DE GRUYTER OPEN

JAISCR, 2016, Vol. 6, No. 4, pp. 219 – 229
10.1515/jaiscr-2016-0016

MINKOWSKI'S INEQUALITY BASED SENSITIVITY ANALYSIS OF FUZZY SIGNATURES

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Abstract

Fuzzy signatures were introduced as special tools to describe and handle complex systems without their detailed mathematical models. The input parameters of these systems naturally have uncertainties, due to human activities or lack of precise data. These uncertainties influence the final conclusion or decision about the system. In this paper we discuss the sensitivity of the weigthed general mean aggregation operator to the uncertainty of the input values, then we analyse the sensitivity of fuzzy signatures equipped with these aggregation operators. Finally, we apply our results to a fuzzy signature used in civil enginnering.

Keywords: aggregation operators, generalized mean, sensitivity analysis, fuzzy signatures, building diagnostics

1 Introduction

The problem of modelling and classification of complex objects and systems often arises in many fields of science and technology. Frequent difficulty is that there appear not well-known or hidden interdependencies between the variables. Moreover, in many cases there are not known accurate mathematical models, and because of lack of reproducibility the statistical tools can be used with large limitations to check the assumptions about the system.

Fuzzy signatures are possible tools to describe such complex systems and objects. In this kind of approach, complex systems are described by a set of qualitative measures, which are also arranged into a hierarchical framework expressing interconnections and dependencies, and modelling the human approach to the problem.

The fuzzy signature based modelling technique can be applied for very different problems, for example in economy, in the medical field [1], and in several fields of engineering and informatics, for example robotics [2], data mining [3] and civil engineering [4].

In a mathematical point of view, fuzzy signatures are hierarchical representations of data structuring into vectors of fuzzy values [5]. A fuzzy signature is defined as a special multidimensional fuzzy data structure, which is a generalization of vector valued fuzzy sets [6]. Vector valued fuzzy sets are special cases of L-fuzzy sets which were introduced in [7]. A fuzzy signature is defined by

$$A\colon X \to S^{(n)}, \tag{1}$$

where X is the universe of discourse, $1 \leq n$ and

$$S^{(n)} = \times_{i=1}^{n} S_i, \qquad S_i = \begin{cases} [0,1] \\ S^{(m)}. \end{cases} \tag{2}$$

We can represent a fuzzy signature by nested vector value fuzzy sets and also by a tree graph (see Figure 1), which is much more understandable [6].

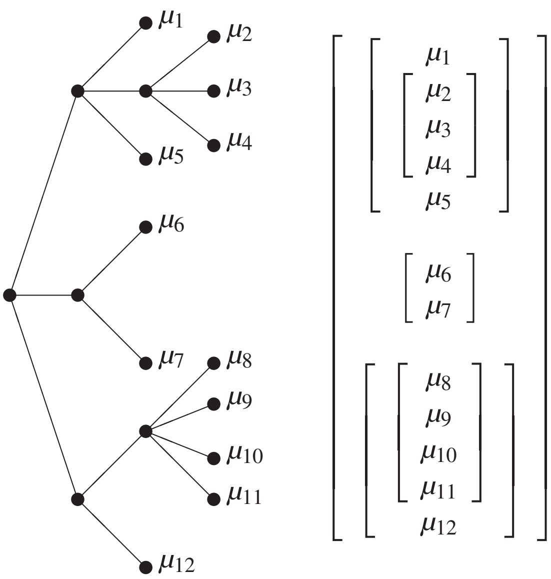

Figure 1. A fuzzy signature graph and the corresponding nested vectors.

The input values (μ-s) of the fuzzy signature usually given by human experts or estimation methods. The final output (at the root of the tree graph) is computed from the inputs applying suitable aggregation functions, this is the membership value of the whole fuzzy signature. Since different human experts could assign slightly different values to the same situation, a usable fuzzy signature based model should not be *too* sensitive to the input values.

The goal of this article is to discuss how the membership value of the whole fuzzy set changes if the membership values in the nested vectors change. In other words, if we think of the tree graph representation, how the membership value of the root changes if the membership values of leaves change. To answer this question we have to know how to compute a membership value of a subgraph from the leaves.

In this article we assume that all the operators applied on membership values in the signature are from the class of weighted generalized mean aggregation operators (WGMs). A similar question was investigated in [8], but for different cases and by different mathematical tools.

There are several approaches discussing the sensitivity of aggregation operators with different weights, for example [9] discusses the sensitivity of weighted fuzzy aggregation, [10] and [11] discuss the sensitivity of ordered weighted aggregation operators.

