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COMPARISON AND OSCILLATION THEOREMS FOR SINGULAR STURM-LIOUVILLE OPERATORS

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Abstract. We prove analogues of the classical Sturm comparison and oscillation theorems for Sturm-Liouville operators on a finite interval with real-valued distributional potentials.

Keywords: Sturm-Liouville equation, distributional potential, Sturm comparison and oscillation theorem, Prüfer angle.

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1. INTRODUCTION

In his seminal papers [21, 22] of 1836 Charles Sturm proved several comparison and oscillation results for second order symmetric differential equations on a finite interval that proved fundamental for further development of the spectral theory for differential and abstract operators. In modern language the Sturm oscillation theorem can be stated as follows.

Assume that p, q, and r are real-valued functions on a finite interval $I = [a, b]$ such that $p > 0$ and $r > 0$ a.e. and $1/p$, q, and r are integrable over I. Consider the Sturm-Liouville eigenvalue problem

$$-(py')' + qy = \lambda ry \tag{1.1}$$

subject to e.g. the Dirichlet boundary conditions

$$y(a) = y(b) = 0. \tag{1.2}$$

It then follows from the results of Sturm that the eigenvalues of (1.1)–(1.2) are real, bounded below and form a discrete subset of $\mathbb{R}$ with the only accumulation point at $+\infty$. List these eigenvalues as $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$; then the eigenfunction y_n corresponding to λ_n has precisely n interior zeros which interlace those of y_{n+1}.

Since then the Sturm theory has been extended in many directions, such as for partial differential and/or higher order equations [3], difference equations [28], less regular potentials etc.; see the historical review by Hinton [11] and the account by Simon [26] on important recent progress, as well as the exhaustive reference lists in these two papers. In particular, some results were established in [4, 16, 18, 25] for differential equations on one-dimensional graphs.

In the recent paper [27], the authors developed the Sturm theory for the Sturm-Liouville eigenvalue problem

$$-y'' + qy = \lambda y \tag{1.3}$$

in the case where the potential q is a real-valued distribution from the Sobolev space $W_2^{-1}(I)$. Two different approaches were realized therein: the first one extending the original method of Sturm, and the other one based on the variational principles.

A more general case of the spectral problem (1.1) with uniformly positive $p \in L_\infty(I)$, real-valued distributions q and r belonging to $W_2^{-1}(0,1)$, and arbitrary separated boundary conditions was discussed in [29]. The author rewrote equation (1.1) as the spectral problem for a linear operator pencil, studied the latter via the quadratic forms, and in that way established analogues of the Sturm theorems and the Chebyshov properties of linear combinations of the eigenfunctions. Oscillation properties of solutions to Sturm-Liouville equations with coefficients that are Borel measures were treated in [19].

The main aim of this note is to give an alternative derivation of the Sturm comparison and oscillation theorems for equation (1.3) with real-valued $q \in W_2^{-1}(I)$. The motivation for doing this has stemmed from our study of singular differential operators on quantum trees [12]. To develop the Sturm theory for such operators, one builds upon such a theory for singular operators on a single edge. However, we found that the approach of the paper [27] does not allow direct generalization to graphs. Like in [27], we also employ the Prüfer angle technique here, but define the Prüfer angle in a different manner. Apart from deriving the analogues of the Sturm theorems, we study in detail properties of the Prüfer angle that prove essential for developing the Sturm theory for quantum trees in the forthcoming paper [12].

The paper is organized as follows. In the next section, we define rigorously the differential equation to be studied and discuss some properties of its solutions. In Section 3 the Prüfer angle is introduced and its properties are established by analyzing the corresponding Carathéodory equation. Finally, these results are used in Section 4 to develop generalizations of the Sturm theory to the case of distributional Sturm-Liouville equations (1.3).

2. DEFINITIONS

Assume that $q \in W_2^{-1}(0,1)$ is a real-valued distribution and consider the Sturm-Liouville differential expression

$$\tau y := -y'' + qy$$

on the interval $(0, 1)$. As multiplication in the space of distributions is not well defined (see, however, [17] and an interesting recent development in [13]), some care should be taken while defining the expression τ. In fact, τ and the corresponding differential operator can be introduced in several equivalent ways, e.g., via the quadratic forms or by approximating q by regular potentials. One of the most efficient definitions uses the regularization by quasi-derivative technique that was first suggested by Atkinson *et al.* [2] for the particular case $q(x) = 1/x$ on the interval $(-1, 1)$ and then developed by Savchuk and Shkalikov [23, 24] for general $q \in W_2^{-1}(0, 1)$. We also mention that important generalizations were recently suggested by Goriunov and Mikhailets [8, 9]; a detailed treatment of the most general differential Sturm-Liouville operators was performed by Eckhardt, Gesztesy, Nichols, and Teschl in their recent fundamental work [6].

In this regularization approach, one takes a real-valued function $u \in L_2(0, 1)$ such that $q = u'$ in the sense of distributions and for every absolutely continuous y denotes by $y^{[1]} := y' - uy$ its *quasi-derivative*; then τ acts via

$$\tau y = -\big(y^{[1]}\big)' - uy^{[1]} - u^2 y \tag{2.1}$$

on its domain

$$\operatorname{dom} \tau := \{y \in L_2(0, 1) \mid y, y^{[1]} \in AC(0, 1),\ \tau y \in L_2(0, 1)\}. \tag{2.2}$$

It is straightforward to see that $\tau y = -y'' + qy$ in the sense of distributions, so that (2.1)–(2.2) gives a natural generalization of the Sturm-Liouville differential expression.

As follows from the definition τ, the equality $\tau y = \lambda y + f$ can be interpreted as the first-order system

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} u & 1 \\ -u^2 - \lambda & -u \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -f \end{pmatrix}$$

for $y_1 = y$ and the quasi-derivative $y_2 = y^{[1]} = y' - uy$. This is a linear system with an integrable matrix coefficient; therefore if $f \in L_1(0, 1)$, then for every point $x_0 \in [0, 1]$ and for every $c_1, c_2 \in \mathbb{C}$ the above system possesses a unique solution $(y_1, y_2)^{\mathrm{t}}$ satisfying the conditions $y_1(x_0) = c_1$ and $y_2(x_0) = c_2$, see [20, Ch. 2]. Equivalently, under the same assumptions the equation $\tau y = \lambda y + f$ possesses a unique solution y satisfying the conditions $y(x_0) = c_1$ and $y^{[1]}(x_0) = c_2$. Observe also that this solution is absolutely continuous along with its quasi-derivative $y^{[1]}$; the usual derivative $y' = y^{[1]} + uy$, on the contrary, need not be continuous.

