A computational scheme for decentralized time-optimal resource allocation in a sequence of projects of activities under constrained resource

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Abstract. We consider a sequence of projects of independent activities; each project composed of activities available for realization at the same time. It is assumed that the activities are continuous dynamical systems whose dynamics depend continuously on the allotted amounts of the resource, and the initial and terminal states are fixed. The problem is to allocate a renewable, continuously divisible resource (e.g., power, fuel flow, money per time unit, approximate manpower) to the activities in order to minimize the performance time of the sequence of projects under the assumption that the allowable level of the total usage of the resource is constant. Although the solution to this problem is known in the literature, nevertheless there is a lack of effective computational algorithms for the time-optimal resource allocation, especially in the case of really large projects. In this paper a decentralized two-level control scheme using the time-decomposition is proposed to find the time-optimal resource allocation in a sequence of projects. The price mechanism is applied to coordinate the lower level tasks of the optimal resource allocation in the successive time intervals determined by the moments at which the successive projects are available for realization. Necessary and sufficient conditions to ensure the determination of the optimal resource allocation according to the method proposed are stated. The problems connected with the numerical realization of the scheme are discussed and the resulting computer algorithm is outlined.
Key words: project of activities, resource allocation, time-optimal control, dynamical system, decentralized control

INTRODUCTION

The concept of the control of a project of activities was introduced by Burkov [4] and Lerner and Tejman [20] as a special approach to resource allocation in PERT networks. Its distinctive feature is that the activities are continuous dynamical systems relating at any time the performance speeds of the activities to the allotted amounts of a resource, and that the initial and terminal states are fixed. By optimal control of a project of activities we shall mean an assignment of constrained resources to activities such that to minimize a performance index of the whole project, e.g. time or cost. The approach naturally employs the fact that every activity is characterized by an amount of work to be performed with controllable rate which depends on the amount of resource. Two resource categories can be distinguished from the viewpoint of resource divisibility: discrete (i.e., discretely divisible) and continuous (i.e., continuously divisible) ones [2,9,14]. Examples of discrete resources include: machines, tools, workers. The most typical continuous resources are: power, energy, liquids and money.

There exist an intensive and ever-increasing literature dealing with various aspects of resource allocation control and scheduling of multi-project of activities [1,2,9,13-15,25-28] and its applications in different areas (see, for example, [2,16,21,24]). During a last four decades a variety of different algorithms for resource allocation and project scheduling have been proposed. The results were obtained for general cases, where all the models relating performance speeds of activities to resource amounts were arbitrary continuous increasing functions [4,9,11,12,14,26], as well as for multi-projects of activities described by specific concave and convex functions [9,11,12,26,27], and even for a special case of simple linear with respect to the resource amount models [10,15,18,19]. The project duration optimization [9,11,12,19,26], minimizing inventory, backlog and production related costs over a production horizon [17], minimizing maximum lateness or just-in-time [13] criteria were used as the objective functions. Optimal resource allocation policies were proposed both for singly [9,11,12] and for doubly constrained resources [9,18]. The most typical renewable continuous resource, which is power, is also, most often, doubly constrained, since its consumption, i.e. energy, is also limited. For the survey of the problems of resource allocation among activieties which can be processed using resources of various categories and types see [1,2,28]. Further explanations, theory and examples on control of project of activities can be fund in [21,25], see also [9,11,12].

The paper deals with a class of time-optimal resource allocation problems in which the total usage of renewable and continuously divisible resource is constrained at every time of the project performance.

A sequence of projects is considered, each project composed of activities which are available for realization at the same time. This assumption, essential for the idea of the time-decomposition applied, seems to be a realistic assumption in most real processes. No specific assumptions concerning the models of activities, like linearity, convexity or concavity are made here. The problem is to find an admissible control minimizing the performance time of the sequence of projects under the assumption that the allowable level of the total usage of the resource is constant during the projects duration. This problem has been solved in [12], where necessary and sufficient optimality conditions are stated in terms of the performance time and the existence of the optimal control is proved. However, the solution concept proposed in [12] leads to the reduction of the primary dynamic optimization problem to some static programming task, the computational complexity of this task grows, in general, faster than the dimension of the problem, i.e., the number of activities and the number of the moments at which the successive projects are available for realization. The paper discusses hierarchical decentralized method for solving this class of optimal resource allocation problems.

TIME-OPTIMAL CONTROL OF A SEQUENCE OF PROJECTS

We consider a sequence of k projects of independent activities (this means that no precedence relations exists among them), each project composed of activities available for realization at the same time, described by the equations:

$$\left.\begin{array}{ll}\dot{x}_{ri}(t) = f_{ri}\big(u_{ri}(t)\big), & t \ge 0 \\ x_{ri}(t) = 0, & t \le t_r\end{array}\right\} i = 1, \dots, n_r, r = 1, \dots, k, \tag{1}$$

where: $x_{ri}(t)$ and $u_{ri}(t)$ are, respectively, the state of the i-th activity in the r-th project and the amount of resource allotted to this activity at the time t. Here t_r is the specified, fixed time at which the r-th project is available for realization, $t_{r+1} \ge t_r$, where $t_1 = 0$, and n_r is the number of activities in the r-th project, $r = 1, \dots, k$. We also assume that for any $i = 1, \dots, n_r$ and $r = 1, \dots, k$ the functions $f_{ri}: \mathcal{R}_+ \to \mathcal{R}_+$ describing the speed (i.e., the rate of activity performance):

$$v_{ri}(t) = \dot{x}_{ri}(t) = f_{ri}\big(u_{ri}(t)\big)$$

are continuous, increasing and such that $f_{ri}(0) = 0$, $\mathcal{R}_+$ is the set of nonnegative real numbers. The assumption means that the greater the assigned resource $u_{ri}(t)$ is, the higher is the speed $v_{ri}(t)$ of execution of the activity.

We say that the activity (r, i) is completed if its state has attained a given terminal state $w_{ri} > 0$. We also assume that the amount of a renewable and continously divisible resource, e.g. power or budget available per time unit for management, is constant and equal to N for any time $t \ge 0$. By the performance time T of a sequence of projects we mean the time at which all the activities are completed, i.e.:

$$T = min\{\tau > t_k : x_{ri}(\tau) = w_{ri}, i = 1, \dots, n_r, r = 1, \dots, k\}.$$

We introduce, for convenience, the following vector notation:

$$\boldsymbol{x}_r = [x_{r1} \;\; \cdots \;\; x_{rn_r}]^T, \quad \boldsymbol{x} = [\boldsymbol{x}_1^T \;\; \dots \;\; \boldsymbol{x}_k^T]^T,$$
$$\boldsymbol{u}_r = [u_{r1} \;\; \cdots \;\; u_{rn_r}]^T, \quad \boldsymbol{u} = [\boldsymbol{u}_1^T \;\; \dots \;\; \boldsymbol{u}_k^T]^T.$$

The vectors $\boldsymbol{v}_r$ and $\boldsymbol{v}$ of speeds of the r-th project activities and all activities, respectively, as well as the vectors of final states $\boldsymbol{w}_r$ and $\boldsymbol{w}$ are defined by analogy. The vector-functions $\boldsymbol{f}_r(\boldsymbol{u}_r)$ and $\boldsymbol{f}(\boldsymbol{u})$ are defined as follows:

$$\boldsymbol{f}_r(\boldsymbol{u}_r) = \big[f_{r1}(u_{r1}) \;\; \dots \;\; f_{rn_r}\big(u_{rn_r}\big)\big]^T,$$
$$\boldsymbol{f}(\boldsymbol{u}) = [\boldsymbol{f}_1(\boldsymbol{u}_1)^T \;\; \dots \;\; \boldsymbol{f}_k(\boldsymbol{u}_k)^T]^T.$$