The paper is organized as follows: in Section 2 we recall some mathematical tools, in Section 3 the sensitivity of WGM is discussed, in Section 4 we examine the sensitivity of fuzzy signatures in general and in special cases, and finally in Section 5 we discuss the sensitivity of a concrete fuzzy signature used in civil engineering.

2 Mathematical Background

The generalized mean and its generalization, the weighted generalized mean form a very large class of aggregation operators. Their various special cases often arise also in theoretical and practical problems.

Definition 1 (Generalized mean) *(see for example [12] or [13]) Let $x_1, \ldots, x_n$ be nonnegative real numbers and $p \in \mathbb{R}$ ($p \neq 0$). Then their generalized mean with parameter p:*

$$M_p(x_1, \ldots, x_n) = \left[\frac{1}{n} \sum_{k=1}^{n} x_k^p \right]^{\frac{1}{p}} \tag{3}$$

Some special cases in p:

- $p = 1$ arithmetic mean
- $p = 2$ quadratic mean
- $p = -1$ harmonic mean

Definition 2 (Weighted generalized mean; WGM) *Let $x_1,\dots,x_n$ and $w_1,\dots,w_n$ be nonnegative real numbers, $w_i \geq 0, \sum_{i=1}^{n} w_i = 1$ and $p \in \mathbb{R}$ ($p \neq 0$). Then the weighted generalized mean of $x_1,\dots,x_n$ with weights $w_1,\dots,w_n$ and with parameter p:*

$$M_p^w(x_1,\dots,x_n) = \left[\sum_{k=1}^{n} w_k x_k^p\right]^{\frac{1}{p}}. \quad (4)$$

We note here that the weighted generalized mean sometimes refered as *'scaled norm'*. We do not use this terminology because of the possible missunderstanding: the properties of the norm are fulfilled only when $p \geq 1$, but the WGM is defined for every $p \in \mathbb{R}$.

The generalized mean is a special case of the weighted generalized mean with weights $w_k = \frac{1}{n}$. The limits at $\pm\infty$ regardless to the weight:

$$\lim_{p\to\infty}\left[\sum_{k=1}^{n} w_k x_k^p\right]^{\frac{1}{p}} = \max(x_i), \quad (5)$$

$$\lim_{p\to-\infty}\left[\sum_{k=1}^{n} w_k x_k^p\right]^{\frac{1}{p}} = \min(x_i). \quad (6)$$

The limit if $p \to 0$ is the weighted geometric mean:

$$\lim_{p\to 0}\left[\sum_{k=1}^{n} w_k x_k^p\right]^{\frac{1}{p}} = \prod_{i=1}^{n} x_i^{w_i}. \quad (7)$$

Our aim is to give upper bound on the change of the weighted generalized mean if we know the change of the input values $x_1,\dots,x_n$. In Section 3 we search for such a bound for $|\Delta M|$ which depends on $\Delta\underline{x}$ or on a kind of vector norm of $\Delta\underline{x}$, and on the basis of these results we give upper bounds for the change of the whole fuzzy signature in Section 4.

First we recall the definition of the p-norm (see for example [14]).

Definition 3 (p-norm) *Let $p \geq 1$ a real number and $\underline{x} = (x_1,\dots,x_n) \in \mathbb{R}^n$. Then the p-norm of $\underline{x}$*

$$\|\underline{x}\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}. \quad (8)$$

Some widely used p-norms:

- $p = 1$ (taxicab norm) $\|\underline{x}\|_1 = |x_1| + \ldots + |x_n|$,
- $p = 2$ (euclidean norm) $\|\underline{x}\|_2 = \sqrt{x_1^2 + \ldots + x_n^2}$,
- $p = \infty$ (maximum norm), $\|\underline{x}\|_\infty = \max(|x_1|,\dots,|x_n|)$.

Two important properties of the p-norm:

- If $1 \leq p \leq q \leq \infty$ then $\|\underline{x}\|_q \leq \|\underline{x}\|_p$,
- If $1 \leq p \leq q \leq \infty$ then $\|\underline{x}\|_p \leq \|\underline{x}\|_q \cdot n^{1/p-1/q}$.

We will use the generalization of the triangular inequality, the so called Minkowski's inequality.

