jet INTL JOURNAL OF ELECTRONICS AND TELECOMMUNICATIONS, 2013, VOL. 59, NO. 3, PP. 301–306
Manuscript received September 2, 2013; revised September, 2013. DOI: 10.2478/eletel-2013-0036

Noncircularity of Demodulated Radar Signal

Rafał Rytel-Andrianik

***Abstract*—The output of the quadrature demodulator is generally regarded as a complex signal that is *circular* if only the demodulator is well balanced. In the paper we analyze properties of quadrature demodulators, particularly digital one, and show that the output is noncircular also if the input signal is nonstationary. Then we find sources of this noncircularity and show that they stem from transients of the low-pass demodulator filter. This is quite important because nonstationary inputs are quite typical in radar where power of echoes depends strongly on range. In the first sections we also review complex random signals and properties of circularity and properness.**

***Keywords*—quadrature demodulator, complex radar signals, properness, circularity**

I. INTRODUCTION

THE backbone of modern radar systems are detection and estimation theories. The algorithms that stem from these theories allow to discriminate target echo form noise and correctly measure its parameters. In many cases the processed signals are *complex-valued*, generated by quadrature demodulator. Until recently, in virtually all the literature corresponding to statistical signal processing of complex radar signals it was assumed that processed signals are circular. Currently there is a growing interest in *noncircular* signals. It was already demonstrated that the exploitation of noncircularity can lead to improved detection and estimation algorithms, particularly in the communications. What is interesting, there is little literature available concerned with noncircularity of *radar signals*. In the paper [1] an idea of a radar transmitting noncircular signals was briefly presented, and in [2] the authors mention that "second order noncircularity interferences (are) omnipresent in applications such as radar, (...)". This claim is evidently in contrast with assumptions adopted in the classic literature on the radar signal processing. If this claim were true the assumptions of many commonly used algorithms would not be met and these algorithms should be adopted to noncircular signals. This means that it is necessary to take a closer look at radar signals and see if they are circular or noncircular. It is also a good occasion to briefly review the concept of complex random variables, which we do in the next section.

II. REVIEW OF COMPLEX VARIABLES, CIRCULARITY AND PROPRIETY

A. Complex Random Variables and Vectors

Let us define the complex random variable as

$$z = z_R + jz_I \tag{1}$$

R. Rytel-Andrianik is with the Institute of Electronic Systems, Warsaw University of Technology, 00-665 Warszawa, ul. Nowowiejska 15/19, Poland (e-mail: r.rytel@elka.pw.edu.pl).

where z_R and z_I are real random variables and $j = \sqrt{-1}$. The complex random variable (1) can be described either by the use of short complex notation, where CDF (cumulative distribution function) is expressed directly as a function of a complex variable z, or by the use of the well understood real notation where complex variable z is represented as a vector $[z_R, z_I]^T$ composed of its real and imaginary parts, and described by a bivariate CDF: $F(z_R, z_I)$. These two descriptions are equivalent, therefore the complex CDF should be interpreted as a concise notation of a real bivariate CDF, that is:

$$F(z) \underset{\text{def.}}{=} F(z_R, z_I). \tag{2}$$

The real bivariate PDF (probability density function) is defined as $f(z_R, z_I) = \frac{\partial^2 F(z_R,z_I)}{\partial z_R \partial z_I}$ and, just like with the CDF, the PDF of a complex variable should be interpreted as a concise notation for the real bivariate PDF:

$$f(z) \underset{\text{def.}}{=} f(z_R, z_I). \tag{3}$$

In a similar way, the PDF for complex random *vectors* is defined as a joint PDF of its real and imaginary parts.

1) Complementary covariance matrix: The covariance matrix of the random vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\mathbf{C}_{\mathbf{x},\mathbf{y}} = \mathrm{E}[(\mathbf{x} - \mathbf{m}_\mathbf{x})\,(\mathbf{y} - \mathbf{m}_\mathbf{y})^H], \tag{4}$$

which can be expressed in terms of real matrices as:

$$\mathbf{C}_{\mathbf{x},\mathbf{y}} = \mathbf{C}_{\mathbf{x}_R,\mathbf{y}_R} + \mathbf{C}_{\mathbf{x}_I,\mathbf{y}_I} + j\{\mathbf{C}_{\mathbf{x}_I,\mathbf{y}_R} - \mathbf{C}_{\mathbf{x}_R,\mathbf{y}_I}\} \tag{5}$$

