta_1n^2\pi^2+k_{I0}^2],\\
&B_\omega=\alpha_4k_I[\alpha_3k_{I0}n^2\pi^2+\beta_2\alpha_3n^4\pi^4-k_Nk_{I0}\alpha_4n^2\pi^2],\\
&C_\omega=\alpha_3n^4\pi^4(k_N\alpha_4-\alpha_3\beta_3n^2\pi^2),\\
&\alpha_0=1+k_{\mu0}n^2\pi^2,\quad \alpha_1=1+k_{\mu1}n^2\pi^2,\quad \alpha_2=k_{\mu0}+k_{\mu1}+k_{\mu0}k_{\mu1}n^2\pi^2,\\
&\alpha_3=\alpha_1+k_l\alpha_0n^2\pi^2,\quad \alpha_4=1+\alpha_2n^2\pi^2.
\end{aligned} \tag{6.9}$$

The smaller root (the eigenvalue $\lambda_n = \omega_n^2$) of Eq. (6.8) is

$$\omega_n^2 = \frac{-B_\omega + \sqrt{B_\omega^2 - 4A_\omega C_\omega}}{2A_\omega}. \tag{6.10}$$

By setting the fundamental frequency ω_1 to zero, we find the critical buckling temperature ΔT_{cr} (for $n = 1$). This condition is satisfied if the coefficient $C_\omega = 0$. After a simple transformation, we come to a relation among the parameters of the system that meets the required condition

$$k_l = \frac{1 + k_{\mu 1} n^2\pi^2}{1 + k_{\mu 0} n^2\pi^2} \frac{k_N(1 + k_{\mu 0} n^2\pi^2) - \beta_3 n^2\pi^2}{\beta_3 n^4\pi^4}. \tag{6.11}$$

It is interesting that the parameters of the system (6.9) can be simplified to certain interesting cases:

A. (Eringen's nonlocal continuum theory – ENCT). In the case where the strain gradient length scale ($l = 0$) and the nonlocal parameter ($\mu_1 = 0$) are zero the parameters of system (55) are

$$\begin{aligned} A_\omega &= -k_I^2(1 + k_{\mu 0} n^2\pi^2)^2[(k_{I0}k_{I2} - k_{I1}^2)n^2\pi^2 + k_{I0}^2], \\ B_\omega &= k_I n^2\pi^2(1 + k_{\mu 0} n^2\pi^2) \\ &\quad \times [(k_{I2} - 2k_B k_{I1} + k_D k_{I0})n^2\pi^2 - k_N k_{I0}(1 + k_{\mu 0} n^2\pi^2) + k_{I0}], \\ C_\omega &= n^4\pi^4[k_N(1 + k_{\mu 0} n^2\pi^2) - (k_C - k_B^2)n^2\pi^2]. \end{aligned} \tag{6.12}$$

By setting the strain gradient length scale ($l = 0$) and the nonlocal parameter ($\mu_1 = 0$) to zero we can find the critical temperature ΔT_{cr} from Eq. (6.11) as follows

$$k_{\mu 0} = \frac{n^2\pi^2(k_C - k_B^2) - k_N}{k_N n^2\pi^2}. \tag{6.13}$$

In a special case of the homogenous beam

$$k_{I0} = 1, \quad k_{I1} = 0, \quad k_{I2} = k_C = k_I, \quad k_B = 0, \tag{6.14}$$

where the smaller root is

$$\omega_n^2 = \frac{1}{1 + k_{\mu 0} n^2\pi^2} \frac{n^2\pi^2\left(n^2\pi^2 - k_N \frac{AL^2}{I}\right)}{1 + n^2\pi^2 \frac{I}{AL^2}}. \tag{6.15}$$

B. (Classical continuum theory). In the case where the strain gradient length scale ($l = 0$) and the nonlocal parameters ($\mu_0 = 0$, $\mu_1 = 0$) are zero the parameters of the system are

$$\begin{aligned} A_\omega &= -k_I^2[(k_{I0}k_{I2} - k_{I1}^2)n^2\pi^2 + k_{I0}^2], \\ B_\omega &= k_I n^2\pi^2[(k_{I2} - 2k_B k_{I1} + k_C k_{I0})n^2\pi^2 - k_N k_{I0} + k_{I0}], \\ C_\omega &= n^4\pi^4[k_N - (k_C - k_B^2)n^2\pi^2]. \end{aligned} \tag{6.16}$$

By setting $\omega_1 = 0$, the critical buckling temperature ΔT_{cr} can be found if the coefficient $C_\omega = 0$. In this case

$$k_N = n^2\pi^2(k_C - k_B^2). \tag{6.17}$$

In a special case of the homogenous beam, based on expressions (4.10), (4.20) and (4.25), one can obtain,

$$k_{I0} = 1, \quad k_{I1} = 0, \quad k_{I2} = k_C = k_I, \quad k_B = 0, \tag{6.18}$$

and the frequencies of the beam can be derived as follows

$$\omega_n^2 = \frac{n^2\pi^2\left(n^2\pi^2 - k_N\frac{AL^2}{I}\right)}{1 + n^2\pi^2\frac{I}{AL^2}}. \tag{6.19}$$