Define for the successive time intervals determined by the times t_r, $r = 1, \dots, k$, the sets od admissible values of resource allocation:

$$\boldsymbol{U}^r = \{\boldsymbol{u} \in \mathcal{R}_+^n : \boldsymbol{u}_s = 0,\; s \ge r+1, \textstyle\sum_{s=1}^{r} \sum_{i=1}^{n_s} u_{si} \le N \},$$

where:

$$n = \textstyle\sum_{r=1}^{k} n_k$$

is the number of all activities. The sets of feasible performing speeds of operations within the r-th time interval will be denoted by:

$$\boldsymbol{V}^r = \{\boldsymbol{v} \in \mathcal{R}_+^n : \boldsymbol{v} = \boldsymbol{f}(\boldsymbol{u}),\; \boldsymbol{u} \in \boldsymbol{U}^r \}.$$

Note, that $\boldsymbol{U}^r$ and $\boldsymbol{V}^r$ are compact subsets of $\mathcal{R}^n$. The following condition is satisfied:

$$\boldsymbol{v}^{r,1} \in \boldsymbol{V}^r,\;\; \mathbb{0}_n \le \boldsymbol{v}^{r,2} \le \boldsymbol{v}^{r,1} \;\Rightarrow\; \boldsymbol{v}^{r,2} \in \boldsymbol{V}^r, \tag{2}$$

here $\mathbb{0}_n$ is zero vector of dimension n. The resource allocation (control):

$$\boldsymbol{u}(t) = [\boldsymbol{u}_1^T(t) \;\; \dots \;\; \boldsymbol{u}_k^T(t)]^T,$$

where:

$$\boldsymbol{u}_r(t) = [u_{r1}(t) \;\; \dots \;\; u_{rn_r}(t)]^T$$

and the functions:

$$u_{r1}: [0, T] \to \mathcal{R}_+ \text{ for } i = 1, \dots, n_r, r = 1, \dots, k,$$

is said to be admissible for the sequence of projects, if the following conditions are satisfied:

(i) $\boldsymbol{u}(t) \in \boldsymbol{U}^r$ for $t_r \le t < t_{r+1}$, $r = 1, \dots, k$, where $t_{k+1} = \infty$,

(ii) $\boldsymbol{u}(t)$ is piecewise continuous function,

(iii) $\int_0^T \boldsymbol{f}\big(\boldsymbol{u}(t)\big) = \boldsymbol{w}$.

The problem is to find an admissible control $\boldsymbol{u}(t)$ for which all the projects are completed as soon as possible, i.e. a control minimizing the performance time T of the sequence of projects.

OPTIMALITY CONDITIONS

In the paper [12] a solution to the above problem is presented, based on the notion of the set of reachable states. This approach leads to the reduction of the primary dynamic optimization problem to the static programming task. The following result follows immediately from [12; Theorem 2].

Theorem 1. T^* is the minimum performance time if and only if T^* and $\boldsymbol{v}^{1*}, \dots, \boldsymbol{v}^{k*}$ are the solutions of the following optimization task:

$$T \to min \tag{3}$$

subject to:

$$\boldsymbol{w} = \tau_1 \boldsymbol{v}^1 + \dots + \tau_{k-1}\boldsymbol{v}^{k-1} + (T - t_k)\boldsymbol{v}^k,$$
$$\boldsymbol{v}^r \in conv(\boldsymbol{V}^r), r = 1, \dots, k, \tag{4}$$

where: $conv(\boldsymbol{V}^r)$ is the convex hull of $\boldsymbol{V}^r$ (the smallest convex set containing $\boldsymbol{V}^r$, i.e., the set of all convex combinations of points in $\boldsymbol{V}^r$) and $\tau_r = t_{r+1} - t_r$ for $r = 1, \dots, k-1$.

The problem (3), (4) is static convex programming task. However, the above task can be solved by using standard convex numerical optimization techniques [3], the computational complexity of this problem grows, in general, faster than the dimension of the problem, i.e., the number of activities and the number of the time moments t_r at which the successive projects are available for realization. Thus, in order to reduce the computational and storage requirements the following two-level scheme is proposed. Taking into account the successive time-intervals structure of both the original dynamic optimal resource allocation problem [12] as well as the static optimization task (3), (4), the time-decomposition approach is applied.

DECENTRALIZED TWO-LEVEL SCHEME FOR RESOURCE ALLOCATION

Note first, that for any positive constant ρ the optimization problem (3), (4) is equivalent to the minimization of the modified strictly convex index:

$$T + \rho T^2 \to min \tag{5}$$

subject to the constraints (4). By introducing a vector of prices $\boldsymbol{\lambda} \in \mathcal{R}^n$, we can define the Lagrangian for the optimization task (5), (4):

$$L(T, \boldsymbol{v}^1, \dots, \boldsymbol{v}^k, \boldsymbol{\lambda}) = T + \rho T^2 + \langle \boldsymbol{\lambda}, \boldsymbol{w} - \tau_1 \boldsymbol{v}^1 + \cdots$$
$$-\tau_{k-1}\boldsymbol{v}^{k-1} - (T - t_k)\boldsymbol{v}^k \rangle, \tag{6}$$

where: ρ is some positive constant and $\langle \cdot,\cdot \rangle$ denotes the inner product. Notice that $L(T, \boldsymbol{v}^1, \dots, \boldsymbol{v}^k, \boldsymbol{\lambda})$ is continous with respect to all arguments and can be expressed as:

$$L(T, \boldsymbol{v}^1, \dots, \boldsymbol{v}^k, \boldsymbol{\lambda}) = \textstyle\sum_{r=1}^{k-1} \tau_r L_r(\boldsymbol{v}^r, \boldsymbol{\lambda}) +$$
$$L_k(T, \boldsymbol{v}^k, \boldsymbol{\lambda}) + \langle \boldsymbol{\lambda}, \boldsymbol{w} \rangle, \tag{7}$$

where for $r = 1, \dots, k-1$ the local Lagrangians are equal:

$$L_r(\boldsymbol{v}^r, \boldsymbol{\lambda}) = -\langle \boldsymbol{\lambda}, \boldsymbol{v}^r \rangle, \tag{8}$$

while for the last time interval we have:

$$L_k(T, \boldsymbol{v}^k, \boldsymbol{\lambda}) = T + \rho T^2 - (T - t_k)\langle \boldsymbol{\lambda}, \boldsymbol{v}^k \rangle. \tag{9}$$

Consider the following two-level scheme.

Infimal Problem (IP). Given $\boldsymbol{\lambda} \in \mathcal{R}^n$, for $r = 1, \dots, k-1$ find the performance speed vectors $\hat{\boldsymbol{v}}^r(\boldsymbol{\lambda})$ such that:

$$L_r(\hat{\boldsymbol{v}}^r(\boldsymbol{\lambda}), \boldsymbol{\lambda}) = min_{\boldsymbol{v}^r \in conv(\boldsymbol{V}^r)}\ L_r(\boldsymbol{v}^r, \boldsymbol{\lambda}), \tag{10}$$

and the pair $\left(\hat{\boldsymbol{v}}^k(\boldsymbol{\lambda}), \hat{T}(\boldsymbol{\lambda})\right)$ such that:

$$L_k\left(\hat{T}(\boldsymbol{\lambda}), \hat{\boldsymbol{v}}^k(\boldsymbol{\lambda}), \boldsymbol{\lambda}\right) = min_{T, \boldsymbol{v}^k \in conv(\boldsymbol{V}^k)}\, L_k(T, \boldsymbol{v}^k, \boldsymbol{\lambda}). \tag{11}$$