where

$$\begin{aligned}
\mathbf{C}_{\mathbf{x}_R,\mathbf{y}_R} &= \mathrm{E}[(\mathbf{x}_R - \mathbf{m}_{\mathbf{x}_R})(\mathbf{y}_R - \mathbf{m}_{\mathbf{y}_R})^T],\\
\mathbf{C}_{\mathbf{x}_I,\mathbf{y}_I} &= \mathrm{E}[(\mathbf{x}_I - \mathbf{m}_{\mathbf{x}_I})(\mathbf{y}_I - \mathbf{m}_{\mathbf{y}_I})^T],\\
\mathbf{C}_{\mathbf{x}_I,\mathbf{y}_R} &= \mathrm{E}[(\mathbf{x}_I - \mathbf{m}_{\mathbf{x}_I})(\mathbf{y}_R - \mathbf{m}_{\mathbf{y}_R})^T],\\
\mathbf{C}_{\mathbf{x}_R,\mathbf{y}_I} &= \mathrm{E}[(\mathbf{x}_R - \mathbf{m}_{\mathbf{x}_R})(\mathbf{y}_I - \mathbf{m}_{\mathbf{y}_I})^T]
\end{aligned}$$

and $\mathbf{x} = \mathbf{x}_R + j\mathbf{x}_I$, $\mathbf{y} = \mathbf{y}_R + j\mathbf{y}_I$. We see that the complex matrix $\mathbf{C}_{\mathbf{x},\mathbf{y}}$ alone is not sufficient to uniquely determine values of all these four real matrices. In [3] it was shown that we additionally need a correlation matrix of vectors $\mathbf{x}$ and $conj\{\mathbf{y}\}$, that is:

$$\tilde{\mathbf{C}}_{\mathbf{x},\mathbf{y}} = \mathrm{E}[(\mathbf{x} - \mathbf{m}_\mathbf{x})(\mathbf{y} - \mathbf{m}_\mathbf{y})^T]. \tag{6}$$

This matrix is called the *complementary covariance matrix* [4], the *relation matrix* [5] or pseudo covariance matrix [3]. The complementary covariance matrix $\tilde{\mathbf{C}}_{\mathbf{x},\mathbf{y}}$ can be expressed in terms of real matrices as:

$$\tilde{\mathbf{C}}_{\mathbf{x},\mathbf{y}} = \mathbf{C}_{\mathbf{x}_R,\mathbf{y}_R} - \mathbf{C}_{\mathbf{x}_I,\mathbf{y}_I} + j\{\mathbf{C}_{\mathbf{x}_I,\mathbf{y}_R} + \mathbf{C}_{\mathbf{x}_R,\mathbf{y}_I}\}. \tag{7}$$

Combining equations (5) and (7) we see that the covariance $\mathbf{C}_{\mathbf{x},\mathbf{y}}$ and complementary covariance $\tilde{\mathbf{C}}_{\mathbf{x},\mathbf{y}}$ matrices together are sufficient to describe all four real covariance matrices:

$$\begin{aligned}
\mathbf{C}_{\mathbf{x}_R,\mathbf{y}_R} &= \frac{1}{2}\,\mathfrak{Re}\{\mathbf{C}_{\mathbf{x},\mathbf{y}} + \tilde{\mathbf{C}}_{\mathbf{x},\mathbf{y}}\},\\
\mathbf{C}_{\mathbf{x}_I,\mathbf{y}_I} &= \frac{1}{2}\,\mathfrak{Re}\{\mathbf{C}_{\mathbf{x},\mathbf{y}} - \tilde{\mathbf{C}}_{\mathbf{x},\mathbf{y}}\},\\
\mathbf{C}_{\mathbf{x}_I,\mathbf{y}_R} &= \frac{1}{2}\,\mathfrak{Im}\{\mathbf{C}_{\mathbf{x},\mathbf{y}} + \tilde{\mathbf{C}}_{\mathbf{x},\mathbf{y}}\},\\
\mathbf{C}_{\mathbf{x}_R,\mathbf{y}_I} &= -\frac{1}{2}\,\mathfrak{Im}\{\mathbf{C}_{\mathbf{x},\mathbf{y}} - \tilde{\mathbf{C}}_{\mathbf{x},\mathbf{y}}\}.
\end{aligned}$$