In a special case without thermal environment ($k_N = 0$), it gets

$$\omega_n^2 = \frac{n^4\pi^4}{1 + n^2\pi^2\frac{I}{AL^2}}. \tag{6.20}$$

C. In the case where the strain gradient length scale (l=0) is zero the parameters of the system are

$$\begin{aligned}
A_\omega &= -k_I^2(1 + k_{\mu 0}n^2\pi^2)^2(1 + k_{\mu 1}n^2\pi^2)^2[(k_{I0}k_{I2} - k_{I1}^2)n^2\pi^2 + k_{I0}^2],\\
B_\omega &= k_I n^2\pi^2(1 + k_{\mu 0}n^2\pi^2)(1 + k_{\mu 1}n^2\pi^2)^2\\
&\quad \times [(k_{I2} - 2k_B k_{I1} + k_C k_{I0})n^2\pi^2 - k_N k_{I0}(1 + k_{\mu 0}n^2\pi^2) + k_{I0}],\\
C_\omega &= n^4\pi^4(1 + k_{\mu 1}n^2\pi^2)^2[k_N(1 + k_{\mu 0}n^2\pi^2) - (k_C - k_B^2)n^2\pi^2].
\end{aligned} \tag{6.21}$$

It is noticeable that in this case the natural frequency does not depend on the nonlocal parameters μ_1, but only on the parameter μ_0. By setting the strain gradient length scale ($l = 0$) to zero we can find the critical temperature ΔT_{cr} from Eq. (6.11) as follows

$$k_{\mu 0} = \frac{n^2\pi^2(k_C - k_B^2) - k_N}{k_N n^2\pi^2}. \tag{6.22}$$

It is noticeable that in this case the critical temperature ΔT_{cr} does not depend on the nonlocal parameters μ_1, but only on the parameter μ_0, and is the same as the one we can determine from conditions (6.13) in case A.

D. (Lower-order nonlocal strain gradient theory – LONSGT). In the case where the nonlocal parameters are equal ($\mu_1 = \mu_0$) the parameters of the system are

$$\begin{aligned}
A_\omega &= -k_I^2(1+k_{\mu 0}n^2\pi^2)^4[(k_{I0}k_{I2}-k_{I1}^2)n^2\pi^2+k_{I0}^2],\\
B_\omega &= k_I n^2\pi^2(1+k_{\mu 0}n^2\pi^2)^3[(k_{I2}-2k_Bk_{I1}+k_Ck_{I0})(1+k_l n^2\pi^2)n^2\pi^2\\
&\quad - k_N k_{I0}(1+k_{\mu 0}n^2\pi^2)+k_{I0}(1+k_l n^2\pi^2)],\\
C_\omega &= n^4\pi^4(1+k_{\mu 0}n^2\pi^2)^2(1+k_l n^2\pi^2)\\
&\quad \times [k_N(1+k_{\mu 0}n^2\pi^2)-(k_C-k_B^2)(1+k_l n^2\pi^2)n^2\pi^2].
\end{aligned} \tag{6.23}$$

By setting $\mu_1 = \mu_0$we can find the critical temperature ΔT_{cr} from Eq. (6.11) as follows

$$k_l = \frac{k_N(1+k_{\mu 0}n^2\pi^2)-\beta_3 n^2\pi^2}{\beta_3 n^4\pi^4}. \tag{6.24}$$

In this case, when the ratio is $l_\mu = l^2/\mu_0 = l^2/\mu_1 = 1$, the parameters of the system are

$$\begin{aligned}
A_\omega &= -k_I^2(1+k_{\mu 0}n^2\pi^2)^4[(k_{I0}k_{I2}-k_{I1}^2)n^2\pi^2+k_{I0}^2],\\
B_\omega &= k_I n^2\pi^2(1+k_{\mu 0}n^2\pi^2)^4[(k_{I2}-2k_Bk_{I1}+k_Ck_{I0})n^2\pi^2-k_Nk_{I0}+k_{I0}],\\
C_\omega &= n^4\pi^4(1+k_{\mu 0}n^2\pi^2)^4[k_N-(k_C-k_B^2)n^2\pi^2],
\end{aligned} \tag{6.25}$$

and the natural frequency does not depend on the nonlocal parameters μ_0 and μ_1, but also not on the strain gradient length scale l. The natural frequencies of the system are equal to the frequencies for the case of the classical continuum theory.

E. (Strain gradient theory). In the case where the nonlocal parameters are zero ($\mu_1 = \mu_0 = 0$) the parameters of the system are

$$\begin{aligned}
A_\omega &= -k_I^2[(k_{I0}k_{I2}-k_{I1}^2)n^2\pi^2+k_{I0}^2],\\
B_\omega &= k_I n^2\pi^2[(k_{I2}-2k_Bk_{I1}+k_Ck_{I0})(1+k_l n^2\pi^2)n^2\pi^2\\
&\quad - k_Nk_{I0}+k_{I0}(1+k_l n^2\pi^2)],\\
C_\omega &= n^4\pi^4(1+k_l n^2\pi^2)[k_N-(k_C-k_B^2)(1+k_l n^2\pi^2)n^2\pi^2].
\end{aligned} \tag{6.26}$$

By setting $\mu_1 = \mu_0 = 0$ we can find the critical temperature ΔT_{cr} from Eq. (6.11) as follows

$$k_l = \frac{k_N-\beta_3 n^2\pi^2}{\beta_3 n^4\pi^4}. \tag{6.27}$$

7. Results and discussion

This section examines the influence of temperature change, FG distribution and small-scale effect on the nondimensional natural frequencies. Varying amounts of small-scale parameters are observed and the variations of the critical buckling temperature and nondimensional natural frequencies with respect to the variations of small-scale parameters are discussed. The functionally graded nanobeam is composed of metal (SUS304) and ceramic (Si_3N_4), where its bottom surface is pure metal and top surface is pure ceramic Nitride. The considered beam has the following dimensions: length $L = 10$ nm, width $b = 1$ nm and thickness h varies.