By virtue of the compactness of the sets $conv(\boldsymbol{V}^r)$ and the continouity of the Lagrangians $L_r(\boldsymbol{v}^r, \boldsymbol{\lambda})$ with respect to $\boldsymbol{v}^r$, the solutions of (10) exist for any $\boldsymbol{\lambda} \in \mathcal{R}^n$, $r = 1, \dots, k-1$, on the basis of the well-known Weierstrass's theorem which asserts the existence of continuous function extrema on compact sets [29; Theorem 7.1]. Compactness of $conv(\boldsymbol{V}^k)$ together with the strict conxexity of the continous local Lagrangian $L_k(T, \boldsymbol{v}^k, \boldsymbol{\lambda})$ with respect to T implies the existence of (11) solution. If the solution of (10) with respect to $\boldsymbol{v}^r$ is not unique, take any:

$$\hat{\boldsymbol{v}}^r(\boldsymbol{\lambda}) \in \hat{\boldsymbol{V}}^r(\boldsymbol{\lambda}),$$

$\hat{\boldsymbol{V}}^r(\boldsymbol{\lambda})$ being the set of speed vectors minimizing $L_r(\boldsymbol{v}^r, \boldsymbol{\lambda})$ on $conv(\boldsymbol{V}^r)$, $r = 1, \dots, k-1$. Similarly, if the solution to (11) is not unique take any $\hat{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \in \hat{\boldsymbol{V}}^k(\boldsymbol{\lambda})$ minimizing $L_k(T, \boldsymbol{v}^k, \boldsymbol{\lambda})$. For given $\hat{\boldsymbol{v}}^k(\boldsymbol{\lambda})$ locally optimal $\hat{T}(\boldsymbol{\lambda})$ is unique due to $L_k(T, \boldsymbol{v}^k, \boldsymbol{\lambda})$ strict convexity with respect to T – for details see the next section.

Coordination Problem (CP). Find $\hat{\boldsymbol{\lambda}} \in \mathcal{R}^n$ such that:

$$L_D(\hat{\boldsymbol{\lambda}}) = max_{\boldsymbol{\lambda} \in \mathcal{R}^n}\, L_D(\boldsymbol{\lambda}), \tag{12}$$

where the dual function $L_D(\boldsymbol{\lambda})$ is defined as follows:

$$L_D(\boldsymbol{\lambda}) = L\left(\hat{T}(\boldsymbol{\lambda}), \hat{\boldsymbol{v}}^1(\boldsymbol{\lambda}), \dots, \hat{\boldsymbol{v}}^k(\boldsymbol{\lambda}), \boldsymbol{\lambda}\right), \tag{13}$$

and take $\hat{\boldsymbol{v}}^r = \hat{\boldsymbol{v}}^r(\hat{\boldsymbol{\lambda}})$, $r = 1, \dots, k$ as the vectors of the optimal performance speeds of the activities in the successive time intervals.

PROPERTIES OF IP AND CP

Let us consider first the IP tasks (10). Since $\mathbb{O}_n \in conv(\boldsymbol{V}^r)$, taking into account (8) we see that for any $r = 1, \dots, k-1$ the solutions of (10) are such that:

$$L_r(\widehat{\boldsymbol{v}}^r(\boldsymbol{\lambda}), \boldsymbol{\lambda}) = -\max_{v^r \in conv(\boldsymbol{V}^r)} \langle \boldsymbol{\lambda}, \boldsymbol{v}^r \rangle \leq 0. \quad (14)$$

Consider an arbitrary $\boldsymbol{\lambda} \in \mathcal{R}^n$. It is obvious that for any positive constant α we have:

$$L_r(\widehat{\boldsymbol{v}}^r(\alpha\boldsymbol{\lambda}), \alpha\boldsymbol{\lambda}) = -\alpha \max_{v^r \in conv(\boldsymbol{V}^r)} \langle \boldsymbol{\lambda}, \boldsymbol{v}^r \rangle = \alpha L_r(\widehat{\boldsymbol{v}}^r(\boldsymbol{\lambda}), \boldsymbol{\lambda}).$$

Since the compact set $conv(\boldsymbol{V}^r) \cap \mathcal{R}^{p_r}$ is an absorbing set for the subspace $\mathcal{R}_+^{p_r}$, taking into account property (2) it is easily seen that the solutions of (10) are such that:

$$\widehat{\boldsymbol{v}}^r(\boldsymbol{\lambda}) \in bd[conv(\boldsymbol{V}^r)],$$

where $bd[conv(\boldsymbol{V}^r)]$ is boundary of the set $conv(\boldsymbol{V}^r)$. Here:

$$p_r = \sum_{s=1}^{r} n_s$$

is the number of operations accessible for realization in the r-th time interval, $p_k = n$.

Let us consider now the last IP task (11). According to the parametric approach of successive optimization [7] applied to (11) the following equivalence holds:

$$L_k(\widehat{T}(\boldsymbol{\lambda}), \widehat{\boldsymbol{v}}^k(\boldsymbol{\lambda}), \boldsymbol{\lambda}) = \min_{v^k \in conv(V^k)} [\min_T L_k(T, \boldsymbol{v}^k, \boldsymbol{\lambda})]. \quad (15)$$

The unique solution of the internal optimization task of the right-hand side of (15) is given by:

$$T = \frac{1}{2\rho}[\langle \boldsymbol{\lambda}, \boldsymbol{v}^k \rangle - 1].$$

Whence, the external task of the right-hand side of (15) takes the form:

$$\min_{v^k \in conv(V^k)} \bar{L}_k(\boldsymbol{v}^k, \boldsymbol{\lambda}), \quad (16)$$

with the function $\bar{L}_k(\boldsymbol{v}^k, \boldsymbol{\lambda})$ defined by:

$$\bar{L}_k(\boldsymbol{v}^k, \boldsymbol{\lambda}) = -\frac{1}{4\rho}[\langle \boldsymbol{\lambda}, \boldsymbol{v}^k \rangle]^2 + \left(\frac{1}{2\rho} + t_k\right)\langle \boldsymbol{\lambda}, \boldsymbol{v}^k \rangle - \frac{1}{4\rho}. \quad (17)$$

Let $\boldsymbol{\lambda} \neq \mathbb{O}_n$ and $\bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \in conv(\boldsymbol{V}^k)$ and $\bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \in conv(\boldsymbol{V}^k)$ be such that:

$$\min_{v^k \in conv(V^k)} \langle \boldsymbol{\lambda}, \boldsymbol{v}^k \rangle = \langle \boldsymbol{\lambda}, \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \rangle, \quad (18)$$

$$\max_{v^k \in conv(V^k)} \langle \boldsymbol{\lambda}, \boldsymbol{v}^k \rangle = \langle \boldsymbol{\lambda}, \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle. \quad (19)$$

Taking into account the properties of the square function (17) of (variable) $\langle \boldsymbol{\lambda}, \boldsymbol{v}^k \rangle$ in (16), as easily check that, the solution $\widehat{\boldsymbol{v}}^k(\boldsymbol{\lambda})$ of (16) for $\boldsymbol{\lambda} \neq \mathbb{O}_n$ is given by:

$$\widehat{\boldsymbol{v}}^k(\boldsymbol{\lambda}) = \begin{cases} \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}) & \eta\langle \boldsymbol{\lambda}, \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \rangle - [\langle \boldsymbol{\lambda}, \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \rangle]^2 < \eta\langle \boldsymbol{\lambda}, \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle - [\langle \boldsymbol{\lambda}, \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle]^2 \\ \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \text{ or } \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) & \eta\langle \boldsymbol{\lambda}, \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \rangle - [\langle \boldsymbol{\lambda}, \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \rangle]^2 = \eta\langle \boldsymbol{\lambda}, \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle - [\langle \boldsymbol{\lambda}, \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle]^2 \\ \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) & \text{otherwise} \end{cases},$$

where: $\eta = 2 + 4\rho t_k$. Note, that since for any $\boldsymbol{\lambda} \neq \mathbb{O}_n$ at least one of inner products $\langle \boldsymbol{\lambda}, \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle$ and $\langle \boldsymbol{\lambda}, \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \rangle$ differs from zero, i.e.:

$$\langle \boldsymbol{\lambda}, \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle - \langle \boldsymbol{\lambda}, \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \rangle > 0,$$

the inequality related to $\bar{\boldsymbol{v}}^k(\boldsymbol{\lambda})$ in the curly brackets in the above formula is equivalent to $\varphi(\boldsymbol{\lambda}) < \eta$, where:

$$\varphi(\boldsymbol{\lambda}) = \langle \boldsymbol{\lambda}, \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle + \langle \boldsymbol{\lambda}, \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \rangle.$$