The following lemma is well known (cf. [14] or [27]), and we give its short proof just for the sake of completeness. We say that a function y strictly increases (resp. decreases) through a point x_0 if there exists a neighbourhood $\mathcal{O}(x_0)$ of x_0 such that $(x - x_0)\big(y(x) - y(x_0)\big) > 0$ (resp., $(x - x_0)\big(y(x) - y(x_0)\big) < 0$) for all $x \in \mathcal{O}(x_0) \setminus \{x_0\}$.

Lemma 2.1. *Assume that y is a solution to the equation $\tau y = \lambda y$ with $\lambda \in \mathbb{R}$ such that $y(x_0) = 0$ and $y^{[1]}(x_0) = c$ for some $x_0 \in [0, 1]$ and some real c. Then the following holds:*

(i) *if* $c = 0$*, then* $y \equiv 0$ *over* $[0, 1]$*,*
(ii) *if* $c > 0$*, then* y *strictly increases through* x_0*,*
(iii) *if* $c < 0$*, then* y *strictly decreases through* x_0*.*

In particular, every zero of a nontrivial solution y *of the equation* $\tau y = \lambda y$ *is an isolated point in* $[0, 1]$*.*

Proof. Part (i) follows from the uniqueness arguments preceding the lemma. To show (ii), we note that y is real-valued and set $z(x) := y(x) \exp\{\int_x^{x_0} u(t)\,dt\}$. Then z is real-valued and absolutely continuous along with y and, moreover, $z'(x) = y^{[1]}(x) \exp\{\int_x^{x_0} u(t)\,dt\}$. Since $y^{[1]}$ remains positive in some ε-neighbourhood of the point x_0, z strictly increases in this neighbourhood thus yielding the result. Part (iii) is established analogously. □

3. THE PRÜFER ANGLE AND ITS PROPERTIES

Fix a real λ and consider a real-valued solution $y(\cdot) = y(\cdot\,; \lambda)$ of the equation $\tau y = \lambda y$. Similarly to the classical theory, we introduce the polar coordinates r and θ via $y(x) = r(x) \sin\theta(x)$ and $y^{[1]}(x) = r(x) \cos\theta(x)$ and call θ the *Prüfer angle* of y. The function θ is defined only modulo 2π; we can, however, single out a continuous branch of θ determined e.g. by the condition $\theta(0) \in [0, 2\pi)$. Differentiating the relation $\cot\theta = y^{[1]}/y$, we get the differential equation[1)]

$$\theta' = (u \sin\theta + \cos\theta)^2 + \lambda \sin^2\theta. \tag{3.1}$$

As u only belongs to $L_2(0, 1)$ and does not possess any additional smoothness, the right-hand side of this equation is not in general continuous. In fact, (3.1) belongs to the class of Carathéodory equations defined as follows.

We call a differential equation

$$y'(x) = f(x, y(x)) \tag{3.2}$$

the *Carathéodory equation* in a domain D of the (x, y)-plane if f satisfies the following conditions in D:

(i) for almost all x, $f(x, y)$ is well defined and continuous in y,
(ii) for every y, the function $f(x, y)$ is measurable in x,
(iii) there exists an integrable function $m(x)$ such that, for all $(x, y) \in D$, $|f(x, y)| \le m(x)$.

The Carathéodory existence theorem [7, Theorem 1.1] asserts that the Carathéodory equation (3.2) possesses a (local) solution subject to the condition $y(x_0) = y_0$, for every point (x_0, y_0) of the interior of D. The solution is understood in the integral sense, i.e., as a continuous function satisfying the equality

$$y(x) = y_0 + \int_{x_0}^{x} f\big(t, y(t)\big)\,dt \tag{3.3}$$

[1)] Note that if θ satisfies (3.1), then so does $\theta + \pi$. Since $\theta + \pi$ is the Prüfer angle of the solution $-y$, this suggests that θ and $\theta + \pi$ should be identified and thus θ becomes defined only modulo π.

in a neighbourhood of x_0. If, in addition, f satisfies

(iv) there exists an integrable function $l(x)$ such that, for all (x, y_1) and (x, y_2) in D, $|f(x, y_1) - f(x, y_2)| \le l(x)|y_1 - y_2|$,

then the above solution is unique in D [7, Theorem 1.2]. We refer to [1] and the monographs [5, Ch. 2], [7, Ch. 1], [10, Ch. 2], and [20, Ch. 2] for further details of the theory.

Clearly, the right-hand side

$$f(x, y) := \big(u(x) \sin y + \cos y\big)^2 + \lambda \sin^2 y$$

of equation (3.1) satisfies (i)–(iv) in the domain $D := [0, 1] \times \mathbb{R}$ with $m(x) = (|u(x)| + 1)^2 + |\lambda|$ and $l(x) = 2m(x)$. We note that every solution of (3.1) is global (i.e., defined over the whole interval $[0, 1]$) and absolutely continuous, see [7, Ch. 1].

Further we observe that if $\theta(x_*) = 0 \mod \pi$ (i.e. if $\sin\theta(x_*) = 0$), equation (3.1) yields the equality $\theta'(x_*) = 1$, and one expects that θ strictly increases through x_*, just as in the classical case of integrable q. However, the fact that θ' is discontinuous does not allow to deduce this property from the mere fact that $\theta'(x_*) = 1$; instead, Lemma 2.1 becomes helpful.

Corollary 3.1. *The function θ strictly increases through every point x_*, where $\theta(x_*) = 0 \mod \pi$ (i.e. through every zero of the corresponding solution y of $\tau y = \lambda y$).*

Proof. Parts (ii) and (iii) of Lemma 2.1 imply that $\cot\theta = y^{[1]}/y$ assumes negative values in some left neighbourhood of x_* and positive values in some right neighbourhood of x_*, thus yielding the claim. □

Since the right-hand side of (3.1) increases with λ, one expects that the solution $\theta(x; \lambda)$ also increases in λ. However, the standard proofs of this fact rely on continuity of the right-hand side f and thus are not applicable to the Carathéodory equations. Below, we justify a weaker monotonicity property for generic Carathéodory equations and then refine it for the particular case of equation (3.1).

Lemma 3.2. *Assume that D is a rectangular domain $[0, 1] \times K$ of the (x, y)-plane, with $K = [a, b]$, $-\infty < a < b < \infty$. Assume further that functions f_1 and f_2 defined on D satisfy the conditions* (i)–(iv) *and that $f_1(x, y) \le f_2(x, y)$ a.e. in D. Let also y_1 and y_2 be the global solutions of the corresponding Carathéodory equations $y_j' = f_j\big(x, y_j(x)\big)$ satisfying the initial conditions $a < y_1(0) \le y_2(0) < b$. Then $y_1(x) \le y_2(x)$ for all $x \in [0, 1]$.*

Proof. This lemma is well known for continuous f_j, see [5, Corollary III.4.2]. Its extension to Carathéodory functions f_j can be obtained by approximating them by continuous functions and establishing continuous dependence of the solutions y_j on the functions f_j. The details are given below; for convenience we divide the proof into several steps.