The validity of the proposed method is confirmed by comparing the obtained results with the results from the literature [30]. A comparison of the dimensionless natural frequencies for the SS nanobeam is shown in Table 2 for different values of $k_{\mu_0} = 0$, 0.01, 0.02, temperature $\Delta T = 0, 20, 40$ and for different values of the gradient index $p = 0.2, 1, 5$.

Table 2. Comparison with [30] of the nondimensional fundamental frequency for a SS FG nanobeam with various nonlocal parameters μ_0, temperature and p.

k_{μ_0}	ΔT [K]	p=0.2		p=1		p=5	
		[30]	Present	[30]	Present	[30]	Present
0	0	8.6845	8.6846	7.0638	7.0638	6.0496	6.0497
	20	8.3092	8.3151	6.6661	6.6708	5.6474	5.6514
	40	7.9105	7.9157	6.2332	6.2374	5.2019	5.2054
0.01	0	8.2853	8.2853	6.7390	6.7391	5.7715	5.7716
	20	7.8910	7.8966	6.3209	6.3254	5.3484	5.3522
	40	7.4700	7.4750	5.8629	5.8668	4.8759	4.8792
0.02	0	7.9365	7.9365	6.4553	6.4554	5.5286	5.5286
	20	7.5239	7.5292	6.0175	6.0218	5.0853	5.0889
	40	7.0812	7.0859	5.5346	5.5382	4.5859	4.5890

The validation results of the present work are given in Table 2, which contains the comparison with the results from paper [30]. The results are obtained from Eq. (6.13), considering $l = 0$ and $\mu_1 = 0$. Papers, Ebrahimi *et al.* [8] and Ebrahimi *et al.* [10], give the results that were solved for the case of variation gradient indices p and nonlocal parameter μ_0 on the critical buckling temperature ΔT_{cr} and dimensionless natural frequencies for a simply supported beam. This paper shows the influence of the higher order nonlocal parameter μ_1 and the strain gradient length scale l.

The results presented in Tables 3–6. are the critical buckling temperature of the simply supported FG nanobeam, with the results being presented through the variation of the nonlocal parameters and strain gradient scale ($\mu_0 = 0, 1, 2, 3$; $\mu_1 = 0, 1, 2, 3$; $l^2 = 0, 1, 2, 3$), with two different temperature rises: linear (LTR) and nonlinear (NLTR). The results in the mentioned Tables show that an increase in the strain gradient length scale influences an increase in the critical buckling temperature.

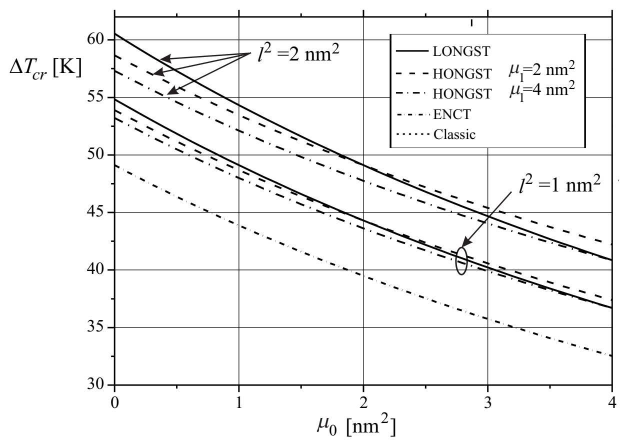

Fig. 2. Variation of the critical buckling temperature of the SS FG nanobeam with respect to the nonlocal parameter μ_0 for different values of the nonlocal parameter μ_1, the strain length scale l and LNR ($p = 1$, $L/h = 50$).

Figure 2 shows the dependence of the critical buckling temperature on the nonlocal parameter μ_0, for different values of the higher order nonlocal parameter μ_1 and the strain gradient length scale l and LONGST. It can also be concluded that with an increase in the nonlocal parameter μ_0, the critical buckling load decreases. Figure 3 shows the dependence of the critical buckling temperature on the nonlocal parameter μ_1 for different values of the higher order nonlocal parameter μ_0 and LONGST. It can also be concluded that with an increase in the nonlocal parameter μ_1, the critical buckling load decreases. To have a better understanding of this issue, variations of the critical buckling temperature of the FG nanobeam are plotted in Fig. 4 with respect to increasing the strain length scale and different values of the nonlocal parameters μ_0 and μ_1. Observing the same figure, it can be noticed that the results are presented for different combinations of the values of the nonlocal parameters μ_0 and μ_1. It can be concluded that the highest value of the critical buckling force was obtained for the lowest values of the nonlocal parameter ($\mu_0 = 0$, $\mu_1 = 0$), with the increasing strain gradient length scale. It should also be noted that the results from Figs. 2–4 are obtained for the gradient index $p = 1$. For the same reason, Fig. 4 presents the variations in the critical buckling temperature ΔT_{cr} with respect to the new scale factor

$$l_\mu = \frac{l^2}{\mu}, \tag{7.1}$$

for different values of the nonlocal parameter where $\mu = \mu_0 = \mu_1$. It can be concluded that the critical buckling temperature is smaller than the result of the classical solution when the nonlocal parameter is smaller than the strain