Theorem 1 (Minkowski's inequality) *(see for example [12] or [13]) Let $\underline{a}, \underline{b} \in \mathbb{R}^n$, $p \geq 1$, then the following inequality holds:*

$$\|\underline{a} + \underline{b}\|_p \leq \|\underline{a}\|_p + \|\underline{b}\|_p. \quad (9)$$

The generalization of the reverse triangular inequalty also holds:

Corollary 1 *If $\underline{a}, \underline{b} \in \mathbb{R}^n$, $p \geq 1$, then,*

$$\left|\|\underline{a}\|_p - \|\underline{b}\|_p\right| \leq \|\underline{a} - \underline{b}\|_p. \quad (10)$$

3 Sensitivity of the Weighted General Mean for $p \geq 1$

In this Section we analyse the change of the WGM under the change of its input vector. Note that we examine the case $p \geq 1$. Let us use the following notations:

$$\underline{w}^{1/p} = \left(w_1^{1/p},\dots,w_n^{1/p}\right), \quad (11)$$

$$\underline{w}^{1/p} \cdot \underline{x} = \left(w_1^{1/p} \cdot x_1,\dots,w_n^{1/p} \cdot x_n\right). \quad (12)$$

If the input vector is $\underline{x} = (x_1,\dots,x_n)$, the vector if the weights is $\underline{w} = (w_1,\dots,w_n)$, then the weighted generalized mean with parameter p is

$$M = \left[\sum_{i=1}^{n} w_i x_i^p\right]^{\frac{1}{p}} = \left[\sum_{i=1}^{n}\left(w_i^{1/p} x_i\right)^p\right]^{\frac{1}{p}}, \quad (13)$$

$$= \left\|\underline{w}^{1/p} \cdot \underline{x}\right\|_p. \quad (14)$$

If the new (maybe perturbed) input vector is $\underline{x}^* = (x_1^*,\dots,x_n^*)$, then the new output is $M^* = \left\|\underline{w}^{1/p} \cdot \underline{x}^*\right\|_p$. So the change of the input is $\Delta\underline{x} =$

$\underline{x}^* - \underline{x}$, the change of the output is $\Delta M = M^* - M$. In the following we give upper estimations for $|\Delta M|$.

$$|\Delta M| = \left| \left\| \underline{w}^{1/p} \cdot \underline{x}^* \right\|_p - \left\| \underline{w}^{1/p} \cdot \underline{x} \right\|_p \right|, \tag{15}$$

$$\leq \left\| \underline{w}^{1/p} \cdot \underline{x}^* - \underline{w}^{1/p} \cdot \underline{x} \right\|_p, \tag{16}$$

$$= \left\| \underline{w}^{1/p} \cdot (\underline{x}^* - \underline{x}) \right\|_p = \left\| \underline{w}^{1/p} \cdot \Delta \underline{x} \right\|_p, \tag{17}$$

$$= \left[\sum_{i=1}^{n} \left(w_i^{1/p} |\Delta x_i| \right)^p \right]^{\frac{1}{p}} = \left[\sum_{i=1}^{n} w_i \cdot |\Delta x_i|^p \right]^{\frac{1}{p}}. \tag{18}$$

We can use this formula when the precision of the inputs are known. For example if we know that the absolute value of the change is less than ε for all i ($|\Delta x_i| < \varepsilon$) then we have

$$|\Delta M| \leq \left[\sum_{i=1}^{n} \left(w_i^{1/p} \varepsilon \right)^p \right]^{\frac{1}{p}} = \left[\sum_{i=1}^{n} w_i \varepsilon^p \right]^{\frac{1}{p}}, \tag{19}$$

$$= \varepsilon \cdot \left[\sum_{i=1}^{n} w_i \right]^{\frac{1}{p}} = \varepsilon, \tag{20}$$

so in this case the output value is less than ε also. Another way when we give upper bounds with the norm of the change of the input vector. Based on the previous upper estimation we get that

$$|\Delta M| \leq \left[\sum_{i=1}^{n} w_i \cdot |\Delta x_i|^p \right]^{\frac{1}{p}}, \tag{21}$$

$$\leq \left[\sum_{i=1}^{n} \left(\max(w_i)^{1/p} |\Delta x_i| \right)^p \right]^{\frac{1}{p}}, \tag{22}$$

$$= \max(w_i)^{1/p} \cdot \|\Delta \underline{x}\|_p, \tag{23}$$

$$= \|\underline{w}^{1/p}\|_\infty \cdot \|\Delta \underline{x}\|_p. \tag{24}$$

We note that in this case some information is lost because only the norm of the change is used, but not the whole vector. As in the previous example if we know that the absolute value of the change is less then ε for all i ($|\Delta x_i| < \varepsilon$), now we get weaker estimation:

$$|\Delta M| \leq \left[\sum_{i=1}^{n} \varepsilon^p \right]^{\frac{1}{p}} = [n \cdot \varepsilon^p]^{\frac{1}{p}} = n^{1/p} \cdot \varepsilon. \tag{25}$$

If the parameter of the aggregation operator is p, but we would like to measure the change of the input vector in q norm, then we have to switch form p to q using the properties of p-norm. We handle the two kind of upper estimations on $|\Delta M|$ as different cases.