Thus $\widehat{\boldsymbol{v}}^k(\boldsymbol{\lambda})$ solving (16) for $\boldsymbol{\lambda} \neq \mathbb{O}_n$ can be rewritten in compact form as follows:

$$\widehat{\boldsymbol{v}}^k(\boldsymbol{\lambda}) = \begin{cases} \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}) & \varphi(\boldsymbol{\lambda}) < \eta \\ \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \text{ or } \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) & \varphi(\boldsymbol{\lambda}) = \eta \\ \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) & \text{otherwise} \end{cases}. \quad (20)$$

Thus, the optimum:

$$L_k(\widehat{T}(\boldsymbol{\lambda}), \widehat{\boldsymbol{v}}^k(\boldsymbol{\lambda}), \boldsymbol{\lambda}) = \min\{\bar{L}_k(\bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}), \boldsymbol{\lambda}), \bar{L}_k(\bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}), \boldsymbol{\lambda})\}, \quad (21)$$

can be rewritten in the compact form as:

$$L_k(\widehat{T}(\boldsymbol{\lambda}), \widehat{\boldsymbol{v}}^k(\boldsymbol{\lambda}), \boldsymbol{\lambda}) = -\frac{1}{4\rho}[\langle \boldsymbol{\lambda}, \widehat{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \rangle]^2 + \left(\frac{1}{2\rho} + t_k\right)\langle \boldsymbol{\lambda}, \widehat{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \rangle - \frac{1}{4\rho}, \quad (22)$$

with $\widehat{\boldsymbol{v}}^k(\boldsymbol{\lambda})$ given by (20). Since $\bar{\boldsymbol{v}}^k(\boldsymbol{\lambda})$ and $\bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda})$ are boundary points of the set $conv(\boldsymbol{V}^k)$, we also have:

$$\widehat{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \in bd[conv(\boldsymbol{V}^k)].$$

Let $\mathbb{O}_n \neq \boldsymbol{\lambda} \in \mathcal{R}^n$. It is obvious, due to affine with respect to $\boldsymbol{\lambda}$ form of the inner product in (18) and (19), that for any positive constant α we have:

$$\bar{\boldsymbol{v}}^k(\alpha\boldsymbol{\lambda}) = \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \text{ and } \bar{\bar{\boldsymbol{v}}}^k(\alpha\boldsymbol{\lambda}) = \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}).$$

Whence, if $\varphi(\boldsymbol{\lambda}) > \eta$, then for any α such that $\alpha > \eta/\varphi(\boldsymbol{\lambda})$, thus in particular for any $\alpha \geq 1$, we have $\widehat{\boldsymbol{v}}^k(\alpha\boldsymbol{\lambda}) = \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda})$, and for any $\alpha < \eta/\varphi(\boldsymbol{\lambda})$, we have $\widehat{\boldsymbol{v}}^k(\alpha\boldsymbol{\lambda}) = \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda})$. When $\varphi(\boldsymbol{\lambda}) < \eta$, for any α such that $\alpha < \eta/\varphi(\boldsymbol{\lambda})$, we have $\widehat{\boldsymbol{v}}^k(\alpha\boldsymbol{\lambda}) = \bar{\boldsymbol{v}}^k(\boldsymbol{\lambda})$ and for any α such that $\alpha > \eta/\varphi(\boldsymbol{\lambda})$, we have $\widehat{\boldsymbol{v}}^k(\alpha\boldsymbol{\lambda}) = \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda})$. In both cases for $\alpha = \eta/\varphi(\boldsymbol{\lambda})$ each of $\bar{\boldsymbol{v}}^k(\boldsymbol{\lambda})$ and $\bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda})$ can be chosen. When $\varphi(\boldsymbol{\lambda}) = \eta$, i.e., both $\bar{\boldsymbol{v}}^k(\boldsymbol{\lambda})$ and $\bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda})$ are locally optimal for $\boldsymbol{\lambda}$, for $\alpha\boldsymbol{\lambda}$ we have $\hat{\boldsymbol{v}}^k(\alpha\boldsymbol{\lambda}) = \overline{\boldsymbol{v}}^k(\boldsymbol{\lambda})$ for $\alpha < 1$, and $\hat{\boldsymbol{v}}^k(\alpha\boldsymbol{\lambda}) = \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda})$ for $\alpha > 1$. The above results in:

$$\hat{\boldsymbol{v}}^k(\alpha\boldsymbol{\lambda}) = \begin{cases} \overline{\boldsymbol{v}}^k(\boldsymbol{\lambda}) = \hat{\boldsymbol{v}}^k(\boldsymbol{\lambda}) & \varphi(\boldsymbol{\lambda}) < \eta,\ \alpha < \eta/\varphi(\boldsymbol{\lambda}) \\ \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) & \varphi(\boldsymbol{\lambda}) < \eta,\ \alpha > \eta/\varphi(\boldsymbol{\lambda}) \\ \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) = \hat{\boldsymbol{v}}^k(\boldsymbol{\lambda}) & \varphi(\boldsymbol{\lambda}) > \eta,\ \alpha > \eta/\varphi(\boldsymbol{\lambda}) \\ \overline{\boldsymbol{v}}^k(\boldsymbol{\lambda}) & \varphi(\boldsymbol{\lambda}) > \eta,\ \alpha < \eta/\varphi(\boldsymbol{\lambda})\,. \\ \overline{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \text{ or } \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) & \alpha = \eta/\varphi(\boldsymbol{\lambda}) \\ \overline{\boldsymbol{v}}^k(\boldsymbol{\lambda}) & \varphi(\boldsymbol{\lambda}) = \eta,\ \alpha < 1 \\ \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) & \varphi(\boldsymbol{\lambda}) = \eta,\ \alpha > 1 \end{cases} \quad (23)$$

Then, we get the following result.

Proposition 1. Let $\boldsymbol{\lambda} \in \mathcal{R}^n$ and $\boldsymbol{\lambda} \neq \mathbb{O}_n$. The solutions of IP tasks are such that:

(i) $\hat{\boldsymbol{v}}^r(\boldsymbol{\lambda}) \in bd[conv(\boldsymbol{V}^r)]$ for $r = 1, \ldots, k$,

(ii) $\hat{\boldsymbol{v}}^r(\alpha\boldsymbol{\lambda}) = \hat{\boldsymbol{v}}^r(\boldsymbol{\lambda})$ for any $\alpha > 0, r = 1, \ldots, k-1$,

(iii) $\hat{\boldsymbol{v}}^k(\alpha\boldsymbol{\lambda})$ is given by (23) for any $\alpha > 0$.