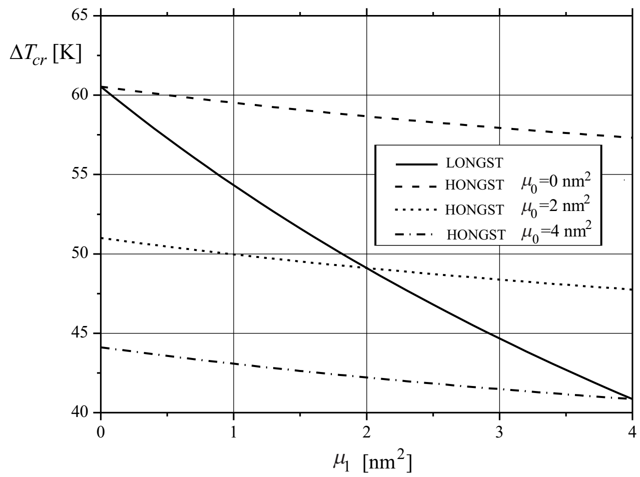

FIG. 3. Variation of the critical buckling temperature of the SS FG nanobeam with respect to the nonlocal parameter μ_1 for different values of nonlocal parameter μ_0 and LNR ($l^2 = 2$ nm^2, $p = 1$, $L/h = 50$).

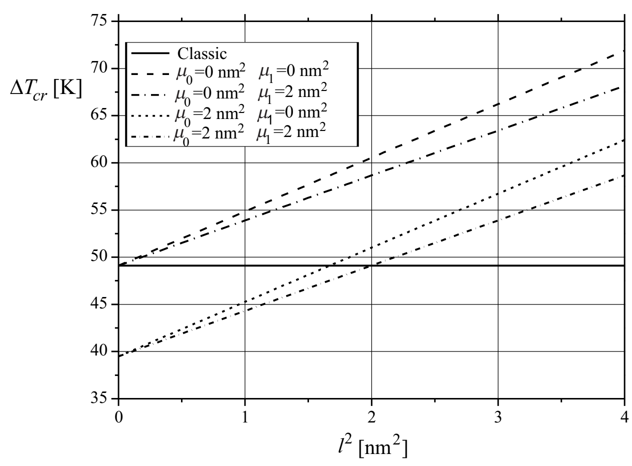

FIG. 4. Variation of the critical buckling temperature of the SS FG nanobeam with respect to the strain length scale l for different values of the nonlocal parameter μ_0, nonlocal parameter μ_1 and LNR ($p = 1$, $L/h = 50$).

length scale ($l_\mu < 1$); the critical buckling temperature is larger than the result of the classical solution when the nonlocal parameter is larger than the strain gradient length scale ($l_\mu > 1$). When the nonlocal parameter is equal to the strain gradient length scale $l_\mu = 1$, the critical buckling temperature is equal to that of the classical solution. Also, when $l_\mu = 0$, the results are equal to those from the nonlocal elasticity theory.

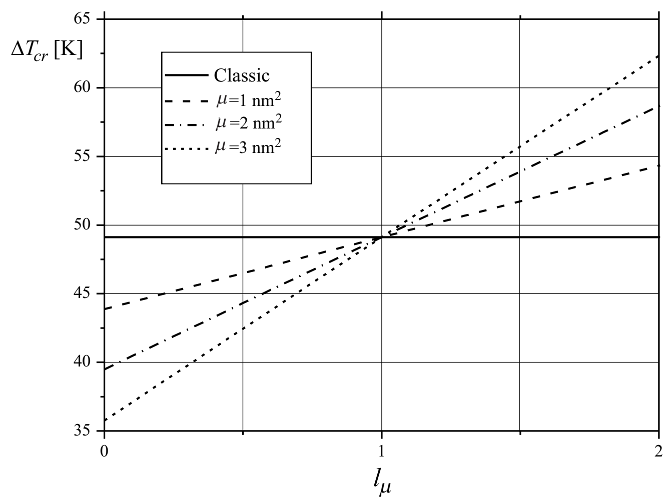

Fig. 5. Variation of the critical buckling temperature of the SS FG nanobeam with respect to ratio l_μ, strain length scale for different values of the nonlocal parameters and LNR ($p = 1$, $L/h = 50$, $\mu_0 = \mu_1 = \mu$).

Also, from Tables 3–6 it can be concluded that the critical buckling temperature decreases with an increase in the strain gradient length scale and the higher order nonlocal parameter μ_1. In the case where the strain gradient length scale is zero, the higher order parameter does not have an effect on the critical buckling temperature ΔT_{cr}.

As it can be seen, Tables 7–14 present the results of the nondimensional natural frequencies of the simply supported FG nanobeam, with Tables 7–10 presenting the results with the linear temperature rise (LTR), while Tables 11–14 contain the results with the nonlinear temperature rise (NLTR). As was the case with the critical buckling force, here presented is the dependence of the nondimensional frequency change on the nonlocal parameters μ_0 and μ_1 as well as the strain gradient length scale l. It should be noted that all results from the tables are the results of the first natural frequencies, with a different gradient index ($p = 0.1, 1$), and with different values of temperature ($\Delta T = 20, 60$ [K]). It can be concluded that an increase in the strain gradient length scale l leads to an increase in the nondimensional natural frequencies. Also, for small values

Table 3. Nonlocality parameters and strain gradient length scale effects on the critical buckling temperature ΔT_{cr} [K] of the SS FG nanobeam in the LTR case when $p = 0.1$ and $L/h = 50$.