Step 1. Set
$$x^* := \sup\{x' \in [0,1] \mid y_1(x) \le y_2(x) \quad \text{on} \quad [0,x']\};$$
we shall prove that $x^* = 1$. Since $x^* \ge 0$ and $y_1(x^*) \le y_2(x^*)$, it is sufficient to prove the following local version of the lemma: for every $x_0 \in [0,1)$ with the property that $a < y_1(x_0) \le y_2(x_0) < b$ there exists a $d > 0$ such that $y_1(x) \le y_2(x)$ for all $x \in [x_0, x_0+d]$.

Step 2. First we show that under the above assumptions (i)–(iv) a unique solution to equation (3.2) subject to the initial condition $y(x_0) = y_0$, with $(x_0,y_0) \in [0,1)\times(a,b)$, can be constructed locally using the Banach fixed point theorem.

To this end, we set $c := \min\{y_0 - a, b - y_0\}$, take $d > 0$ such that
$$\int_{x_0}^{x_0+d} m(x)\,dx < \frac{c}{2}, \qquad \int_{x_0}^{x_0+d} l(x)\,dx < \frac{1}{2}, \tag{3.4}$$
and introduce the space $C := C[x_0, x_0+d]$ of functions continuous over $[x_0, x_0+d]$ with the norm
$$\|y\|_C := \max_{x\in[x_0,x_0+d]} |y(x)|.$$
Next, consider in the space C the nonlinear operator T defined via
$$Ty(x) := y_0 + \int_{x_0}^{x} f(t, y(t))\,dt \tag{3.5}$$
for $y \in C$ such that $(t, y(t)) \in D$ for all $t \in [x_0, x_0+d]$. Then the solution of the equation $y' = f(x,y)$ on $[x_0, x_0+d]$ subject to the initial condition $y(x_0) = y_0$ is a fixed point of the operator T.

Since
$$|Ty_1(x) - Ty_2(x)| \le \int_{x_0}^{x} l(t)|y_1(t) - y_2(t)|\,dt \le \tfrac{1}{2}\|y_1 - y_2\|_C,$$
the operator T is a contraction; moreover, the ball[2)]
$$B(y_0) := \{y \in C \mid \|y - y_0\|_C \le c\}$$
belongs to the domain of T and is mapped into itself, as follows from the estimates
$$\|Ty_0 - y_0\|_C \le \int_{x_0}^{x_0+d} m(t)\,dt < \frac{c}{2}$$

[2)] Slightly abusing the notation, we shall use y_0 both for the real number in the initial condition and for the constant function equal to y_0.

and

$$\|Ty - y_0\|_C \le \|Ty - Ty_0\|_C + \|Ty_0 - y_0\|_C < \tfrac{1}{2}\|y - y_0\|_C + \tfrac{c}{2} \le c$$

for all $y \in B(y_0)$.

Therefore the Banach fixed point theorem gives the unique solution of the equation $y = Ty$ in $C = C[x_0, x_0 + d]$ as the limit of $T^n y_0$ as $n \to \infty$. This fixed point satisfies (3.3) and thus is a solution to the Carathéodory differential equation (3.2) satisfying the required initial condition.

Step 3. We next show that the above fixed point of the operator T depends continuously on f in some special sense. Assume that functions f_1 and f_2 defined on the domain D satisfy there conditions (i)–(iv) with integrable functions m_j and l_j, $j = 1, 2$.

Given any $x_0 \in [0, 1)$ and $y_0 \in (a, b)$, we define c as on Step 2 and take $\delta \in (0, 1-x_0]$ so that (3.4) holds with m and l replaced by m_j and l_j, $j = 1, 2$. Denote by T_1 and T_2 the operators defined as T on Step 2 but with f_1 and f_2 instead of f, and denote by y_1 and y_2 the fixed points of these operators on $[x_0, x_0 + d]$. Then $y_1 - y_2$ can be estimated in the space $C := C[x_0, x_0 + \delta]$ via

$$\|y_1 - y_2\|_C = \|T_1 y_1 - T_2 y_2\|_C \le \|T_1 y_1 - T_1 y_2\|_C + \|T_1 y_2 - T_2 y_2\|_C \le$$
$$\le \tfrac{1}{2}\|y_1 - y_2\|_C + \int_{x_0}^{x_0+d} |f_1(t, y_2(t)) - f_2(t, y_2(t))|\, dt,$$

so that

$$\|y_1 - y_2\|_C \le 2 \int_{x_0}^{x_0+d} \sup_{y\in K} |f_1(t, y) - f_2(t, y)|\, dt. \tag{3.6}$$

Step 4. Next we show that, given a Carathéodory function f on D possessing the properties (i)–(iv), there is a net f_ε of continuous functions on D satisfying (i)–(iv) and such that

$$\int_0^1 \sup_{y\in K} |f_\varepsilon(t, y) - f(t, y)|\, dt \to 0, \qquad \varepsilon \to 0. \tag{3.7}$$

Take an arbitrary continuous function ϕ of compact support such that $0 \le \phi(x) \le 1$ for all $x \in \mathbb{R}$ and $\int \phi = 1$, and put $\phi_\varepsilon(x) := \varepsilon^{-1}\phi(x/\varepsilon)$. We then mollify f by ϕ_ε to get f_ε, viz.

$$f_\varepsilon(x, y) := \int_{\mathbb{R}} \phi_\varepsilon(x - \xi) f(\xi, y)\, d\xi.$$

Denote by m_ε and l_ε the analogous mollifications of the functions m and l; then m_ε and l_ε are continuous over $[0, 1]$ (and thus integrable) and converge to m and l respectively in the topology of the space $L_1(0, 1)$ [15, Theorem VI.1.10]. Next we find that

$$|f_\varepsilon(x, y)| \le \int_{\mathbb{R}} \phi_\varepsilon(x - \xi) m(\xi)\, d\xi = m_\varepsilon(x)$$

and

$$|f_\varepsilon(x,y_1) - f_\varepsilon(x,y_2)| \le |y_1 - y_2| \int_{\mathbb{R}} \phi_\varepsilon(x-\xi) l(\xi)\, d\xi = |y_1 - y_2| l_\varepsilon(x),$$

so that f_ε satisfy the Carathéodory properties (iii) and (iv). Moreover, the functions f_ε are continuous on D by virtue of the relations

$$\begin{aligned}|f_\varepsilon(x_1,y_1) - f_\varepsilon(x_2,y_2)| \le |f_\varepsilon(x_1,y_1) - f_\varepsilon(x_2,y_1)| + |f_\varepsilon(x_2,y_1) - f_\varepsilon(x_2,y_2)| \le \\ \le \int_{\mathbb{R}} |\phi_\varepsilon(x_1-\xi) - \phi_\varepsilon(x_2-\xi)| m(\xi)\, d\xi + \\ + |y_1 - y_2| \int_{\mathbb{R}} \phi_\varepsilon(x_2-\xi) l(\xi)\, d\xi.\end{aligned}$$