From (7), (10), (11) and (13) as a straightforward conclusion we see that IP tasks can expressed in compact form as:

$$\min_{\substack{T, \boldsymbol{v}^r \in conv(\boldsymbol{V}^r), \\ r=1,\ldots,k}} L(T, \boldsymbol{v}^1, \ldots, \boldsymbol{v}^k, \boldsymbol{\lambda}) = L_D(\boldsymbol{\lambda}),$$

whence the equivalent sum-form of the dual function follows:

$$L_D(\boldsymbol{\lambda}) = \sum_{r=1}^{k-1} \tau_r L_r(\hat{\boldsymbol{v}}^r(\boldsymbol{\lambda}), \boldsymbol{\lambda}) + L_k(\hat{T}(\boldsymbol{\lambda}), \hat{\boldsymbol{v}}^k(\boldsymbol{\lambda}), \boldsymbol{\lambda}) + \langle \boldsymbol{\lambda}, \boldsymbol{w} \rangle. \quad (24)$$

Since the local Lagrangians (8) are continuous with respect to all arguments, from the compactness of $conv(\boldsymbol{V}^r)$ using the known results concerning the continuity of minimum of continuous functions on compact sets [29; Theorem 7.2] we conclude that the functions $L_r(\hat{\boldsymbol{v}}^r(\boldsymbol{\lambda}), \boldsymbol{\lambda})$ defined by (10) are continuous on $\mathcal{R}^n$, $r = 1, \ldots, k-1$. In view of the same result the right hand size of (18) and (19) are continuous functions of $\boldsymbol{\lambda}$, whence the continuity of $L_k(\hat{T}(\boldsymbol{\lambda}), \hat{\boldsymbol{v}}^k(\boldsymbol{\lambda}), \boldsymbol{\lambda})$ defined by (22), (23) follows. Thus the continuity of $L_D(\boldsymbol{\lambda})$ is resolved. Dual function $L_D(\boldsymbol{\lambda})$ as minimum of the weighted sum (24) of local Lagrangians plus affine component $\langle \boldsymbol{\lambda}, \boldsymbol{w} \rangle$ is concave function, for details see proof of Theorem 2.16 in [7]. The next result is valid

Proposition 2. The dual function $L_D(\boldsymbol{\lambda})$ (13) is continuous concave function in the space $\mathcal{R}^n$.

Having already proved that the dual function is continuous on the space $\mathcal{R}^n$, we wish to show that the solution to the CP there exists. The proof is based on the observation that the maximization of the continuous dual function on $\mathcal{R}^n$ can be restricted to some compact ball $\boldsymbol{\mathcal{B}}(\mathbb{O}_n, \alpha_{max}) \subset \mathcal{R}^n$, the radius α_{max} is defined by (27) and (28) below.

Note that on the basis of (24), (14) and (22) for $\boldsymbol{\lambda} = \mathbb{O}_n$ we have $L_D(\mathbb{O}_n) = -1/4\rho$. Let us consider an arbitrary $\mathbb{O}_n \neq \boldsymbol{\lambda} \in \mathcal{R}^n$. Since, in view of (14), each term in the first sum of (24) is non-positive, taking into account (22) we have:

$$L_D(\boldsymbol{\lambda}) \leq L_k(\hat{T}(\boldsymbol{\lambda}), \hat{\boldsymbol{v}}^k(\boldsymbol{\lambda}), \boldsymbol{\lambda}) + \langle \boldsymbol{\lambda}, \boldsymbol{w} \rangle. \quad (25)$$

Obviously, for any $\boldsymbol{\lambda} \neq \mathbb{O}_n$ at least one of the inner products $\langle \boldsymbol{\lambda}, \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle$ and $\langle \boldsymbol{\lambda}, \overline{\boldsymbol{v}}^k(\boldsymbol{\lambda}) \rangle$ differs from zero. Without the loss of generality we assume that $\langle \boldsymbol{\lambda}, \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle > 0$. In view of (20), this means that $\varphi(\boldsymbol{\lambda}) \geq \eta$, i.e., $\eta/\varphi(\boldsymbol{\lambda}) \leq 1$. Due to (21):

$$L_k(\hat{T}(\boldsymbol{\lambda}), \hat{\boldsymbol{v}}^k(\boldsymbol{\lambda}), \boldsymbol{\lambda}) \leq \bar{L}_k(\overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}), \boldsymbol{\lambda}),$$

which, in view of (25) and (17), yield:

$$L_D(\boldsymbol{\lambda}) \leq -\frac{1}{4\rho}[\langle \boldsymbol{\lambda}, \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle]^2 + \left(\frac{1}{2\rho} + t_k\right)\langle \boldsymbol{\lambda}, \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle + -\frac{1}{4\rho} + \langle \boldsymbol{\lambda}, \boldsymbol{w} \rangle. \quad (26)$$

Let us now prove that for any $\boldsymbol{\lambda} \in \boldsymbol{\mathcal{A}}(\mathbb{O}_n, \varepsilon, 1)$, where $\boldsymbol{\mathcal{A}}(\mathbb{O}_n, \varepsilon, 1)$ is closed annulus in $\mathcal{R}^n$, i.e., a region bounded by two concentric circles of radiuses equal ε and 1:

$$\boldsymbol{\mathcal{A}}(\mathbb{O}_n, \varepsilon, 1) = \{\boldsymbol{\lambda} \in \mathcal{R}^n : \varepsilon \leq \langle \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \leq 1\},$$

there exists a positive coefficient $\alpha_0(\boldsymbol{\lambda})$ such that for any $\alpha > \alpha_0(\boldsymbol{\lambda})$ the inequality:

$$L_D(\alpha\boldsymbol{\lambda}) < L_D(\mathbb{O}_n)$$

holds. Let:

$$\alpha_0(\boldsymbol{\lambda}) = max\left\{1, \frac{(2+4\rho t_k)\langle \boldsymbol{\lambda}, \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle + 4\rho\|\boldsymbol{\lambda}\|\|\boldsymbol{w}\|}{[\langle \boldsymbol{\lambda}, \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle]^2}\right\}, \quad (27)$$

where: $\|\cdot\|$ denotes the Euclidean norm in $\mathcal{R}^n$. For any $\alpha > \alpha_0(\boldsymbol{\lambda})$ we have, in particular, that:

$$\alpha[\langle \boldsymbol{\lambda}, \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle]^2 > (2 + 4\rho t_k)\langle \boldsymbol{\lambda}, \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle + 4\rho\|\boldsymbol{\lambda}\|\|\boldsymbol{w}\|,$$

and the inequality:

$$\alpha > 1 > \eta/\varphi(\boldsymbol{\lambda}) \text{ or } \eta/\varphi(\boldsymbol{\lambda}) = 1 < \alpha$$

from the third and seventh rows of (23), respectively, is satisfied. Thus, taking into account Proposition 1 (iii), the Schwarz inequality and (26) it is easy to check that:

$$L_D(\alpha\boldsymbol{\lambda}) \leq -\alpha^2\frac{1}{4\rho}[\langle \boldsymbol{\lambda}, \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle]^2 + \alpha\left(\frac{1}{2\rho} + t_k\right)\langle \boldsymbol{\lambda}, \overline{\overline{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle - \frac{1}{4\rho} + \alpha\langle \boldsymbol{\lambda}, \boldsymbol{w} \rangle < -\frac{1}{4\rho}.$$

The assumed positivity of $\langle \boldsymbol{\lambda}, \bar{\bar{\boldsymbol{v}}}^k(\boldsymbol{\lambda}) \rangle$ combined with its continuity and compactness of the set $\boldsymbol{\mathcal{A}}(\mathbb{O}_n, \varepsilon, 1)$ implies that there exists:

$$\alpha_{max} = max_{\boldsymbol{\lambda} \in \boldsymbol{\mathcal{A}}(\mathbb{O}_n, \varepsilon, 1)} \alpha_0(\boldsymbol{\lambda}) > 1. \quad (28)$$

From the above it follows that for any $\boldsymbol{\lambda}$ such that $||\boldsymbol{\lambda}|| > \alpha_{max}$, we have:

$$L_D(\alpha \boldsymbol{\lambda}) < L_D(\mathbb{O}_n).$$

Thus the dual function maximization in (12) can be restricted to closed ball $\boldsymbol{\mathcal{B}}(\mathbb{O}_n, \alpha_{max})$ of radius α_{max}, i.e., the equivalence hols:

$$L_D(\hat{\boldsymbol{\lambda}}) = max_{\boldsymbol{\lambda} \in \mathcal{R}^n} L_D(\boldsymbol{\lambda}) = max_{\boldsymbol{\lambda} \in \boldsymbol{\mathcal{B}}(\mathbb{O}_n, \alpha_{max})} L_D(\boldsymbol{\lambda}), \quad (29)$$

and the existence of $\hat{\boldsymbol{\lambda}}$ is guaranteed by continuity of $L_D(\boldsymbol{\lambda})$. Thus, we proved the next result.