μ_0 [nm^2]	μ_1 [nm^2]	l^2 [nm^2]			
		0	1	2	3
0	0	68.9262	76.4676	83.9636	91.4150
	1	68.9262	75.7920	82.6202	89.4112
	2	68.9262	75.2276	81.4971	87.7354
	3	68.9262	74.7488	80.5443	86.3130
1	0	62.0220	69.6055	77.1428	84.6348
	1	62.0220	68.9262	75.792	82.6202
	2	62.0220	68.3586	74.6628	80.9352
	3	62.0220	67.8772	73.7048	79.5051
2	0	56.2260	63.8453	71.4177	78.9441
	1	56.2260	63.1628	70.0607	76.9203
	2	56.2260	62.5925	68.9262	75.2276
	3	56.2206	62.1089	67.9637	73.7908
3	0	51.2912	58.9412	66.5437	74.0996
	1	51.2912	58.2559	65.1813	72.0679
	2	51.2912	57.6834	64.0423	70.3686
	3	51.2912	57.1978	63.0760	68.9262

Table 4. Nonlocality parameters and strain gradient length scale effects on the critical buckling temperature ΔT_{cr} [K] of the SS FG nanobeam in the LTR case when $p = 1$ and $L/h = 50$.

μ_0 [nm^2]	μ_1 [nm^2]	l^2 [nm^2]			
		0	1	2	3
0	0	49.1109	54.8391	60.5418	66.2192
	1	49.1109	54.3256	59.5191	64.6917
	2	49.1109	53.8966	58.6644	63.4145
	3	49.1109	53.5328	57.9394	62.3309
1	0	43.8746	49.6265	55.3524	61.0528
	1	43.8746	49.1109	54.3256	59.5191
	2	43.8746	48.6801	53.4674	58.2368
	3	43.8746	48.3148	52.7395	57.1488
2	0	39.4848	45.2567	51.0024	56.7221
	1	39.4848	44.7393	49.9720	55.1832
	2	39.4848	44.3070	49.1109	53.8966
	3	39.4848	43.9404	48.3804	52.8049
3	0	35.7514	41.5405	47.3031	53.0395
	1	35.7514	41.0216	46.2697	51.4961
	2	35.7514	40.5880	45.4060	50.2057
	3	35.7514	40.2204	44.6735	49.1109

Table 5. Nonlocality parameters and strain gradient length scale effects on the critical buckling temperature ΔT_{cr} [K] of the SS FG nanobeam in the NLTR case when $p = 0.1$ and $L/h = 50$.

μ_0 [nm^2]	μ_1 [nm^2]	l^2 [nm^2]			
		0	1	2	3
0	0	69.8527	77.5239	85.1543	92.7448
	1	69.8527	76.8365	83.7864	90.7032
	2	69.8527	76.2621	82.643	88.9959
	3	69.8527	75.775	81.673	87.547
1	0	62.8347	70.5435	78.211	85.8378
	1	62.8347	69.8527	76.8365	83.7864
	2	62.8347	69.2756	75.6875	82.071
	3	62.8347	68.7861	74.7128	80.6152
2	0	56.9468	64.6876	72.3865	80.0441
	1	56.9468	63.994	71.0064	77.9845
	2	56.9468	63.4145	69.8527	76.2621
	3	56.9468	62.923	68.8741	74.8004
3	0	51.9362	59.7046	67.4304	75.1145
	1	51.9362	59.0085	66.0455	73.0478
	2	51.9362	58.4269	64.8878	71.3195
	3	51.9362	57.9337	63.9058	69.8527

Table 6. Nonlocality parameters and strain gradient length scale effects on the critical buckling temperature ΔT_{cr} [K] of the SS FG nanobeam in the NLTR case when $p = 1$ and $L/h = 50$.

μ_0 [nm^2]	μ_1 [nm^2]	l^2 [nm^2]			
		0	1	2	3
0	0	51.4254	57.4813	63.5221	69.5481
	1	51.4254	56.9379	62.4380	67.9256
	2	51.4254	56.4800	61.5321	66.5698
	3	51.4254	56.0992	60.7640	65.4198
1	0	45.9001	51.9700	58.0246	64.0641
	1	45.9001	51.4254	56.9379	62.4380
	2	45.9001	50.9704	56.0301	61.0791
	3	45.9001	50.5847	55.2602	59.9266
2	0	41.2757	47.3575	53.4237	59.4747
	1	41.2757	46.8118	52.3350	57.8455
	2	41.2757	46.3560	51.4254	56.4840
	3	41.2757	45.9695	50.6540	55.3293
3	0	37.3485	43.4404	49.5167	55.5774
	1	37.3485	42.8938	48.4261	53.9456
	2	37.3485	42.4373	47.5150	52.5820
	3	37.3485	42.0501	46.7424	51.4254

Table 7. Nonlocality parameters and strain gradient length scale effects on the first nondimensional frequency ω_1 in the LTR case when $p = 0.1$, $L/h = 20$, $\Delta T = 20$ [K]. (Data [8]).