The complex random vectors $\mathbf{x}$ and $\mathbf{y}$ are *uncorrelated* if all real correlation matrices are zeros, that is if: $\mathbf{C}_{\mathbf{x}_R,\mathbf{y}_R} = \mathbf{C}_{\mathbf{x}_I,\mathbf{y}_I} = \mathbf{C}_{\mathbf{x}_R,\mathbf{y}_I} = \mathbf{C}_{\mathbf{x}_I,\mathbf{y}_R} = \mathbf{0}$, which in terms of complex correlation and complementary correlation matrices is equivalent to $\mathbf{C}_{\mathbf{x},\mathbf{y}} = \mathbf{0}$ and $\tilde{\mathbf{C}}_{\mathbf{x},\mathbf{y}} = \mathbf{0}$.

2) Circular distributions: The complex random variable z is said to be circular if PDF of z is the same as PDF of $ze^{j\alpha}$ for any real α. As a result, the PDF of $z = Ae^{j\phi}$ is:

$$f_z(A;\phi) = \frac{1}{2\pi} f_A(A) \tag{8}$$

which means that the amplitude is independent of the phase which is uniformly distributed on $[-\pi, \pi)$.

For *vectors* a few different definitions of circularity were proposed in [6], among which:

a) Marginal Circularity: A complex random vector $\mathbf{z}$ is marginally circular if its components z_k are scalar, complex and circular random variables.

b) Weak Circularity: A random vector $\mathbf{z}$ is weakly circular if $\mathbf{z}$ and $\mathrm{e}^{j\alpha}\mathbf{z}$ both have the same probability distribution for any α.

The PDF of a weakly circular random vector can be expressed as:

$$f(\mathbf{A};\phi_1,\phi_2,...,\phi_m) = f(\mathbf{A};\phi_1+\alpha,\phi_2+\alpha,...,\phi_m+\alpha) \tag{9}$$

Weak circularity implies marginal circularity.

3) Proper complex random vectors: The term *proper* in relation to complex random vectors was introduced in [3]. The complex random vector is called *proper* if the complementary covariance matrix vanishes, that is if

$$\tilde{\mathbf{C}}_{\mathbf{z},\mathbf{z}} = \mathbf{0}. \tag{10}$$

This condition is met if a random vector is weakly circular, hence proper random vectors are also called *second order circular*. In terms of real correlation matrices, the properness means that:

$$\mathbf{C}_{\mathbf{z}_R,\mathbf{z}_R} = \mathbf{C}_{\mathbf{z}_I,\mathbf{z}_I}, \quad \text{and} \quad \mathbf{C}_{\mathbf{z}_R,\mathbf{z}_I} = -\mathbf{C}_{\mathbf{z}_I,\mathbf{z}_R}. \tag{11}$$

Noting that $\mathbf{C}_{\mathbf{z}_R,\mathbf{z}_I} = \mathbf{C}^T_{\mathbf{z}_I,\mathbf{z}_R}$ the second equation can be also expressed as

$$\mathbf{C}^T_{\mathbf{z}_R,\mathbf{z}_I} = -\mathbf{C}_{\mathbf{z}_R,\mathbf{z}_I}, \quad \text{or} \quad \mathbf{C}^T_{\mathbf{z}_I,\mathbf{z}_R} = -\mathbf{C}_{\mathbf{z}_I,\mathbf{z}_R}, \tag{12}$$

which means that matrices $\mathbf{C}_{\mathbf{z}_R,\mathbf{z}_I}$ and $\mathbf{C}_{\mathbf{z}_I,\mathbf{z}_R}$ are skew-symmetric. Hence:

- the real and imaginary parts of the proper complex vector have equal covariance matrices, and
- real and imaginary parts can be correlated, but only if they correspond to different time samples (the main diagonal of an antisymmetric matrix is zero).

a) Jointly Proper Vectors ([3]).: Two complex random vectors $\mathbf{x}$ i $\mathbf{y}$ are jointly proper if the vector $[\mathbf{x}^T, \mathbf{y}^T]^T$ is proper.