Indeed, the first summand on the right-hand side of the above inequalities tends to zero as $|x_1 - x_2| \to 0$ uniformly in $y_1, y_2 \in K$ due to the uniform continuity of ϕ_ε, while the second term is bounded by $\varepsilon^{-1}\|l\|_{L_1}|y_1 - y_2|$, with $\|l\|_{L_1}$ denoting the norm of l in $L_1(0,1)$, and tends to zero as $|y_1 - y_2| \to 0$ uniformly in $x_2 \in [0,1]$. Therefore f_ε enjoys properties (i) and (ii) as well.

Now we set $g_\varepsilon := f_\varepsilon - f$ and note that for each fixed $y \in K$ we get

$$\int_0^1 |g_\varepsilon(x,y)|\, dx \to 0$$

as $\varepsilon \to 0$ [15, Theorem VI.1.10]. Since K is a compactum, for every $\delta > 0$ it possesses a finite δ-net K_δ. Now for every $y \in K$ we can find $y^* \in K_\delta$ such that $|y - y^*| \le \delta$, so that

$$|g_\varepsilon(x,y)| \le |g_\varepsilon(x,y^*)| + |g_\varepsilon(x,y) - g_\varepsilon(x,y^*)| \le \sum_{y' \in K_\delta} |g_\varepsilon(x,y')| + \delta(l(x) + l_\varepsilon(x))$$

and

$$\limsup_{\varepsilon\to 0} \int_0^1 \sup_{y\in K} |g_\varepsilon(x,y)|\, dx \le \lim_{\varepsilon\to 0} \Big[\sum_{y'\in K_\delta} \int_0^1 |g_\varepsilon(x,y')|\, dx + \delta\|l\|_{L_1} + \delta\|l_\varepsilon\|_{L_1} \Big] = 2\delta\|l\|_{L_1}.$$

As $\delta > 0$ was arbitrary, (3.7) follows.

Step 5. Now, given two functions f_1 and f_2 as in the assumption of the lemma, we construct their mollifications $f_{1,\varepsilon}$ and $f_{2,\varepsilon}$ as on Step 4 and denote by $y_{j,\varepsilon}$ the solutions of the equations

$$y' = f_{j,\varepsilon}\big(x, y(x)\big)$$

subject to the initial conditions $y_{j,\varepsilon}(x_0) = y_j(x_0)$. Then

$$\|y_{j,\varepsilon} - y_j\|_C \le 2 \int_{x_0}^{x_0+d} \sup_{y\in K} |f_{j,\varepsilon}(x,y) - f_j(x,y)|\, dx \to 0$$

as $\varepsilon \to 0$ by (3.6) and (3.7). Moreover,

$$f_{2,\varepsilon}(x,y) - f_{1,\varepsilon}(x,y) = \int_{\mathbb{R}} \phi_\varepsilon(x-\xi)[f_2(\xi,y) - f_1(\xi,y)]\,d\xi \geq 0$$

a.e. in D. By [5, Corollary III.4.2], $y_{1,\varepsilon}(x) \leq y_{2,\varepsilon}(x)$ for all $x \in [x_0, x_0+d]$, and thus

$$y_1(x) = \lim_{\varepsilon\to 0} y_{1,\varepsilon}(x) \leq \lim_{\varepsilon\to 0} y_{2,\varepsilon}(x) = y_2(x), \qquad x \in [x_0, x_0+d].$$

The lemma is proved. $\square$

Remark 3.3. There is a "backward" version of this lemma claiming that $y_1(x) \geq y_2(x)$ for $x \in [0,1)$ as soon as $y_1(1) \geq y_2(1)$. It can be derived from the "forward" version by reversing the direction of x (i.e., by replacing x with $1-x$).

We are now in position to prove monotonicity of the Prüfer angle θ with respect to the variable λ.

Lemma 3.4. *Assume that $\lambda_1 < \lambda_2$ and that $\theta(\cdot;\lambda_1)$ and $\theta(\cdot;\lambda_2)$ are solutions of equation* (3.1) *satisfying the condition $\theta(0;\lambda_1) \leq \theta(0;\lambda_2)$. Then for every $x \in (0,1]$ the inequality $\theta(x;\lambda_1) < \theta(x;\lambda_2)$ holds. Likewise, if $\theta(1;\lambda_1) \geq \theta(1;\lambda_2)$, then $\theta(x;\lambda_1) > \theta(x;\lambda_2)$ for all $x \in [0,1)$.*

Proof. We shall only establish the first part of the lemma, the second one being completely analogous. The functions

$$f_j(x,y) := \big(u(x)\sin y + \cos y\big)^2 + \lambda_j \sin^2 y, \qquad j = 1,2,$$

satisfy the assumptions of Lemma 3.2 for a compact set $K = [0,\pi]$. We observe that the fact that K is compact was only used in the proof of that lemma to derive (3.7). Since f_j (and thus their mollifications $f_{j,\varepsilon}$) are periodic in the variable y with period π, the conclusion of Lemma 3.2 holds for the above f_j with a noncompact set $K = \mathbb{R}$. As a result, no restrictions on the initial values of θ are needed and we get the inequality $\theta(x;\lambda_1) \leq \theta(x;\lambda_2)$ for all $x \in [0,1]$. It remains to prove that this inequality is strict for all nonzero x.

First of all we prove that the set S of all $x \in [0,1]$ such that $\theta(x;\lambda_1) = \theta(x;\lambda_2)$ is nowhere dense in $[0,1]$. Indeed, S is closed; should it contain an interval $[a,b]$, then the following equality would hold:

$$\int_a^b f_1(t,\theta(t;\lambda_1))\,dt = \int_a^b f_2(t,\theta(t;\lambda_1))\,dt.$$

This would yield the relation $\sin\theta(t;\lambda_1) \equiv 0$ for all $t \in [x_1, x_0]$ and thus $\theta(t;\lambda_1) \equiv \pi k$, $k \in \mathbb{Z}$, for such x, but this is impossible in view of Corollary 3.1.