μ_0 [nm^2]	μ_1 [nm^2]	l^2 [nm^2]			
		0	1	2	3
0	0	8.4722 (8.4634)	8.9058	9.3193	9.7152
	1	8.4722	8.8678	9.2464	9.6101
	2	8.4722	8.8358	9.1851	9.5215
	3	8.4722	8.8087	9.1327	9.4457
1	0	8.0573 (8.0488)	8.5121	8.9438	9.3556
	1	8.0573	8.4722	8.8678	9.2464
	2	8.0573	8.4388	8.8038	9.1542
	3	8.0573	8.4103	8.7492	9.0754
2	0	7.6936 (7.6854)	8.1687	8.6176	9.0443
	1	7.6936	8.1272	8.5387	8.9313
	2	7.6936	8.0923	8.4722	8.8358
	3	7.6936	8.0626	8.4155	8.7541
3	0	7.4699 (7.3633)	7.8659	8.3311	8.7717
	1	7.4699	7.8227	8.2495	8.6552
	2	7.4699	7.7865	8.1806	8.5567
	3	7.4699	7.7557	8.1219	8.4722

Table 8. Nonlocality parameters and strain gradient length scale effects on the first nondimensional frequency ω_1 in the LTR case when $p = 1$, $L/h = 20$, $\Delta T = 20$ [K]. (Data [8]).

μ_0 [nm^2]	μ_1 [nm^2]	l^2 [nm^2]			
		0	1	2	3
0	0	5.7117 (5.7114)	6.0106	6.2953	6.5676
	1	5.7117	5.9844	6.2451	6.4953
	2	5.7117	5.9624	6.2028	6.4344
	3	5.7117	5.9436	6.1668	6.3822
1	0	5.4254 (5.4251)	5.7392	6.0367	6.3202
	1	5.4254	5.7117	5.9844	6.2451
	2	5.4254	5.6887	5.9403	6.1816
	3	5.4254	5.6691	5.9026	6.1273
2	0	5.1742 (5.1737)	5.5024	5.8120	6.1059
	1	5.1742	5.4737	5.7576	6.0281
	2	5.1742	5.4496	5.7117	5.9624
	3	5.1742	5.4291	5.6726	5.9060
3	0	4.9514 (4.9508)	5.2933	5.6144	5.9182
	1	4.9514	5.2635	5.5581	5.8379
	2	4.9514	5.2384	5.5106	5.7699
	3	4.9514	5.2171	5.4701	5.7117

Table 9. Nonlocality parameters and strain gradient length scale effects on the first nondimensional frequency ω_1 in the LTR case when $p = 0.1$, $L/h = 20$, $\Delta T = 60$ [K].

μ_0 [nm^2]	μ_1 [nm^2]	l^2 [nm^2]			
		0	1	2	3
0	0	8.1133	8.5669	8.9978	9.4089
	1	8.1133	8.5272	8.9219	9.2999
	2	8.1133	8.4938	8.8580	9.2078
	3	8.1133	8.4655	8.8035	9.1291
1	0	7.6771	8.1551	8.6065	9.0354
	1	7.6771	8.1133	8.5272	8.9219
	2	7.6771	8.0782	8.4604	8.8259
	3	7.6771	8.0484	8.4033	8.7438
2	0	7.2929	7.7945	8.2656	8.7113
	1	7.2929	7.7507	8.1830	8.5935
	2	7.2929	7.7140	8.1133	8.4938
	3	7.2929	7.6828	8.0538	8.4084
3	0	6.9506	7.4751	7.9652	8.4268
	1	6.9506	7.4295	7.8794	8.3050
	2	6.9506	7.3912	7.8070	8.2018
	3	6.9506	7.3586	7.7452	8.1133

Table 10. Nonlocality parameters and strain gradient length scale effects on the first nondimensional frequency ω_1 in the LTR case when $p = 1$, $L/h = 20$, $\Delta T = 60$ [K].

μ_0 [nm^2]	μ_1 [nm^2]	l^2 [nm^2]			
		0	1	2	3
0	0	5.3737	5.6917	5.9929	6.2796
	1	5.3737	5.6639	5.9399	6.2037
	2	5.3737	5.6405	5.8953	6.1395
	3	5.3737	5.6207	5.8572	6.0846
1	0	5.0670	5.4031	5.7194	6.0192
	1	5.0670	5.3737	5.6639	5.9399
	2	5.0670	5.3491	5.6171	5.8729
	3	5.0670	5.3281	5.5771	5.8154
2	0	4.7958	5.1496	5.4806	5.7928
	1	4.7958	5.1188	5.4226	5.7103
	2	4.7958	5.0929	5.3737	5.6405
	3	4.7958	5.0709	5.3319	5.5807
3	0	4.5532	4.9245	5.2697	5.5936
	1	4.5532	4.8923	5.2094	5.5082
	2	4.5532	4.8652	5.1584	5.4358
	3	4.5532	4.8422	5.1149	5.3737

Table 11. Nonlocality parameters and strain gradient length scale effects on the first nondimensional frequency ω_1 in the NLTR case when $p = 0.1$, $L/h = 20$, $\Delta T = 20$ [K]. (Data [8]).

μ_0 [nm^2]	μ_1 [nm^2]	l^2 [nm^2]			
		0	1	2	3
0	0	8.4743 (8.4674)	8.9078	9.3212	9.7170
	1	8.4743	8.8697	9.2483	9.4119
	2	8.4743	8.8378	9.1869	9.5233
	3	8.4743	8.8106	9.1346	9.4475
1	0	8.0594 (8.0532)	8.5141	8.9457	9.3574
	1	8.0594	8.4743	8.8697	9.2483
	2	8.0594	8.4408	8.8057	9.1561
	3	8.0594	8.4124	8.7512	9.0773
2	0	7.6959 (7.6902)	8.1708	8.6196	9.0462
	1	7.6959	8.1293	8.5407	8.9332
	2	7.6959	8.0944	8.4743	8.8378
	3	7.6959	8.0648	8.4175	8.7561
3	0	7.3736 (7.3685)	7.8681	8.3332	8.7737
	1	7.3736	7.8250	8.2516	8.6572
	2	7.3736	7.7887	8.1828	8.5587
	3	7.3736	7.7579	8.1240	8.4743

Table 12. Nonlocality parameters and strain gradient length scale effects on the first nondimensional frequency ω_1 in the NLTR case when $p = 1$, $L/h = 20$, $\Delta T = 20$ [K]. (Data [8]).