Theorem 2. The solution $\hat{\boldsymbol{\lambda}}$ of the CP (12) there exists.

APPLICABILITY OF THE SCHEME

By a simple reasoning it is obvious that the scheme is applicable to the time-optimal control of the sequence of projects, i.e. ensures the determination of the optimal resource allocation, if and only if for every $\hat{\boldsymbol{\lambda}}$ being the solution of the dual problem (12):

$$\left(\hat{T}(\hat{\boldsymbol{\lambda}}), \hat{\boldsymbol{v}}^1(\hat{\boldsymbol{\lambda}}), \dots, \hat{\boldsymbol{v}}^k(\hat{\boldsymbol{\lambda}})\right)$$

is the solution to the original optimization task (5), (4), i.e., $\left(\hat{\boldsymbol{v}}^1(\hat{\boldsymbol{\lambda}}), \dots, \hat{\boldsymbol{v}}^k(\hat{\boldsymbol{\lambda}})\right)$ is the vector of the optimal performance speeds of the activities in the successive time intervals. It is well known that if the saddle point of the Lagrangian $L(T, \boldsymbol{v}^1, \dots, \boldsymbol{v}^k, \boldsymbol{\lambda})$ (6) there exists, then the dual approach can be successfully applied to solve (5), (4). If the solutions of IP tasks (10) and (11) are unique for any $\boldsymbol{\lambda} \in \mathcal{R}^n$, then the existence of a saddle point of the Lagrangian follows immediately from [8; Theorem 1, (ii) and (iii)]. Thus, the following theorem is valid.

Theorem 3. If the solutions of the IP tasks (10) and (11) are unique for any $\boldsymbol{\lambda} \in \mathcal{R}^n$, then the two-level scheme is applicable to the problem of time-optimal control of a sequence of projects of activities.

The uniqueness of IP solutions is guaranteed for any $\boldsymbol{\lambda} \in \mathcal{R}^n$, for example, if the sets $conv(\boldsymbol{V}^r)$, $r = 1, \dots, k$, are strictly convex, so for example when the functions f_{ri} in (1) are strictly concave [9,11,28], see also [12]. Note, hat in view of (29) the uniqueness requirement from Theorem 3 can be restricted to $\boldsymbol{\lambda} \in \boldsymbol{\mathcal{B}}(\mathbb{O}_n, \alpha_{max})$.Using [7; Lemmas 2.9 and 2.10] we can state for the case less restrictive than the uniqueness requirement from Theorem 3, the following condition.

Theorem 4. If the set $\boldsymbol{\Omega}(\hat{\boldsymbol{\lambda}})$, where:

$$\boldsymbol{\Omega}(\boldsymbol{\lambda}) = \left\{\boldsymbol{\omega}(\boldsymbol{\lambda}) = \sum_{r=1}^{k-1} \tau_r \hat{\boldsymbol{v}}^r(\boldsymbol{\lambda}) + \left(\hat{T}(\boldsymbol{\lambda}) - t_k\right)\hat{\boldsymbol{v}}^k(\boldsymbol{\lambda}) : \hat{\boldsymbol{v}}^r(\boldsymbol{\lambda}) \in \hat{\boldsymbol{V}}^r(\boldsymbol{\lambda})\right\},$$

consists of a single point for any $\hat{\boldsymbol{\lambda}}$ being the solution to CP task (12), then the scheme is applicable to optimal resource allocation problem.

Notice, that the last result requires only the uniqueness of the state vector $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}(\hat{\boldsymbol{\lambda}})$ achieved for the performance speeds $\hat{\boldsymbol{v}}^r(\hat{\boldsymbol{\lambda}})$, and the uniqueness of the vectors $\hat{\boldsymbol{v}}^r(\hat{\boldsymbol{\lambda}})$ is not required here. Thus, the optimal resource allocation does not have to be unique, if only the vector $\hat{\boldsymbol{\omega}}$ of final states is uniquely determined. Also, the uniqueness of the dual problem solution is not necessary. Note finally, that the applicability of the scheme requires, after all, the existence of a sadle point of the Lagrangian (6). However, this is the crucial necessary applicability condition for most decentralized schemes, c.f., [7,8].

COMPUTATION OF THE OPTIMAL RESOURCE ALLOCATION

Assume that the applicability conditions are satisfied and the two-level scheme results in:

$$\left(\hat{T}, \hat{\boldsymbol{v}}^1, \dots, \hat{\boldsymbol{v}}^k\right),$$

where: $\hat{T} = \hat{T}(\hat{\boldsymbol{\lambda}})$ and $\hat{\boldsymbol{v}}^r = \hat{\boldsymbol{v}}^r(\hat{\boldsymbol{\lambda}})$ for $r = 1, \dots, k$. Note, that in view of Proposition 1 by the known Caratheodory's theorem, which asserts that any boundary point of the convex hull of compact set in $\mathcal{R}^n$ is a convex combination of its extreme points, the vectors $\hat{\boldsymbol{v}}^r$ can be expressed as:

$$\hat{\boldsymbol{v}}^r = \sum_{s=1}^{p_r} \alpha_r^s \, \hat{\boldsymbol{v}}^{r,s}, \quad (30)$$

where: $\hat{\boldsymbol{v}}^{r,s} \in \boldsymbol{V}^r$ for $s = 1, \dots, p_r$, and the nonnegative constants α_s^r are such that $\sum_{s=1}^{p_r} \alpha_r^s = 1$ for $r = 1, \dots, k$.

Let us introduce the vector function:

$$\hat{\boldsymbol{u}}(t) = \begin{cases} \hat{\boldsymbol{u}}^{r,s} & t_r + \tau_r^{s-1} \le t < t_r + \tau_r^s, \; s = 1, \dots, p_r, r = 1, \dots, k \\ \hat{\boldsymbol{u}}^{k,n} & t = \hat{T} \end{cases}, \quad (31)$$

where $\hat{\boldsymbol{u}}^{r,s} \geq \mathbb{O}_n$ are such that:

$$\hat{\boldsymbol{v}}^{r,s} = \boldsymbol{f}(\hat{\boldsymbol{u}}^{r,s})$$

and

$$\tau_r^s = \sum_{q=1}^{s} \alpha_r^q \, \tau_r \text{ for } s = 1, \dots, p_r, r = 1, \dots, k-1,$$

and for the last time interval:

$$\tau_k^s = \sum_{q=1}^{s} \alpha_k^q \left(\hat{T} - t_k\right), s = 1, \dots, n,$$

with $\tau_r^0 = 0$, $r = 1, \dots, k$. If the activities models f_{ri} are strictly increasing, then:

$$\hat{\boldsymbol{u}}^{r,s} = \boldsymbol{f}^{-1}(\hat{\boldsymbol{v}}^{r,s}),$$

where $\boldsymbol{f}^{-1}$ denotes the inverse function. By virtue of (30) and the definitions of the sets $\boldsymbol{V}^k$ and $\boldsymbol{U}^k$, the conditions (*i*), (*ii*) are satisfied for $\hat{\boldsymbol{u}}(t)$ in (31), in view of (4) the given terminal state $\boldsymbol{w}$ is reached, thus $\hat{\boldsymbol{u}}(t)$ (31) is admissible resource allocation (control), and in view of Theorems 1 and 3 or 4 its optimality is guaranteed. The control (31) is generally not unique in the set of time-optimal resource allocations. In view of the above, there is at least one piecewise constant time-optimal resource allocation with at most $\sum_{r=1}^{k} p_r - 1$ discontinuity points.

COMPUTATIONAL ALGORITHM

The computations should be arranged hierarchically in two-level structure, i.e., in each iteration of the maximization procedure of CP level the whole optimization procedures for solving IP independent tasks must be realized (see, Fig. 1).

Step 1: Determine in the following two-level computations the optimal speeds $(\hat{\boldsymbol{v}}^1, \dots, \hat{\boldsymbol{v}}^k)$ in the successive time intervals, $r = 1, \dots, k$, and the optimal performance time $\hat{T}$.