If the starting point is that $|\Delta M| \leq \left\| \underline{w}^{1/p} \cdot \Delta \underline{x} \right\|_p$ then

– if $p \leq q$ then

$$|\Delta M| \leq \left\| \underline{w}^{1/p} \cdot \Delta \underline{x} \right\|_p \leq n^{1/p-1/q} \cdot \left\| \underline{w}^{1/p} \cdot \Delta \underline{x} \right\|_q, \tag{26}$$

$$\leq n^{1/p-1/q} \cdot \left\| \underline{w}^{1/p} \right\|_\infty \cdot \|\Delta \underline{x}\|_q. \tag{27}$$

– if $p > q$ then

$$|\Delta M| \leq \left\| \underline{w}^{1/p} \cdot \Delta \underline{x} \right\|_p \leq \left\| \underline{w}^{1/p} \cdot \Delta \underline{x} \right\|_q, \tag{28}$$

$$\leq \left\| \underline{w}^{1/p} \right\|_q \cdot \|\Delta \underline{x}\|_q. \tag{29}$$

If use the estimation $|\Delta M| \leq \max(w_i)^{1/p} \cdot \|\Delta \underline{x}\|_p$ then

– if $p \leq q$ then

$$|\Delta M| \leq \max(w_i)^{1/p} \cdot \|\Delta \underline{x}\|_p, \tag{30}$$

$$\leq \max(w_i)^{1/p} \cdot n^{1/p-1/q} \cdot \|\Delta \underline{x}\|_q, \tag{31}$$

$$= \|\underline{w}^{1/p}\|_\infty \cdot n^{1/p-1/q} \cdot \|\Delta \underline{x}\|_q. \tag{32}$$

– if $p > q$ then

$$|\Delta M| \leq \max(w_i)^{1/p} \cdot \|\Delta \underline{x}\|_p, \tag{33}$$

$$\leq \max(w_i)^{1/p} \cdot \|\Delta \underline{x}\|_q, \tag{34}$$

$$= \|\underline{w}^{1/p}\|_\infty \cdot \|\Delta \underline{x}\|_q. \tag{35}$$

It is easy to check that the bounds from the second estimation are weaker.

3.1 Special case: equal weights

A special case worth mentioning is when $w_i = 1/n$ for all i. The computations and the final formulas are much more simpler than in general case.

$$|\Delta M| = \left| \left[\sum_{i=1}^{n} \frac{1}{n} x_i^{*p} \right]^{\frac{1}{p}} - \left[\sum_{i=1}^{n} \frac{1}{n} x_i^{p} \right]^{\frac{1}{p}} \right|, \quad (36)$$

$$= \left(\frac{1}{n}\right)^{1/p} \cdot \left| \left[\sum_{i=1}^{n} x_i^{*p} \right]^{\frac{1}{p}} - \left[\sum_{i=1}^{n} x_i^{p} \right]^{\frac{1}{p}} \right|, \quad (37)$$

$$= \left(\frac{1}{n}\right)^{1/p} \cdot \left| \|\underline{x}^*\|_p - \|\underline{x}\|_p \right|, \quad (38)$$

$$\leq \left(\frac{1}{n}\right)^{1/p} \cdot \|\Delta \underline{x}\|_p . \quad (39)$$

If the change of the input vector is measured in other norm (q) then

– if $p \leq q$ then

$$|\Delta M| \leq \left(\frac{1}{n}\right)^{1/p} \cdot \|\Delta \underline{x}\|_p , \quad (40)$$

$$\leq \left(\frac{1}{n}\right)^{1/p} \cdot n^{1/p-1/q} \cdot \|\Delta \underline{x}\|_q , \quad (41)$$

$$= n^{-1/q} \cdot \|\Delta \underline{x}\|_q . \quad (42)$$

– if $p > q$ then

$$|\Delta M| \leq \left(\frac{1}{n}\right)^{1/p} \cdot \|\Delta \underline{x}\|_p \leq n^{-1/p} \cdot \|\Delta \underline{x}\|_q . \quad (43)$$

So the general form is

$$|\Delta M| \leq n^{-\min(1/p,1/q)} \cdot \|\Delta \underline{x}\|_q \quad (44)$$

4 Sensitivity of a Fuzzy Signature

4.1 General case

Applying the results of the previous Section we can analyse the sensitivity of fuzzy signatures in which the values are determined by a WGM operator in every nodes. The sensitivity bound of the whole fuzzy signature can be derived from the bounds of the WGM-s, according to the graph structure of the signature. The whole computation can be carried out from the leaves of the signature to the root.