Assume that the set S contains an $x_0 > 0$. We set $\theta(x_0;\lambda_1) = \theta(x_0;\lambda_2) =: \theta_0$ and denote by $x_1 < x_0$ a point where $\theta(x_1;\lambda_2) > \theta(x_1;\lambda_1)$. Further, set

$$\theta_1 := \tfrac{1}{2}\big(\theta(x_1;\lambda_2) + \theta(x_1;\lambda_1)\big)$$

and denote by $\theta_1(\cdot;\lambda_2)$ the solution of the equation (3.1) with $\lambda = \lambda_2$ subject to the initial condition $\theta(x_1) = \theta_1$. By Lemma 3.2, $\theta_1(x;\lambda_2) \ge \theta(x,\lambda_1)$ for all $x \in [x_1, x_0]$ and, in particular, $\theta_1(x_0;\lambda_2) \ge \theta_0$. On the other hand, since the trajectories of different solutions to the equation (3.1) for $\lambda = \lambda_2$ cannot intersect, $\theta(x_0;\lambda_2) > \theta_1(x_0;\lambda_2) \ge \theta_0$, a contradiction. Therefore the set S does not contain points of $(0,1]$, and the proof is complete. $\square$

More can be said on the Prüfer angle θ if its value at $x = 0$ is fixed.

Theorem 3.5. *Assume that the Prüfer angle $\theta(\cdot;\lambda)$ for the solution $y(\cdot;\lambda)$ of the equation $\tau y = \lambda y$ satisfies the condition $\theta(0,\lambda) \equiv \alpha \in [0,\pi)$ for all $\lambda \in \mathbb{R}$. Then, for every fixed $x \in (0,1]$, $\theta(x;\lambda) \to 0$ as $\lambda \to -\infty$ and $\theta(x;\lambda) \to +\infty$ as $\lambda \to +\infty$.*

Proof. We divide the proof in several steps.
Step 1. First we prove that there exist $K > 0$ and $\delta > 0$ such that $\theta(x;\lambda) < \pi - \delta$ for all $x \in [0,1]$ and all $\lambda \le -K$.

Set $\delta := \frac{1}{2}\min\{\pi - \alpha, \frac{\pi}{2}\}$. As the function $F(x) := \int\limits_0^x \bigl(|u(t)| + 1\bigr)^2\, dt$ is uniformly continuous over $[0,1]$, there exists $\delta_1 > 0$ such that

$$F(x_2) - F(x_1) = \int\limits_{x_1}^{x_2} \bigl(|u(x)| + 1\bigr)^2\, dx < \delta$$

whenever $0 < x_2 - x_1 < \delta_1$. Set now

$$K := \frac{\bigl(\|u\|_{L_2} + 1\bigr)^2}{\delta_1 \sin^2\delta},$$

where $\|u\|_{L_2}$ denotes the norm of u in $L_2(0,1)$; we claim that $\theta(x;-K) < \pi - \delta$ for all $x \in [0,1]$.

Indeed, assume that $x_1 < x_2$ are such that $\theta(x;-K) \in [\delta, \pi - \delta]$ for all $x \in [x_1, x_2]$ and, moreover, that $\theta(x_1;-K) \le \pi - 2\delta$. Upon integrating (3.1) from x_1 to x_2, we find that

$$\theta(x_2;-K) \le \theta(x_1;-K) + \int\limits_{x_1}^{x_2} \bigl(|u(x)| + 1\bigr)^2\, dx - K(x_2 - x_1)\sin^2\delta.$$

If $x_2 - x_1 < \delta_1$, then the integral above is less than δ, and we find that

$$\theta(x_2;-K) < \theta(x_1;-K) + \delta \le \pi - \delta;$$

otherwise

$$\int\limits_{x_1}^{x_2} \bigl(|u(x)| + 1\bigr)^2\, dx - K(x_2 - x_1)\sin^2\delta \le 0$$

so that $\theta(x_2;-K) \le \theta(x_1;-K) < \pi - \delta$. Since $\theta(0;-K) \le \pi - 2\delta$, $\theta(\cdot;-K)$ never reaches the value $\pi - \delta$, thus establishing the claim.

Step 2. For every fixed $x \in (0,1]$ the function $\theta(x;\lambda)$ assumes positive values and decreases in λ. Therefore the limit $\theta_*(x) := \lim_{\lambda\to-\infty}\theta(x;\lambda)$ exists, is non-negative and, moreover, $\theta_*(x) < \pi$ by Step 1. We claim that the function θ_* is non-increasing on $(0,1]$.

Assume it is not; then there are x_1 and x_2, $x_1 < x_2$, such that $\theta_*(x_1) < \theta_*(x_2)$. Take $\delta > 0$ such that $\theta_*(x_2) - \theta_*(x_1) \ge 3\delta$ and introduce δ_1 and K as on Step 1. Without loss of generality we can assume that δ is taken small enough and K large enough so that $\theta(x_1;\lambda) < \theta_*(x_1)+\delta$ and $\theta(x,\lambda) < \pi-\delta$, $x \in [0,1]$, whenever $\lambda < -K$, see Step 1.

Now for every $\lambda < -K$ there exists $x_* \in [x_1, x_2]$ such that $\theta(x_2;\lambda) - \theta(x_*;\lambda) \ge \delta$ and $\theta(x;\lambda) \in [\delta, \pi-\delta]$ for all $x \in [x_*, x_2]$. As on Step 1, we get the inequality

$$\delta \le \theta(x_2,\lambda) - \theta(x_*;\lambda) \le \int_{x_*}^{x_2} \bigl(|u(x)|+1\bigr)^2\,dx - |\lambda|(x_2-x_*)\sin^2\delta$$

for all $\lambda < -K$, which, however, can hold neither if $x_2 - x_* < \delta_1$ due to the choice of δ_1 nor if $x_2 - x_* \ge \delta_1$ due to the choice of K. The contradiction derived shows that no such points x_1 and x_2 as above exist and so θ_* is non-increasing.

Finally, assume that $\theta_*(x_0) > 0$ for some $x_0 \in (0,1]$. Then $\theta_*(x) \ge \theta_*(x_0)$ for all $x \in [0,x_0]$. Choose $\delta \in (0, \theta_*(x_0))$ and $K > 0$ so that $\theta(x;\lambda) < \pi - \delta$ for all $x \in [0,x_0]$ and all $\lambda < -K$. Since also $\theta(x;\lambda) \ge \theta_*(x_0) \ge \delta$ for all $x \in [0,x_0]$ and all $\lambda < -K$, we find that for all such λ

$$\delta \le \theta(x_0;\lambda) \le \alpha + \bigl(\|u\|_{L_2}+1\bigr)^2 - |\lambda| x_0 \sin^2\delta,$$

which is impossible. Therefore $\theta_*(x) = 0$ for all $x \in (0,1]$ as claimed.
Step 3. For every fixed $x \in (0,1]$, the function $\theta(x;\lambda)$ increases in λ, whence the limit

$$\theta^*(x) := \lim_{\lambda\to\infty}\theta(x;\lambda)$$

exists in a generalized sense, i.e., as a finite number or $+\infty$. Observe that for $\lambda > 0$ the function $\theta(x;\lambda)$ is increasing in $x \in [0,1]$ and thus θ^* is non-decreasing.