μ_0 [nm^2]	μ_1 [nm^2]	l^2 [nm^2]			
		0	1	2	3
0	0	5.7185 (5.7124)	6.0170	6.3014	6.5735
	1	5.7185	5.9908	6.2513	6.5013
	2	5.7185	5.9689	6.2091	6.4404
	3	5.7185	5.9502	6.1731	6.3883
1	0	5.4326 (5.4269)	5.7460	6.0431	6.3263
	1	5.4326	5.7185	5.9908	6.2513
	2	5.4326	5.6955	5.9468	6.1879
	3	5.4326	5.6795	5.9092	6.1337
2	0	5.1817 (5.1764)	5.5094	5.8186	6.1123
	1	5.1817	5.4807	5.7643	6.0345
	2	5.1817	5.4567	5.7185	5.9689
	3	5.1817	5.4363	5.6794	5.9126
3	0	4.9591 (4.9541)	5.3006	5.6213	5.9247
	1	4.9591	5.2708	5.5650	5.8445
	2	4.9591	5.2458	5.5176	5.7767
	3	4.9591	5.2245	5.4771	5.7185

Table 13. Nonlocality parameters and strain gradient length scale effects on the first nondimensional frequency ω_1 in the NLTR case when $p = 0.1$, $L/h = 20$, $\Delta T = 60$ [K].

μ_0 [nm^2]	μ_1 [nm^2]	l^2 [nm^2]			
		0	1	2	3
0	0	8.1210	8.5743	9.0048	9.4156
	1	8.1210	8.5346	8.9289	9.3066
	2	8.1210	8.5012	8.8651	9.2147
	3	8.1210	8.4729	8.8107	9.1360
1	0	7.6853	8.1628	8.6138	9.0424
	1	7.6853	8.1210	8.5345	8.9289
	2	7.6853	8.0860	8.4678	8.8331
	3	7.6853	8.0562	8.4107	8.7510
2	0	7.3015	7.8025	8.2732	8.7186
	1	7.3015	7.7588	8.1906	8.6008
	2	7.3015	7.7221	8.1210	8.5012
	3	7.3015	7.6909	8.0615	8.4159
3	0	6.9595	7.4835	7.9731	8.4343
	1	6.9595	7.4379	7.8873	8.3125
	2	6.9595	7.3997	7.8151	8.2094
	3	6.9595	7.3671	7.7532	8.1210

Table 14. Nonlocality parameters and strain gradient length scale effects on the first nondimensional frequency ω_1 in the NLTR case when $p = 1$, $L/h = 20$, $\Delta T = 60$ [K].

μ_0 [nm^2]	μ_1 [nm^2]	l^2 [nm^2]			
		0	1	2	3
0	0	5.3996	5.7162	6.0162	6.3019
	1	5.3996	5.6885	5.9634	6.2262
	2	5.3996	5.6652	5.9190	6.1622
	3	5.3996	5.6454	5.8810	6.1075
1	0	5.0943	5.4288	5.7438	6.0424
	1	5.0943	5.3996	5.6885	5.9634
	2	5.0943	5.3751	5.6419	5.8966
	3	5.0943	5.3542	5.6021	5.8394
2	0	4.8246	5.1765	5.5060	5.8168
	1	4.8246	5.1459	5.4483	5.7347
	2	4.8246	5.1202	5.3996	5.6652
	3	4.8246	5.0983	5.3580	5.6057
3	0	4.5836	4.9527	5.3961	5.6185
	1	4.5836	4.9206	5.2360	5.5335
	2	4.5836	4.8937	5.1854	5.4614
	3	4.5836	4.8708	5.1420	5.3996

of the strain gradient scale, increasing the higher order nonlocal parameter μ_1 has no effect on the change in the natural frequencies, contrary to the nonlocal parameter μ_0, as can be seen for the values of the strain gradient length scale $l = 0$. Increasing the nonlocal parameter μ_0 with the strain gradient length scale l, leads to a decrease in the nondimensional natural frequencies. Also, an increase in the strain gradient scale l and a higher order nonlocal parameter μ_1, have more influence on a decrease in the nondimensional natural frequencies. To have a better understanding of this issue, variations of the frequency ratio

$$k_{\omega n} = \frac{\omega_n}{\omega_{nc}}, \tag{7.2}$$

are plotted in Figs. 6–8 with respect to the nonlocal scale parameters μ_0 for different values of the strain length scale l and the nonlocal parameter μ_1, where ω_n is the nondimensional frequency calculated using the nonlocal theory (for the parameters of system (6.9)) and ω_{nc} is the nondimensional frequency calculated using the classical local theory (for the parameters of system (6.16)). This frequency ratio can be used as an indicator that serves to quantitatively estimate the effects of the nonlocal parameters μ_0 and μ_1 as well as the strain length scale l on the vibration solution. From Figs. 6–8 it can be seen that the frequency ratio is lower when $l = 0$ regardless of the values of the nonlocal parameters μ_0 and μ_1. Furthermore, the frequency ratio has higher values for higher frequencies. It is also obvious that the frequency ratio decreases with an increase in the