Step 1.0: Choose the initial point $\boldsymbol{\lambda}^o$ for numerical procedure applied to solve the coordination problem (12).

Step 1.1: Let $\boldsymbol{\lambda}^m$ be the m-th iterate in the numerical maximization procedure chosen to solve (12). For $\boldsymbol{\lambda} = \boldsymbol{\lambda}^m$ solve $k-1$ independent local resource allocation tasks (10) and the task (11) according to the chosen numerical optimization procedures and determine the performance speeds $\hat{\boldsymbol{v}}^r(\boldsymbol{\lambda}^m)$, $r = 1, \dots, k-1$, and $\left(\hat{T}(\boldsymbol{\lambda}^m), \hat{\boldsymbol{v}}^k(\boldsymbol{\lambda}^m)\right)$.

Step 1.2: Using $\hat{\boldsymbol{v}}^r(\boldsymbol{\lambda}^m)$, $r = 1, \dots, k$ and $\hat{T}(\boldsymbol{\lambda}^m)$ compute, according to the numerical procedure selected to solve CP (12), the new vector of prices $\boldsymbol{\lambda}^{m+1}$ which is the next approximation of $\hat{\boldsymbol{\lambda}}$. If for $\boldsymbol{\lambda}^{m+1}$ the stopping rule of the chosen maximization scheme is satisfied, e.g.:

$$\|\boldsymbol{\lambda}^{m+1} - \boldsymbol{\lambda}^m\| \leq \varepsilon_1$$

or

$$|L_D(\boldsymbol{\lambda}^{m+1}) - L_D(\boldsymbol{\lambda}^m)| \leq \varepsilon_2,$$

where ε_1 and ε_2 are preselected small positives, put:

$$\hat{\boldsymbol{v}}^r = \hat{\boldsymbol{v}}^r(\boldsymbol{\lambda}^m), r = 1, \dots, k$$

as the vectors of the optimal performance speeds and:

$$\hat{T} = \hat{T}(\boldsymbol{\lambda}^m)$$

as the optimal performance time of the multi–project and go to step 2. Otherwise return to step 1.1 and continue the computations for $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{m+1}$.

Step 2: Compute the vector of the optimal control $\hat{\boldsymbol{u}}(t)$ according to (31).

Remark 1. It is known that too large Lagrangian functions may be a drawback when applying numerical optimization, since large Lagrangians cannot completely prevent the constraint from violation. This disadvantage is overcome in the scheme. The equivalence (29) implies that CP can be replaced by dual function maximization with the constrain:

$$\boldsymbol{\lambda} \in \boldsymbol{\mathcal{B}}(\mathbb{O}_n, \alpha_{max})$$

imposed. However, the numerical studies suggest that it is unnecessary in most cases.

Remark 2. The appealing feature of the scheme is that only the values of local Lagrangians $L_r(\hat{\boldsymbol{v}}^r(\boldsymbol{\lambda}^m), \boldsymbol{\lambda}^m)$ and $L_k\left(\hat{T}(\boldsymbol{\lambda}^m), \hat{\boldsymbol{v}}^k(\boldsymbol{\lambda}^m), \boldsymbol{\lambda}^m\right)$, not the IP solutions $\hat{\boldsymbol{v}}^r(\boldsymbol{\lambda}^m)$ and $\hat{T}(\boldsymbol{\lambda}^m)$, are used for $\boldsymbol{\lambda}^m$ in successive iterations of the numerical procedure solving CP (see Fig. 1).

Remark 3. In view of Proposition 1, the vectors resulting from the IP tasks in each iteration of numerical procedure solving CP are such that:

$$\tilde{\boldsymbol{v}}^r = \hat{\boldsymbol{v}}^r(\boldsymbol{\lambda}^m) \in bd[conv(\boldsymbol{V}^r)].$$

Thus $\tilde{\boldsymbol{v}}^r$ can be expressed as a convex combination of the form (30). Hence the respective control $\tilde{\boldsymbol{u}}^r(t)$ of the form (31) exists and can be treated as an approximate solution of the overall problem.

Remark 4. Since $L_D(\boldsymbol{\lambda})$ is continuous concave function on $\mathcal{R}^n$ (Proposition 2), known sub-gradient methods of non-differentiable optimization [3,6] may be implemented for constrained maximization task (12). Also (10) and (11) (equivalently (18) or (19)) are convex optimization tasks and can be solved by using standard convex programming methods [3].

Both in centralized (global) resource allocation by direct T minimization in (3) (or equivalently in (5)) subject to constraints (4) and in decentralized computations of the scheme, there are two types of constraints. The inequality constraints:

$$0 \leq \alpha_r^s \leq 1 \text{ and } \hat{v}_{ri}^{r,s} \geq 0,$$

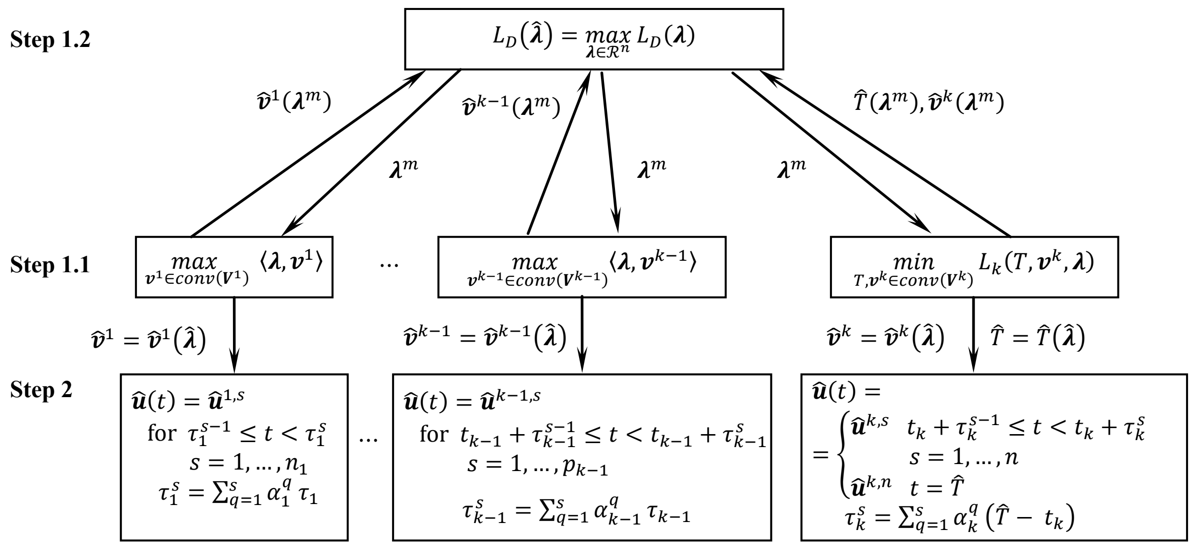

Fig. 1. Two-level decentralized procedure for optimal resource allocation

which are imposed on every α_r^s and $\hat{v}_{ri}^{r,s}$, $i = 1, \dots, n_r$, $s = 1, \dots, p_r$, $r = 1, \dots, k$ in the convex combination representation (30) of $\hat{\boldsymbol{v}}^r$. The set of equality constraints is composed of convex combination coefficients sum:

$$\sum_{s=1}^{p_r+1} \alpha_r^s = 1$$

and the total resource usage constraints imposed on any $\hat{\boldsymbol{v}}^{r,s}$ in (30). If the functions f_{ri} are strictly increasing, i.e. f_{ri}^{-1} exists for each operation, then in view of Proposition 1 the last constraints take the form:

$$\sum_{q=1}^{r} \sum_{i=1}^{n_q} f_{qi}^{-1}(\hat{v}_{qi}^{r,s}) = N, \; s = 1, \dots, p_r, r = 1, \dots, k.$$