We first show that θ^* must strictly increase on every interval where it is finite. Assume therefore that $\theta^*(x_2) < \infty$ for some $x_2 \in (0,1]$; we take an arbitrary $x_1 \in [0,x_2)$ and show that

$$\theta^*(x_2) - \theta^*(x_1) \ge x_2 - x_1. \tag{3.8}$$

Take $\delta > 0$ such that $\theta^*(x_2) - \theta^*(x_1) \le \delta$; then there exists $K > 0$ such that $\theta(x_2;\lambda) - \theta(x_1;\lambda) < 2\delta$ for all $\lambda > K$. In view of (3.1), this yields the inequality

$$\int_{x_1}^{x_2} (u\sin\theta + \cos\theta)^2\,dx + \lambda\int_{x_1}^{x_2} \sin^2\theta\,dx < 2\delta.$$

In particular, for such λ

$$\int_{x_1}^{x_2} \sin^2\theta\,dx < \frac{2\delta}{\lambda},$$

so that

$$\int_{x_1}^{x_2} \cos^2\theta\,dx > x_2 - x_1 - \frac{2\delta}{\lambda}$$

and, due to the Cauchy-Bunyakovsky-Schwarz inequality,

$$\int_{x_1}^{x_2} \bigl|u\sin\theta\cos\theta\bigr|\,dx \le \|u\|_{L_2}\Bigl(\frac{2\delta}{\lambda}\Bigr)^{1/2}.$$

It now follows that

$$\begin{aligned}\theta(x_2;\lambda) - \theta(x_1;\lambda) &\ge \int_{x_1}^{x_2} \cos^2\theta\,dx - 2\int_{x_1}^{x_2} |u\sin\theta\cos\theta|\,dx > \\ &> x_2 - x_1 - \frac{2\delta}{\lambda} - 2\|u\|_{L_2}\Bigl(\frac{2\delta}{\lambda}\Bigr)^{1/2},\end{aligned}$$

for all $\lambda > K$, thus yielding (3.8).

Next we prove that if $\theta^*(x_0) \in (\pi n, \pi(n+1))$ for some $x_0 \in (0,1)$ and some $n \in \mathbb{N}$, then $\theta^*(x_0+0) \ge \pi(n+1)$ and $\theta^*(x_0-0) \le \pi n$. Indeed, for every sufficiently small $\delta > 0$ there exists $K > 0$ such that $\pi n + \delta \le \theta(x_0;\lambda) < \pi(n+1) - \delta$ for all $\lambda > K$. Denote by $\bigl(x_-(\lambda), x_+(\lambda)\bigr)$ the largest open interval in $[0,1]$ containing x_0 such that

$$\theta(x;\lambda) \in (\pi n + \delta; \pi(n+1) - \delta)$$

for all $x \in \bigl(x_-(\lambda), x_+(\lambda)\bigr)$. Then it follows from (3.1) that, for $\lambda > K$,

$$\pi - 2\delta \ge \theta(x_+(\lambda);\lambda) - \theta(x_-(\lambda);\lambda) \ge \lambda(x_+(\lambda) - x_-(\lambda))\sin^2\delta$$

and, as $\lambda \to +\infty$,

$$x_+(\lambda) - x_-(\lambda) \le \frac{\pi - 2\delta}{\lambda\sin^2\delta} \to 0.$$

Thus $x_+(\lambda) \to x_0$ as $\lambda \to +\infty$ and $\theta(x_+(\lambda);\lambda) = \pi(n+1) - \delta$ for all λ large enough. Now for every $\varepsilon > 0$ we find that

$$\theta^*(x_0+\varepsilon) = \lim_{\lambda\to+\infty} \theta(x_0+\varepsilon,\lambda) \ge \lim_{\lambda\to+\infty} \theta(x_+(\lambda);\lambda) = \pi(n+1) - \delta;$$

as a result, $\theta^*(x_0+0) \ge \pi(n+1) - \delta$. Similar arguments show that $\theta^*(x_0-0) \le \pi n + \delta$. As $\delta > 0$ was arbitrary, the claim follows.

Assume now that $\theta^*(x_0) < \infty$ for some $x_0 \in (0,1]$. Combining the above two properties of the function θ^*, we see that $\theta^*(x_2) - \theta^*(x_1) \ge \pi$ whenever $0 < x_1 < x_2 \le x_0$. This is impossible and thus $\theta^*(x) \equiv +\infty$ for all $x \in (0,1]$. The proof of the theorem is complete. $\square$

4. STURM COMPARISON AND OSCILLATION THEOREMS

The Sturm comparison theorem for the singular Sturm-Liouville differential equation can easily be derived from the monotonicity of the Prüfer angle established in Lemma 3.4.

Theorem 4.1. *Assume that $y(\cdot;\lambda_j)$, $j = 1, 2$, are real-valued solutions of the equations $\tau y = \lambda_j y$ and let $\lambda_1 < \lambda_2$. Then $y(\cdot;\lambda_2)$ vanishes at least once between every two zeros of $y(\cdot;\lambda_1)$.*

Proof. Let $x_0 < x_1$ be two successive zeros of $y(\cdot;\lambda_1)$, and assume for the sake of definiteness that $y(x;\lambda_1) > 0$ for $x \in (x_0, x_1)$. By Lemma 2.1, $y^{[1]}(x_0;\lambda_1) > 0$ and $y^{[1]}(x_1;\lambda_1) < 0$, so that

$$\begin{aligned} \frac{y^{[1]}(x;\lambda_1)}{y(x;\lambda_1)} &\to +\infty \quad \text{as} \quad x \to x_0+, \\ \frac{y^{[1]}(x;\lambda_1)}{y(x;\lambda_1)} &\to -\infty \quad \text{as} \quad x \to x_1-. \end{aligned}$$

We now fix the Prüfer angle $\theta(\cdot,\lambda_1)$ corresponding to the solution $y(\cdot;\lambda_1)$ by the condition $\theta(x_0;\lambda_1) = 0$. Then $\theta(x;\lambda_1)$ is positive for $x > x_0$ by Corollary 3.1 and does not take values πn, $n \in \mathbb{Z}$, for $x \in (x_0, x_1)$; therefore $\theta(x;\lambda_1) \in (0,\pi)$ for all $x \in (x_0, x_1)$, and $\theta(x_1, \lambda_1) = \pi$.