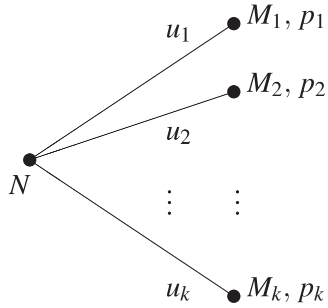

Figure 2. A part of a fuzzy signature.

Let us denote the inputs of M_i by x_{ij}, $j = 1, \ldots, n_i$, and the weights of the inputs by w_{ij}, $j = 1, \ldots, n_i$, so we have

$$M_i = \left[\sum_{j=1}^{n_i} w_{ij} \cdot x_{ij}^{p_i} \right]^{1/p_i} . \quad (45)$$

Then upper estimation of the change of M_i is given by the previous Section:

$$|\Delta M_i| \leq \left[\sum_{j=1}^{n_i} w_{ij} \cdot \Delta x_{ij}^{p_i} \right]^{1/p_i} = \|\underline{w_i}^{1/p_i} \cdot \Delta \underline{x_i}\|_{p_i} \quad (46)$$

Let us denote the minimum of $p_1, p_2, \ldots, p_k$ by p_*. Then, because of the properties of the p-norm the following holds for any $i = 1, 2, \ldots, k$:

$$\|\underline{w_i}^{1/p_i} \cdot \Delta \underline{x_i}\|_{p_i} \leq \|\underline{w_i}^{1/p_i} \cdot \Delta \underline{x_i}\|_{p_*} . \quad (47)$$

Moreover

$$\|\underline{w_i}^{1/p_i} \cdot \Delta \underline{x_i}\|_{p_*} \leq \|\underline{w_i}^{1/p_i}\|_{p_*} \cdot \|\Delta \underline{x_i}\|_{p_*} . \quad (48)$$

Using the above upper estimations we get an upper estimation for the change of the next stage (N, see Figure 2), where

$$\underline{u}^{1/q} = (u_1^{1/q}, \ldots, u_k^{1/q}), \quad (49)$$

$$\Delta \underline{M} = (\Delta M_1, \ldots, \Delta M_k), \quad (50)$$

$$\underline{u}^{1/q} \cdot \Delta \underline{M} = (u_1^{1/q} \cdot \Delta M_1, \ldots, u_k^{1/q} \cdot \Delta M_k). \quad (51)$$

The upper bound is

$$|\Delta N| \leq \|\underline{u}^{1/q} \cdot \Delta \underline{M}\|_q = \left[\sum_{i=1}^{k} u_i \cdot |\Delta M_i|^q\right]^{1/q} \quad (52)$$

$$\leq \left[\sum_{i=1}^{k} u_i \cdot \|\underline{w_i}^{1/p_i} \cdot \Delta \underline{x_i}\|_{p_i}^q\right]^{1/q} \quad (53)$$

$$\leq \left[\sum_{i=1}^{k} u_i \cdot \|\underline{w_i}^{1/p_i} \cdot \Delta \underline{x_i}\|_{p_*}^q\right]^{1/q} \quad (54)$$

$$= \left[\sum_{i=1}^{k} \left(u_i^{1/q} \cdot \|\underline{w_i}^{1/p_i} \cdot \Delta \underline{x_i}\|_{p_*}\right)^q\right]^{1/q} \quad (55)$$

$$= \left\|\underline{u}^{1/q} \cdot \|\underline{w}^{1/p_i} \cdot \Delta \underline{x}\|_{p_*}\right\|_q . \quad (56)$$

Here the last term is the q-norm of a vector whose ith element is

$$u_i^{1/q} \cdot \left[\sum_{j=1}^{n_i} \left(w_{ij}^{1/p_i} \cdot \Delta x_{ij}\right)^{p_*}\right]^{1/p_*} . \quad (57)$$

As we can see, in general case a closed, elegant formula couldn't be given, but only recursive method which can be applied from level to level.

4.2 Special cases

The sensitivity and complexity of a fuzzy signature mostly depend on the structure of the graph and on the aggregation operators applied in the nodes. According to this fact several special cases can be distinguished. In this Section we examine the case when the fuzzy signature is equipped with very similar aggregation operators and the case when the graph is a (maybe not perfect) full n-ary graph.

4.2.1 Homogeneous fuzzy signatures

The sensitivity analysis of a fuzzy signature becomes much more simple if the value of the parameter p is the same for all of the WGM operators applied in the nodes. If this condition holds, the output vale of the signature is the weighted generalized mean of the input values with parameter p, where the weights are the product of the weights form the root to the leaves.