b) Uncorrelated Proper Vectors.: Two jointly proper vectors $\mathbf{x}$ i $\mathbf{y}$ are uncorrelated if $\mathbf{C}_{\mathbf{x},\mathbf{y}} = \mathbf{0}$.

c) Affine Transformations ([3]).: If $\mathbf{x}$ is a complex proper random vector, then

$$\mathbf{A}\mathbf{x} + \mathbf{b} \tag{13}$$

is also proper for any complex matrix $\mathbf{A}$ and any complex vector $\mathbf{b}$.

d) Sum of Proper Vectors ([3]).: If $\mathbf{x}$ and $\mathbf{y}$ are independent proper random vectors then

$$a\mathbf{x} + b\mathbf{y} \tag{14}$$

is also proper for any complex coefficients a and b.

4) Gaussian distribution: The complex random vector is called *gaussian* if its real and imaginary parts are jointly gaussian. The N-dimensional *proper* random vector $\mathbf{z}$ with autocovariance matrix $\mathbf{C}_{\mathbf{z},\mathbf{z}}$ is Gaussian if its PDF is given by the equation:

$$f(\mathbf{z}) = \frac{1}{\pi^N \det(\mathbf{C}_{\mathbf{z},\mathbf{z}})} \mathrm{e}^{-(\mathbf{z}-\mathbf{m})^H \mathbf{C}^{-1}_{\mathbf{z},\mathbf{z}} (\mathbf{z}-\mathbf{m})}. \tag{15}$$

If the gaussian random vector is not proper, its PDF depends also on the complementary covariance matrix $\tilde{\mathbf{C}}_{\mathbf{z},\mathbf{z}}$.

5) Detecting and measuring noncircularity: The impropriety of a random vector can be detected using the GLRT-test described in [7]. The decision statistic is

$$l = 1 - \frac{\det[\underline{\hat{\mathbf{C}}_{\mathbf{z},\mathbf{z}}}]}{\det[\hat{\mathbf{C}}_{\mathbf{z},\mathbf{z}}]} \tag{16}$$

where $\hat{\mathbf{C}}_{\mathbf{z},\mathbf{z}}$ is the estimated covariance matrix, and $\underline{\hat{\mathbf{C}}_{\mathbf{z},\mathbf{z}}}$ is the estimated augmented covariance matrix, which is defined as the covariance matrix of the augmented vector $[\mathbf{z}^T, \mathbf{z}^H]^T$:

$$\underline{\mathbf{C}_{\mathbf{z},\mathbf{z}}} = \begin{pmatrix} \mathbf{C}_{\mathbf{z},\mathbf{z}} & \tilde{\mathbf{C}}_{\mathbf{z},\mathbf{z}} \\ \mathbf{C}^*_{\mathbf{z},\mathbf{z}} & \tilde{\mathbf{C}}^*_{\mathbf{z},\mathbf{z}} \end{pmatrix}. \tag{17}$$

Unfortunately the authors of [7] did not provide an equation to calculate detection threshold. For the one-dimensional (marginal) case the test statistic becomes sample estimate of k_1^2, where k_1 is a non-circularity coefficient (also called non-circularity rate [8]) and is defined as:

$$k_1 = \frac{|\mathrm{E}\{(z-m_z)^2\}|}{\mathrm{E}\{|z-m_z|^2\}}. \tag{18}$$

This coefficient is equal to zero for proper random variable and is equal to one for a maximally *improper* random variable (for example for purely real or purely imaginary noise) and can be used as a measure of impropriety of a scalar random variable or marginal impropriety of a vector random variable.

The test for higher order noncircularity of a non-gaussian random variable, based on higher order moments, is described in [9].

B. Complex Random Signals

The complex random signal is defined in [3] as $z(t) = z_R(t)+jz_I(t)$, where $z_R(t)$ and $z_I(t)$ are real random signals. The mean function is $m_z(t) = \mathrm{E}\left[(z(t))\right]$, the autocovariance function is

$$c_z(\tau,t) = \mathrm{E}\left[(z(t+\tau) - m_z(t+\tau)) \cdot (z(t) - m_z(t))^*\right], \tag{19}$$

and the complementary autocovariance function is

$$\tilde{c}_z(\tau,t) = \mathrm{E}\left[(z(t+\tau) - m_z(t+\tau)) \cdot (z(t) - m_z(t))\right]. \tag{20}$$

Analogously are defined autocorrelation $r_z(\tau,t)$ and complementary autocorrelation $\tilde{r}_z(\tau,t)$ functions.