Let $\theta(\cdot;\lambda_2)$ be the Prüfer angle corresponding to the solution $y(\cdot;\lambda_2)$ and fixed by the condition $\theta(x_0;\lambda_2) \in [0,\pi)$. By Lemma 3.4, we have $\theta(x_1;\lambda_2) > \theta(x_1;\lambda_1) = \pi$ and thus there is $x^* \in (x_0, x_1)$ such that $\theta(x^*;\lambda_2) = \pi$. Then $y(x^*;\lambda_2) = 0$, and the proof is complete. $\square$

Consider now the Sturm-Liouville operator T generated in $L_2(0,1)$ by the differential expression τ and the boundary conditions

$$\sin\alpha\, y^{[1]}(0) - \cos\alpha\, y(0) = \sin\beta\, y^{[1]}(1) - \cos\beta\, y(1) = 0$$

for some $\alpha \in [0,\pi)$ and $\beta \in (0,\pi]$. It is known [23,24] that the operator T is self-adjoint and that its spectrum consists entirely of simple eigenvalues.

We denote by $y(\cdot;\lambda)$ the solution of the equation $\tau y = \lambda y$ normalized by the initial conditions $y(0) = \sin\alpha$ and $y^{[1]}(0) = \cos\alpha$, and let $\theta(\cdot;\lambda)$ be the corresponding Prüfer angle subject to the initial condition $\theta(0;\lambda) = \alpha$.

Lemma 4.2. *The solution $y(\cdot;\lambda)$ has n zeros inside the interval $(0,1)$ if and only if $\pi n < \theta(1;\lambda) \le \pi(n+1)$. In particular, the number of interior zeros of $y(\cdot;\lambda)$ is a non-decreasing function of λ.*

Proof. The number of interior zeros of $y(\cdot;\lambda)$ is equal to the number of interior points x, where $\theta(x;\lambda) = 0 \mod \pi$. Since $\theta(\cdot;\lambda)$ increases through every such point by Corollary 3.1, the lemma follows. $\square$

It follows from the above lemma that the solution $y(\cdot;\lambda)$ has at least n interior zeros for all $\lambda > \lambda_n^*$, with λ_n^* denoting the unique solution of the equation $\theta(1;\lambda) = \pi n$. If we denote the nth zero by x_n, then x_n becomes a function of $\lambda \in (\lambda_n^*, +\infty)$. As in the classical Sturm-Liouville theory, we conclude the following.

Lemma 4.3. *x_n is a continuous and strictly decreasing function of the variable $\lambda \in (\lambda_n^*, +\infty)$.*

Proof. The properties of the Prüfer angle imply that $\theta(x_n(\lambda);\lambda) = \pi n$ for all $\lambda > \lambda_n^*$. Since θ strictly increases in λ, for every λ_1 and λ_2 such that $\lambda_2 > \lambda_1 > \lambda_n^*$ the inequality $\theta(x_n(\lambda_1), \lambda_2) > \theta(x_n(\lambda_1), \lambda_1) = \pi n$ holds. As in Lemma 4.2, this implies that $y(x;\lambda_2)$ has at least n zeros in $(0, x_n(\lambda_1))$, so that $x_n(\lambda_2) < x_n(\lambda_1)$.

By (3.6), the Prüfer angle $\theta(\cdot;\lambda)$ depends continuously on λ in the topology of the space $C[0,1]$ and whence is a continuous function of x and λ. Take an arbitrary $\lambda^* > \lambda_n^*$ and set $x^* = x_n(\lambda^*)$; then by the simplest form of the implicit function theorem, there exist a neighbourhood $\mathcal{O}$ of the point x^* and a continuous function $\lambda(x)$ defined on $\mathcal{O}$ such that $\lambda(x^*) = \lambda^*$ and $\theta(x;\lambda(x)) = \pi n$ for all $x \in \mathcal{O}$. In view of Corollary 3.1 the function $\lambda(x)$ strictly decreases in $\mathcal{O}$, and thus there is a neighbourhood $\mathcal{O}'$ of λ^* and a continuous function $x(\lambda)$ that is inverse to $\lambda(x)$. In particular, $\theta(x(\lambda), \lambda) = \pi n$ for all $\lambda \in \mathcal{O}'$. Therefore $x_n(\lambda) = x(\lambda)$ in $\mathcal{O}$, and x_n is continuous in $\mathcal{O}$ and whence for all $\lambda > \lambda_n^*$. □

As in [21], one can prove the Sturm oscillation principle using the above monotonicity of the zeros $x_n(\lambda)$. We give another proof based directly on the properties of the Prüfer angle.

Theorem 4.4. *The operator T is bounded below and its eigenvalues can be listed as*

$$\lambda_0 < \lambda_1 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$

with the only accumulation point at $+\infty$. Denote by y_n a real-valued eigenfunction corresponding to λ_n; then y_n has n interior zeros, which interlace the zeros of y_{n+1}.

Proof. Clearly, a real λ is an eigenvalue of T if and only if $\theta(1;\lambda) = \beta \mod \pi$. We observe that $\theta(1;\lambda) > 0$ for all $\lambda \in \mathbb{R}$ and that $\theta(1;\lambda) \to 0$ as $\lambda \to -\infty$ by Theorem 3.5; therefore, there is $K > 0$ such that $\theta(1;\lambda) \neq \beta \mod \pi$ if $\lambda < -K$, which yields the bound $\lambda_0 \geq -K$.

As λ increases from $-\infty$ to $+\infty$, $\theta(1;\lambda)$ strictly increases from 0 to $+\infty$. Therefore for every $n \in \mathbb{Z}_+$ there exists a unique λ_n such that $\theta(1;\lambda_n) = \beta + \pi n$. In particular, $\theta(1;\lambda_0) = \beta \leq \pi$, and by Lemma 4.2 the eigenfunction $y_0 := y(\cdot;\lambda_0)$ corresponding to the first eigenvalue λ_0 has no interior zeros. Similarly, as $\theta(1;\lambda_n) \in (\pi n, \pi n + \pi]$, the function $\theta(\cdot;\lambda_n)$ has exactly n interior points x_k, $k = 1, \ldots, n$, at which $\theta(x_k, \lambda_n) = \pi k$.

Further, by Theorem 4.1 each of the intervals (x_1, x_2), …, (x_{n-1}, x_n) contains at least one zero of y_{n+1}. By Lemma 3.4, $\theta(x_1;\lambda_{n+1}) > \theta(x_1;\lambda_n) = \pi$ and thus $\theta(\cdot, \lambda_{n+1})$ assumes the value π inside the interval $(0, x_1)$, i.e., y_{n+1} has a zero in $(0, x_1)$. Next we observe that $\theta(1;\lambda_n) + \pi = \theta(1;\lambda_{n+1})$. Applying the "backward" part of Lemma 3.4 to the solutions $\theta(\cdot;\lambda_n) + \pi$ and $\theta(\cdot;\lambda_{n+1})$ on the interval $(x_n, 1)$, we conclude that

$\theta(x_n;\lambda_n)+\pi > \theta(x_n;\lambda_{n+1})$, i.e., $\theta(x_n;\lambda_{n+1}) < \pi(n+1)$. As $\theta(1;\lambda_{n+1}) > \pi(n+1)$, $\theta(\cdot;\lambda_{n+1})$ assumes the value $\pi(n+1)$ at some point in the interval $(x_n,1)$, and y_{n+1} vanishes at that point.