In the case of centralized resource allocation by direct T minimization the total number of constraints (inequality and equality) is equal to:

$$CN = \sum_{r=1}^{k-1} [1 + 3p_r + p_r^2] + 1 + 4n + n^2,$$

while the number of optimization variables:

$$DV = 1 + \sum_{r=1}^{k-1} [2p_r + p_r^2] + 2n + n^2.$$

For decentralized scheme the number of constraints for IP (10):

$$CN_r = 1 + 3p_r + p_r^2, r = 1, \dots, k-1,$$

and for the last local task (11) is:

$$CN_k = 1 + 3n + n^2.$$

The respective numbers of optimization variables are as follows:

Table 1. The numbers of optimization variables and constraints (inequalities and equalities together) in global and decentralized approaches; the meanings of symbols are explained in the text

Each project composed of $n_r = 2$ operations				
k	2	4	6	8
DV	33	161	449	961
CN	44	192	508	1056
DV_r	8,25	8,24,48,81	8,24,48,80,120,169	8,24,48,80,120,168,224,289
CN_r	11,29	11,29,55,89	11,29,55,89,131,181	11,29,55,89,131,181,239,305
Each project composed of $n_r = 4$ operations				
k	2	4	6	8
DV	105	561	1625	3553
CN	126	620	1738	3736
DV_r	24,81	24,80,168,289	24,80,168,288,440,625	24,80,168,288,440,624,840,1089
CN_r	29,89	29,89,181,305	29,89,181,305,461,649	29,89,181,305,461,649,869,1121

$$DV_r = 2p_r + p_r^2 \text{ for } r = 1, \dots, k-1$$

and

$$DV_k = 1 + 2n + n^2.$$

To compare the computations for decentralized and global approaches the numbers of optimization variables and constraints are summarized in Table 1 and visualized in Fig. 2 and 3 for a few exemplary sequences of projects data. In Fig. 2 and 3 the average numbers of both the optimization variables and constraints for decentralized approach are given. The sequences of $2-8$ projects, each project set composed of 2,4,6 activities, were included in the experiment. One can observe from Table 1 and Fig. 2,3, that the mean numbers of the optimization variables and constraints are evidently reduced for decentralized approach in comparison to the primary global optimization task.

FINAL REMARKS

Efficient time-decomposition algorithm for finding globally optimal resource allocation is proposed and discussed. The Lagrangian based time-decomposition approach is applied. It allows the global solution to be found in decentralized manner. The convexification technique is combined with the price coordination in order to guarantee its applicability. The necessary and sufficient applicability conditions are derived and analyzed. Taking into account the specific properties of IP and CP tasks it is proved that the scheme is applicable to time-optimal resource allocation even if the solutions of lower level and coordination tasks are not unique. The applicability conditions suggest that the scheme can be successfully applied to find the optimal resource allocation for a wide class of activity models, including, in particular the commonly used in project modelling practice models being concave with respect to the resource amount. The considerable decrease on the mean numbers of both the optimization variables and constraints are reported for the scheme. An alternative approach of local optimization of a sequence of project is proposed in [11], according to which the choice of the optimal control is made at every moments t_r of the successive project appearance for all the activities actually at performance and the final states of activities that are equal to given terminal states minus the states so far reached. However, for control determined by this algorithm the performance time is, in general, greater than the globally optimal performance time determined by two-level scheme. A considerable range of technical and economic applications should be pointed out. The main fields are the following: the allocation of working power of building concern among enterprises in the course of building (cf., [9]), the allocation of a financial outlay among projects under constrained intensity of investing [1,2,5,6,9,23], the allocation of a common primary memory among independent tasks appearing at different times in a multiprocessor systems (cf., [1,21,24,27]), can be also applied in decision support intelligent systems [22], where different resources with constrained consumption must be optimized.

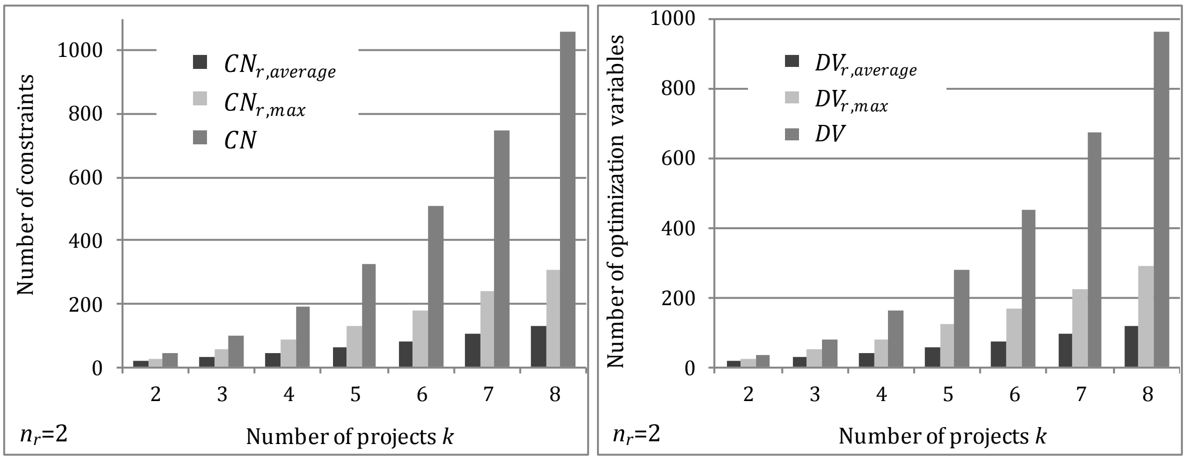

Fig. 2. The numbers of optimization constraints and variables in global and decentralized approach; $CN_{r,max}$, $DV_{r,max}$ and $CN_{r,average}$, $DV_{r,average}$ – maximal and average numbers of constraints and decision variables for IP tasks, CN and DV – the total numbers of constraints and optimization variables for global approach, number of project activities $n_r = 2$

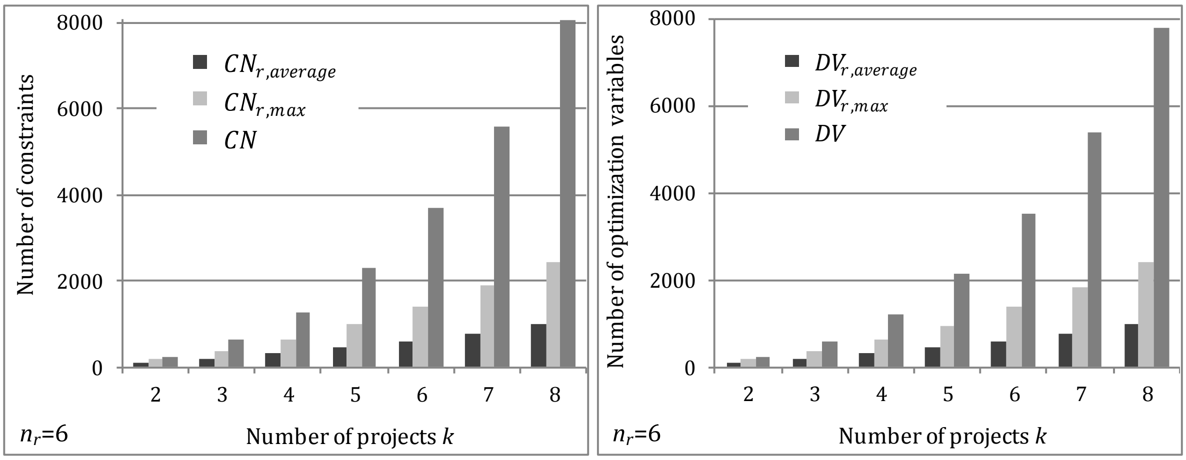

Fig. 3. The numbers of optimization constraints CN, $CN_{r,max}$, $CN_{r,average}$ and variables DV, $DV_{r,max}$, $DV_{r,average}$ in global and decentralized approaches, number of project activities $n_r = 6$

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