A complex stochastic process is *wide-sense stationary* if its real and imaginary parts are jointly wide-sense stationary. This means that the real and complex parts are wide-sense stationary, and additionally their cross-correlation function depends only on the time difference. Hence, a complex stochastic process $z(t)$ is wide-sense stationary if and only if $m_z(t)$, $r_z(\tau,t)$ i $\tilde{r}_z(\tau,t)$ are independent of t.

A random signal $z(t)$ is *proper* if its complementary autocovariance function vanishes, that is if $\tilde{c}_z(\tau,t) = 0$. If this process is also stationary then

$$c_{z_R,z_R}(\tau) = c_{z_I,z_I}(\tau) \qquad c_{z_R,z_I}(\tau) = -c_{z_R,z_I}(-\tau) \tag{21}$$

where the real covariance functions are defined in an usual way.

III. Baseband Signal as an Envelope of the Analytic Signal

In a typical pulse radar the incoming signal, after being received by an antenna and microwave receiver, is demodulated by the use of an analog or digital quadrature demodulator. The obtained complex signal is linearly filtered in different dimensions (beamforming, range compression, Doppler filtering) sometimes with the use of the Fourier Transform. Then detection and estimation are performed.

If the input signal $x_{rf}(t)$ is narrow band, then the output baseband signal will be its complex envelope. The complex envelope can be obtained by firstly creating the analytic signal $z^a_{rf}(t)$ from the input real signal $z_{rf}(t)$

$$z^a_{rf}(t) = z_{rf}(t) + jH_{HT}\left\{z_{rf}(t)\right\} \tag{22}$$

where $H_{HT}\{\cdot\}$ is Hilbert transform operator, and by demodulating the obtained signal as

$$z(t) = z^a_{rf}(t)\exp(-j\omega_c t). \tag{23}$$

The corresponding diagram is depicted in Fig. 1. It is interesting if the complex baseband signal is proper. For the analog case, it was shown in [10] that if the complex baseband signal is formed by the use of the analytic signal and the real input signal $z_{rf}(t)$ is wide sense stationary with zero mean, then the complex baseband signal $z(t)$ is wide sense stationary and *proper*.

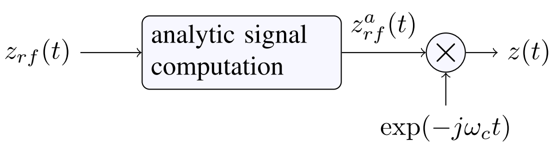


Fig. 1. Obtaining the analogue baseband signal as a demodulated analytic signal.

IV. Analogue Quadrature Demodulator

The analog quadrature demodulator is depicted in the standard form in Fig. 2a. It produces in-phase and quadrature signal components that are treated as real and imaginary parts of a complex baseband signal. Using the complex notation, the quadrature demodulator can be also presented in an equivalent form depicted in Fig. 2b (in fact, this is a slightly more general form because the LP filter coefficients are now not bound to be real).

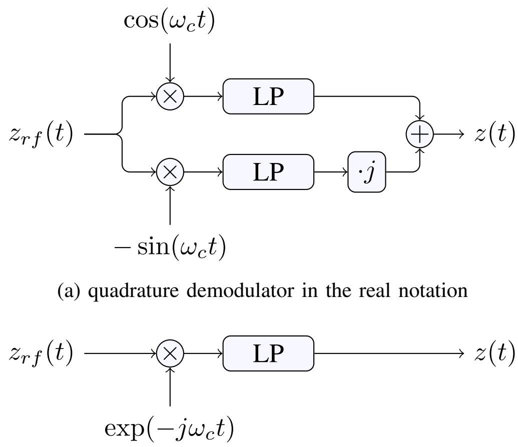


Fig. 2. Obtaining the analogue baseband signal.