Finally, as y_{n+1} has exactly $n+1$ interior zeros, we see that each of the intervals $(0,x_1)$, (x_1,x_2), $\ldots$, $(x_n,1)$ contains exactly one such zero, i.e., the zeros of y_n and y_{n+1} strictly interlace. The proof is complete. □

Combining Theorem 3.5 and Lemma 4.2, we immediately get another form of the Sturm oscillation principle, namely:

Corollary 4.5. *Assume that $\beta = \pi$. Then the number of eigenvalues of T strictly below λ is equal to the number of interior zeros of the solution $y(\cdot;\lambda)$.*

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REFERENCES

[1] A. Alexiewicz, W. Orlicz, *On a theorem of C. Carathéodory*, Ann. Polon. Math. **1** (1955), 414–417.

[2] F.V. Atkinson, W.N. Everitt, A. Zettl, *Regularization of a Sturm-Liouville problem with an interior singularity using quasi-derivatives*, Diff. Integr. Equat. **1** (1988) 2, 213–221.

[3] J. Ben Amara, *Sturm theory for the equation of vibrating beam*, J. Math. Anal. Appl. **349** (2009) 1, 1–9.

[4] G. Berkolaiko, *A lower bound for nodal count on discrete and metric graphs*, Commun. Math. Phys. **278** (2008), 803–819.

[5] E.A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[6] J. Eckhardt, F. Gesztesy, R. Nichols, G. Teschl, *Weyl-Titchmarsh theory for Sturm-Liouville operators with distributional potentials*, Opuscula Math. **33** (2013) 3, 467–563.

[7] A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Nauka Publ., Moscow, 1985 [in Russian]; Engl. transl. in: Kluwer Academic Publishers, Dordrecht, 1988.

[8] A.S. Goriunov, V.A. Mikhailets, *Regularization of singular Sturm-Liouville equations*, Methods Funct. Anal. Topology **16** (2010) 2, 120–130.

[9] A.S. Goriunov, V.A. Mikhailets, *Regularization of binomial differential equations with singular coefficients*, Ukrainian Math. J. **63** (2011) 9, 1190–1205.

[10] P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, Inc., New York-London-Sydney, 1964.

[11] D. Hinton, *Sturm's 1836 oscillation results. Evolution of the theory*, [in:] W. O. Amrein, A. M. Hintz, D. B. Hinton (eds), *Sturm-Liouville Theory: Past and Present*, Birkhäuser Verlag, Basel, 2005; pp. 1–27.

[12] M. Homa, R. Hryniv, *Oscillation properties of singular quantum trees* (submitted).

[13] A. Kamiński, S. Mincheva-Kamińska, *Equivalence of the Mikusiński-Shiraishi-Itano products in $\mathcal{S}'$ for various classes of delta-sequences*, Integral Transforms Spec. Funct. **20** (2009) 3–4, 207–214.

[14] T. Kappeler, P. Perry, M. Shubin, P. Topalov, *The Miura map on the line*, Int. Math. Res. Not. **2005** (2005) 50, 3091–3133.

[15] Y. Katznelson, *An Introduction to Harmonic Analysis*, Third edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2004.

[16] M. Nowaczyk, *Inverse Problems for Graph Laplacians*, PhD thesis, Lund, Sweden, 2007.

[17] M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Equations*, Longman, J. Wiley, Essex, New York, 1992.

[18] Yu.V. Pokornyi, V.J. Pryadiev, A. Al'-Obied, *On the oscillation of the spectrum of a boundary value problem on a graph*, Matem. Zametki **60** (1996) 3, 468–470 [in Russian]; Engl. transl. in: Mathem. Notes, **60** (1996) 3, 351–353.

[19] Yu. V. Pokornyi, M.B. Zvereva, A.S. Ishchenko, S.A. Shabrov, *On an irregular extension of the oscillation theory of the Sturm-Liouville spectral problem*, Matem. Zametki **82** (2007) 4, 578–582 [in Russian]; Engl. transl. in: Mathem. Notes, **82** (2007) 3–4, 518–521.

[20] W.T. Reid, *Ordinary Differential Equations*, John Wiley & Sons, Inc., New York-London-Sydney, 1971.

[21] C. Sturm, *Mémoire sur les Équations différentielles linéaires du second ordre*, J. Math. Pures Appl. **1** (1836), 106–186.

[22] C. Sturm, *Mémoire sur une classe d'Équations à différences partielles*, J. Math. Pures Appl. **1** (1836), 373–444.

[23] A.M. Savchuk, A.A. Shkalikov, Sturm-Liouville operators with singular potentials, Matem. Zametki **66** (1999) 6, 897–912 [in Russian]; Engl. transl. in: Math. Notes **66** (1999) 5–6, 741–753.

[24] A.M. Savchuk, A.A. Shkalikov, *The Sturm-Liouville operators with distributional potential*, Trudy Mosk. Matem Ob-va **64** (2003) 159–212 [in Russian]; Engl. transl. in: Trans. Moscow Math. Soc. **2003** (2003), 143–192.

[25] Ph. Schapotschnikow, *Eigenvalue and nodal properties on quantum graph trees*, Waves Random Complex Media **16** (2006) 3, 167–178.

[26] B. Simon, *Sturm oscillation and comparison theorems*, [in:] W.O. Amrein, A.M. Hintz, D.B. Hinton (eds), Sturm-Liouville Theory: Past and Present, Birkhäuser Verlag, Basel, 2005; pp. 29–43.

[27] A.A. Shkalikov, J. Ben Amara, *Oscillation theorems for Sturm-Liouville problems with distribution potentials*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. (2009) 3, 43–49 [in Russian]; Engl. transl. in: Moscow Univ. Math. Bull. **64** (2009) 3, 132–137.

[28] G. Teschl, *Oscillation theory and renormalized oscillation theory for Jacobi operators*, J. Differential Equations **129** (1996) 2, 532–558.

[29] A.A. Vladimirov, *On the oscillation theory of the Sturm-Liouville problem with singular coefficients*, Zh. Vychisl. Mat. Mat. Fiz. **49** (2009) 9, 1609–1621 [in Russian]; Engl. transl. in: Comput. Math. Math. Phys. **49** (2009) 9, 1535–1546.

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