Definition 4 *A fuzzy signature is called homogeneous if all of the aggregation operators in the nodes are weighted generalized mean operators with the same value of p.*

Lemma 2 *The WGM of $y_1, \dots, y_k$ with weights $v_1, \dots, v_k$ and with parameter p where all of the y_i-s are WGM's of x_{ji}-s with weights $w_{1i}, \dots, w_{n_i i}$ and with the same parameter of p, is the WGM of the x-s with weights $v_i \cdot w_{ji}$*

Proof:

$$\left[\sum_{i=1}^{k} v_i \cdot y_i^p\right]^{\frac{1}{p}} = \left[\sum_{i=1}^{k} v_i \cdot \left[\left[\sum_{j=1}^{n_i} w_{ji} \cdot x_{ji}^p\right]^{\frac{1}{p}}\right]^p\right]^{\frac{1}{p}} \quad (58)$$

$$= \left[\sum_{i=1}^{k} \sum_{j=1}^{n_i} v_i \cdot w_{ji} \cdot x_{ji}^p\right]^{\frac{1}{p}} = \left[\sum_{l=1}^{\sum n_i} c_l \cdot x_l^p\right]^{\frac{1}{p}} . \quad (59)$$

So the sensitivity analysis of a homogeneous fuzzy signature is nothing else but the simple sensitivity analysis of only one weighted generalized mean aggregation operator, which was discussed in details in Section 3.

4.2.2 Full n-ary fuzzy signatures with equal weights

The n-ary fuzzy signature is a fuzzy signature which has an n-ary tree graph representation. An n-ary tree is a tree graph in which each node has no more than n children. A *full* n-ary tree is an n-ary tree where within each level every node has either 0 or n children; a *perfect* n-ary tree is a full n-ary tree in which all leaf nodes are at the same depth (the distance from the leaf to the root is the same).

We consider the case when the weights are equal, so for a full n-ary fuzzy signature the weights are $1/n$ for all of the nodes.

If the signature is homogeneous, then (according to the previous subsection) it can be transformed into one simple weighted generalized mean of the inputs. For example, if we think of the tree graphs from Figure 3 as homogeneous fuzzy signatures, then the weights on l.h.s. are $1/9$ for all of the inputs, and $1/3, 1/9, \dots, 1/9$ on r.h.s.

In general, a perfect homogeneous n-ary fuzzy signature with k levels is equivalent to a weighted general mean where the weights are $1/n^k$ for all of the inputs. Then the upper bound of the change of this WGM is following:

$$|\Delta M| \leq \left(\frac{1}{n}\right)^{k/p} \cdot \|\Delta \underline{x}\|_p. \tag{60}$$

If the tree graph is not perfect (but still full) then an input value x_i has the weight $1/n^{l_i}$, where l_i denotes the level of x_i, the number of inputs is m. Then the upper bound of the change of the corresponding WGM is

$$|\Delta M| \leq \left[\sum_{i=1}^{m} \left(\frac{1}{n}\right)^{l_i} \cdot |\Delta x_i|^p\right]^{1/p} \tag{61}$$

$$\leq \left(\frac{1}{n}\right)^{l_*/p} \cdot \|\Delta \underline{x}\|_p, \tag{62}$$

where l_* denotes the minimum of the levels.

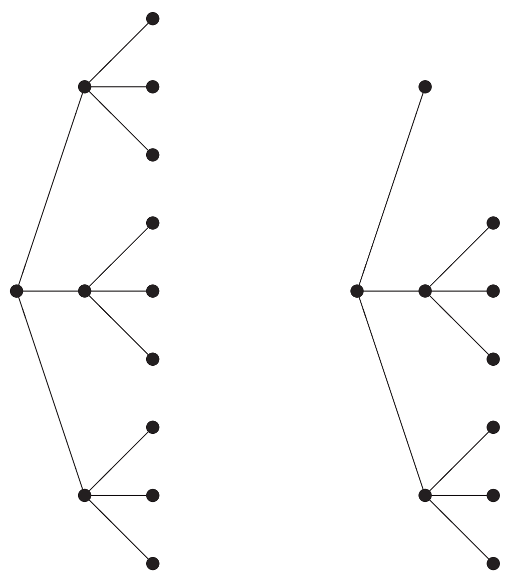

Figure 3. A perfect ternary fuzzy signature graph (left), and a not perfect ternary fuzzy signature graph (right).