If the following conditions hold:

- the input signal $z_{rf}(t)$ is a real wide sense stationary and zero-mean stochastic process with power spectrum $S_{z_{rf}}(\omega)$ and
- the analog filter LP (Fig. 2) is an ideal low-pass filter with impulse response $h(t)$ and passband $[-\Omega_m, \Omega_m]$, and
- the frequency ω_c of the demodulator is greater then one-sided filter passband $\omega_c > \Omega_m$,

then the output signal $z(t)$ is a complex random process with zero mean, autocorrelation function

$$r_z(\tau) = \frac{1}{2\pi}\int_{-\Omega_m}^{\Omega_m} S_{z_{rf}}(\omega+\omega_c)\mathrm{e}^{j\omega\tau}\mathrm{d}\omega, \tag{24}$$

and complementary covariance function equal to zero, hence it is *proper*.

V. Digital Quadrature Demodulator

In many modern radar systems the quadrature demodulator is digital. Its block diagram is depicted in Fig. 3. Now, the real input signal $p(n)$ and the complex modulating harmonic are both discrete. Discrete is also the low-pass filter, transmittance of which we denoted by $H(z)$,

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}, \tag{25}$$

where $h(n)$ is the impulse response of the filter. The output

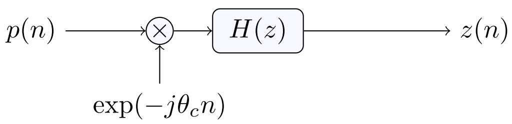


Fig. 3. Quadrature demodulator for discrete signals.

signal is thus the mixed and filtered input signal $p(n)$:

$$z(n) = \sum_{k=-\infty}^{\infty} p(k) e^{-j\theta_c k} h(n-k). \tag{26}$$

A. Wide Sense Stationary Input

Let the real input signal $p(n)$ be zero-mean wide sense stationary with power spectrum $S_p(e^{j\theta})$. Then the output signal $z(n)$ is also wide sense stationary and its spectrum is

$$S_z(e^{j\theta}) = S_p(e^{j(\theta+\theta_c)}) \cdot |H(e^{j\theta})|^2. \tag{27}$$

The autocorrelation function is the inverse Fourier transform of the spectrum and is equal to

$$R_z(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_p(e^{j(\theta+\theta_c)}) \cdot |H(e^{j\theta})|^2 e^{jm\theta} d\theta. \tag{28}$$

To see if the signal $z(n)$ is proper we need to calculate the complementary correlation function

$$\tilde{R}_z(n_1, n_2) = \mathrm{E}\{z(n_1) z(n_2)\}. \tag{29}$$

Using (26) and the fact that the autocorrelation function of the input is

$$\mathrm{E}\{p(k_1)p(k_2)\} = R_p(k_2 - k_1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_p(e^{j\theta}) e^{j\theta n} \mathrm{d}\theta \tag{30}$$

because $p(k)$ is real and stationary, we obtain after some algebra, the final expression for the complementary autocorrelation function:

$$\tilde{R}_z(n_1, n_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_p(e^{j(\theta+\theta_c)}) \cdot H(e^{j\theta}) H(e^{j(-\theta-2\theta_c)}) e^{-jn_1 2\theta_c} e^{j(n_2-n_1)\theta} d\theta. \tag{31}$$

An example of filter characteristics $H(e^{j\theta})$ and $H(e^{j(-\theta-2\theta_c)})$ is given in Fig. 4. It can be noticed that for a reasonable filter which rejects image spectrum component we have $H(e^{j\theta})H(e^{j(-\theta-2\theta_c)}) \approx 0$. The integral in (31) is therefore approximately equal to zero (exactly equal to zero for an ideal low-pass filter), and the complementary function is approximately zero

$$\tilde{R}_z(n_1, n_2) \approx 0 \tag{32}$$

Hence, the output of the discrete quadrature demodulator is *proper* just as for an analogue realization. It must be stressed however that this is true only under the conditions that the input is wide sense stationary and the demodulator is ideal. Next, we consider non-ideal demodulator.

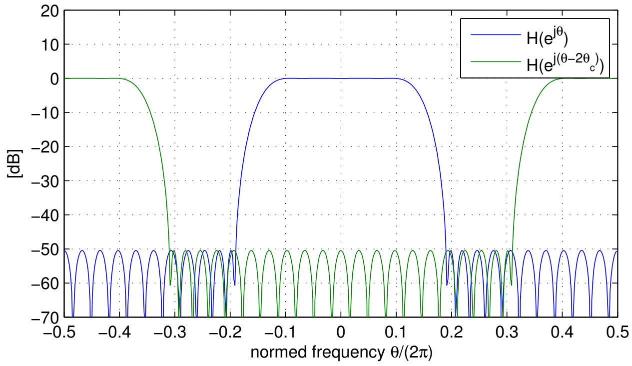


Fig. 4. Example of characteristics of $H(e^{j\theta})$ and $H(e^{j(-\theta-2\theta_c)})$ for $\theta_c = 2\pi 0.25$.