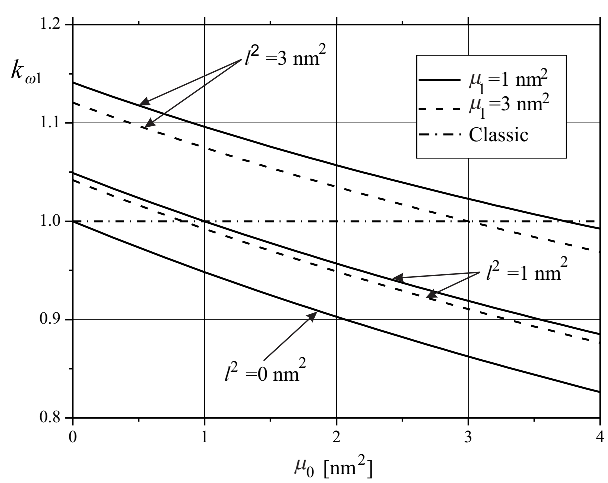

Fig. 6. Variation of the frequency ratio for the first nondimensional frequency of the SS FG nanobeam with respect to the nonlocal parameter μ_0 for different values of μ_1 and l^2 and LNR ($p = 1$, $L/h = 20$).

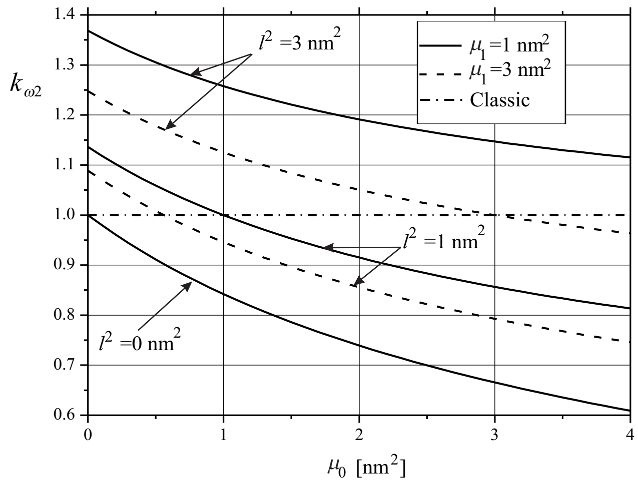

Fig. 7. Variation of the frequency ratio for the second nondimensional frequency of the SS FG nanobeam with respect to the nonlocal parameter μ_0 for different values of μ_1 and l^2 and LNR ($p = 1$, $L/h = 20$).

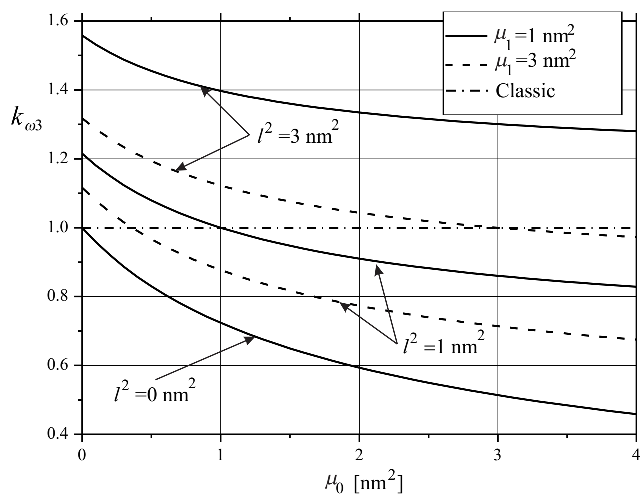

Fig. 8. Variation of the frequency ratio for the third nondimensional frequency of the SS FG nanobeam with respect to the nonlocal parameter μ_0 for different values of μ_1 and l^2 and LNR ($p = 1$, $L/h = 20$).

nonlocal parameters and a decrease in the strain gradient length scale. In the cases when the strain gradient length scale is zero, changing the values of the higher order μ_1 has no effect on the change in the frequency, as shown in the tables.

8. Conclusions

This paper investigates the thermal buckling and vibration of the FG nanobeam subjected to different temperature distributions in the through-thickness direction (LNR and NLTR). By using the variational approach, the equations of motion are obtained based on the Euler–Bernoulli beam theory within the framework of the higher-order nonlocal strain gradient theory. The effect of the nonlocal parameters and strain gradient length scale on the critical buckling temperature and nondimensional frequency is observed. Numerical results are presented for certain characteristics of the rectangular cross-section of the beam. It is concluded that an increase in the nonlocal parameters will decrease the critical buckling temperature and nondimensional frequency, while a decrease in the strain gradient length scale will lead to a decrease in the critical buckling temperature and nondimensional natural frequency. For small values of the strain gradient scale, the dominant influence is exerted by the nonlocal parameter, while for higher values, the dominant influence is shown by the higher-order nonlocal parameter. If the nonlocal parameters are equal, then for the values of the strain gradient scale that are smaller than the nonlocal parameter, the critical buckling temperature and nondimensional frequency are lower than in the classical solution, and for the values of the strain gradient scale that are higher than the nonlocal parameter, the critical buckling temperature and nondimensional frequency are higher than in the classical solution. In the case when the strain gradient length scale is zero, the higher-order nonlocal parameters practically have no effect on the critical buckling temperature and nondimensional frequency.

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