If the fuzzy signature is not homogeneous, then the computation can be carried out as in 4.1, but the weights are everywhere $1/n$:

$$|\Delta N| \leq \left[\sum_{i=1}^{n} \frac{1}{n} \cdot |\Delta M_i|^q\right]^{1/q} \tag{63}$$

$$\leq \left\| \underline{1/n}^{1/q} \cdot \|\underline{1/n}^{1/p_i} \cdot \Delta \underline{x}\|_{p_*} \right\|_q, \tag{64}$$

where the last term is the q-norm of a vector whose ith element is

$$\left(\frac{1}{n}\right)^{1/q} \cdot \left[\sum_{j=1}^{n} \left(\left(\frac{1}{n}\right)^{1/p_i} \cdot \Delta x_{ij}\right)^{p_*}\right]^{1/p_*}, \tag{65}$$

and p_* denotes the minimum of $p_1, \ldots, p_n$.

5 Example From Civil Engineering

In this Section we discuss the sensitivity of a fuzzy signature which was applied for status-determining and ranking buildings of similar age and structural arrangement.

In Budapest city a lot of old residential buildings are available of similar age and structural arrangement. At the end of the 19th and at the beginning of the 20th centuries the number of inhabitants increased from 280000 to 730000. In this time period new city districts were constructed with the application of the technological methods, which were known at that time. A significant part of these residential buildings still constitutes the dominant element of the current townscape. It is one of the most pressing issues of the Hungarian capital that a considerable part of these buildings are in degraded condition. The modernization and renovation of these buildings and their ranking from the aspect of the urgency of their renovation are significant task due to the limited financial possibilities.

A decision-supporting model was created by applying the fuzzy signature [15]. This model is suitable for the ranking and qualification of residential buildings. The model was used for the first time on a database, which is based on expert opinions. After that a tree-structure, necessary for the examination of the load-bearing structures of buildings, were prepared. Primary structures (main load-bearing structures) and secondary structures (so not main load-bearing structures which play an important role in the protection of the main load bearing structures) were differentiate during the research, in this article we deal only with the branch of the primary structures. With the help of this branch it is possible to make a ranking of the load bearing structures of the examined buildings based on their arrangement, materials and conditions. The examined load bearing structures used in the model are as follows: foundation structures, wall structures, floor structures, side corridor structures, step structures and roof structures. The database was prepared on the basis of the research of more than hundred buildings, typical in Budapest, so the results achieved, well reflect the actual conditions of this type of residential buildings.

The structure of the signature is shown in Figure 4. The names and meanings of the input and internal variables are listed below.

The input variables:

x_1 : foundation structures
x_2 : wall structures
x_3 : cellar floor
x_4 : intermediate floor
x_5 : cover floor
x_6 : side corridor structures
x_7 : step structures
x_8 : facade
x_9 : footing
x_{10} : roof structures
x_{11} : roof covering
x_{12} : tin structures
x_{13} : insulation against soil moisture and ground water

The internal variables:

h_1 : floor structures
h_2 : vertical load-bearing structures
h_3 : horisontal load-bearing structures
h_4 : primary structures
h_5 : surface formation
h_6 : secondary structures
h_7 : primary and secondary structures

This is a homogeneous fuzzy signatures with parameter $p=1$ and with the following weights:

$$w_{1,1}=0.75, \quad w_{1,2}=0.25,$$
$$w_{2,1}=0.4, \quad w_{2,2}=\frac{0.6\cdot n}{n+1},$$
$$w_{2,3}=\frac{0.6}{n+1}, \quad w_{2,4}=\frac{0.4}{0.8+0.2\cdot n},$$
$$w_{2,5}=\frac{0.2\cdot n}{0.8+0.2\cdot n}, \quad w_{2,6}=\frac{0.2}{0.8+0.2\cdot n},$$
$$w_{2,7}=\frac{0.2}{0.8+0.2\cdot n}, \quad w_{3,1}=0.55-0.05\cdot n,$$
$$w_{3,2}=0.45+0.05\cdot n, \quad w_{3,3}=\frac{0.65}{0.8+0.2\cdot f},$$
$$w_{3,4}=\frac{0.2\cdot f}{0.8+0.2\cdot f}, \quad w_{3,5}=\frac{0.15}{0.8+0.2\cdot f},$$
$$w_{3,6}=1-\frac{0.5}{n}, \quad w_{3,7}=\frac{0.5}{n},$$

$$w_{4,1}=\frac{0.35\cdot m}{0.2+0.45\cdot(n-1)+0.35\cdot m},$$
$$w_{4,2}=\frac{0.45\cdot(n-1)}{0.2+0.45\cdot(n-1)+0.35\cdot m},$$
$$w_{4,3}=\frac{0.2}{0.2+0.45\cdot(n-1)+0.35\cdot m}.$$

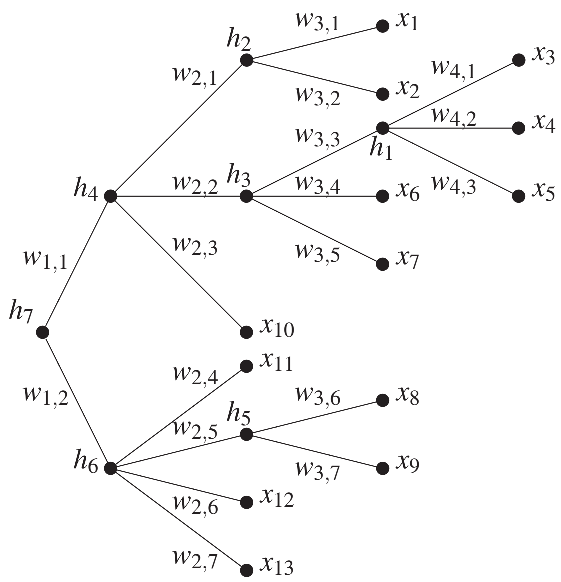

Figure 4. A fuzzy signature for status-determining and ranking buildings.

The possible values of the parameters:

- $n = 2,3,4,5$ (number of the storeys of the building),
- $0 \leq m \leq 1$ (extend of the cellar built),
- $f = 0$ or 1 (building with or without side corridor).

The input values (x_i-s) are real numbers between 0 and 1 according to the opinion of a human expert about the status of the i-th partial structure. The final output is the membership value of h_7. If a building is surveyed by different experts, then their opinion about the status of partial structures may result in different values of h_7. The following question arises: if there are small differences between the ratings given by the experts to the partial structures, then how large can be the deviation between the final scores of a building? In other words, how sensitive is this fuzzy signature to small perturbations?

This is a homogeneous fuzzy signature, so we can analyse it as a simple WGM. From the results of Section 4 it follows that

$$|\Delta h_7| \leq \sum_{i=1}^{13} v_i \cdot |\Delta x_i|, \tag{66}$$

where v_i is the weight of the ith input, computed as the product of the weights from the root to the ith leaf (for example with the notations of Figure 4 $v_1 = w_{1,1} \cdot w_{2,1} \cdot w_{3,1}$). If we measure the sensitivity in one of the well-known vector norms of the change, then

$$|\Delta h_7| \leq \max(v_i) \cdot \|\Delta \underline{x}\|_1, \tag{67}$$

$$|\Delta h_7| \leq \sqrt{13} \cdot \max(v_i) \cdot \|\Delta \underline{x}\|_2, \tag{68}$$

$$|\Delta h_7| \leq 13 \cdot \Delta\|\underline{x}\|_\infty. \tag{69}$$

Actual values of the v_i-s depend on the actual values of the parameters (m, n, f), as it is shown in Table 1.

We can conclude that this signature is not too sensitive, namely a small change in the partial opinions do not yields a large difference between the final conclusions. If the absolute values of the differences of the ratings given by the human experts are than ε for all of the variables, then the difference between the final conclusions are less ε, too.

Table 1. Examples for the weights of the input variables for different values of the parameters (rounded to four decimals).

	$n = 3, f = 1, m = 1$	$n = 4, f = 0, m = 0.5$
x_1	0.1200	0.1050
x_2	0.1800	0.1950
x_3	0.0530	0.0297
x_4	0.1362	0.2289
x_5	0.0303	0.0339
x_6	0.0675	0.0000
x_7	0.0506	0.0675
x_8	0.1125	0.0900
x_9	0.0714	0.0625
x_{10}	0.0893	0.1094
x_{11}	0.0179	0.0156
x_{12}	0.0357	0.0312
x_{13}	0.0357	0.0312

6 Conclusions

The sensitivity of the weighted generalized mean aggregation operator for parameter value $p \geq 1$ was discussed via Minkowski's inequality and in terms of various vector norms of the input vector. Applying these results the sensitivity of fuzzy signatures equipped with WGMs was also discussed. In general case a recursive estimation can be given, but in the special case when the WGMs have the same parameter, the sensitivity analyses of a fuzzy signature simplified to a sensitivity analyses of a single WGM. The complexity of the fuzzy signature is influenced by the structure and the aggregation operators also, but the aggregation operators play the main role: complexity can be highly reduced if we use the same kind of aggregation operators.

A real-life example from civil engineering was also discussed, we analysed the sensitivity of a method for status-determining and ranking buildings. It was concluded that the applied fuzzy signature is not too sensitive: if the opinions of the experts about the partial structures are relatively close to each other, then the final evaluations of the whole building will be close to each other, too.

Acknowledgement

This work was supported by the Hungarian Scientific Research Fund (OTKA) K105529 and K108405.

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