B. Non-Ideal Mixer

Let us now assume that the mixer is not well balanced and multiplies the incoming signal $p(n)$ not by an ideal harmonic $\cos(\theta_c n) + j\sin(\theta_c n)$ but by the signal $\cos(\theta_c n) + j\alpha\sin(\theta_c n + \phi)$. The output signal is then

$$z^u(n) = \sum_{k=-\infty}^{\infty} p(k) \left(\alpha_1 e^{j\theta_c k} + \alpha_2 e^{-j\theta_c k}\right) h(n-k) \tag{33}$$

where

$$\alpha_1 = 0.5[1 + \alpha e^{j\phi}], \tag{34}$$

$$\alpha_2 = 0.5[1 - \alpha e^{-j\phi}]. \tag{35}$$

The complementary covariance of the output signal can be calculated in a similar manner as (31). It is equal to

$$\tilde{R}_{z^u}(m) = 2\alpha_1\alpha_2 \frac{1}{2\pi} \mathfrak{Re} \int_{-\pi}^{\pi} S_p(e^{j(\theta+\theta_c)}) \cdot |H(e^{j\theta})|^2 e^{jm\theta} d\theta. \tag{36}$$

for example the complementary variance is equal to

$$\tilde{R}_{z^u}(0) = \frac{\alpha_1\alpha_2}{\pi} \int_{-\pi}^{\pi} S_p(e^{j(\theta+\theta_c)}) \cdot |H(e^{j\theta})|^2 d\theta. \tag{37}$$

Clearly, for an unbalanced mixer, the complementary autocorrelation of the output signal is not zero and hence this signal is not proper unless $\alpha_1 = 0$ or $\alpha_2 = 0$. This in turn gives the conditions:

$$\alpha = 1, \quad \phi = \pi \quad \text{or} \tag{38}$$

$$\alpha = 1, \quad \phi = 0. \tag{39}$$

which means that only the output of a well balanced mixer is proper.

C. Nonstationary Signals

It was shown that if the input signal $p(n)$ is stationary and the mixer is well balanced, then the output signal $z(n)$ is proper. On the other hand, if the input signal is nonstationary then the quadrature demodulation can give improper baseband signal. To see the degree of improperness computer simulations were performed.

1) Simulation 1: input signal power changing with time: We generated 5000 realizations of a bandpass random nonstationary real signal $p(n)$ with variance depended on time n as $n^{-\gamma}$. For the pulsed radar the dependence of the incoming signal power on time is natural as in radar time corresponds to range and clutter echoes received from greater ranges are weaker. Amplitude dependence $n^{-\gamma}$ corresponds to range dependence of echo power as $R^{-2\gamma}$. The signal was damped exponentially for $n \in [100, 300]$, $n \in [400, 550]$ and $n \in [600, 750]$ with growing exponents γ of 0.5, 1 and 2, correspondingly.

Each signal realization was fed to the input of the discrete demodulator of the form as in Fig. 3. The carrier frequency was $\theta_c = 2\pi \cdot 0.25$ and the bandwidth $2\pi \cdot 0.2$, which means that the power of the input signal is concentrated at normalized frequencies $[0.15, 0.35]$. Based on the obtained complex signal $z(n)$, an improperness statistic was estimated. The averaged signal amplitude as a function of time is depicted in Fig. 5a, and the estimate of non-circularity coefficients $k_1(n)$, defined in (18), is depicted in Fig. 5b. We see that for the time instants n where the signal amplitude changes abruptly, the non-circularity coefficient $k_1(n)$ is far from zero, often close to 1 which indicates strong *impropriety*. We also see that the faster the signal power changes the greater the noncircularity. It is also interesting that for step changes (that is very short changes) of signal power for $n = 900$ and $n = 1300$ the spikes of non-circularity coefficient are not very high. What causes substantial non-circularity are rather long-lasting changes of the signal power.

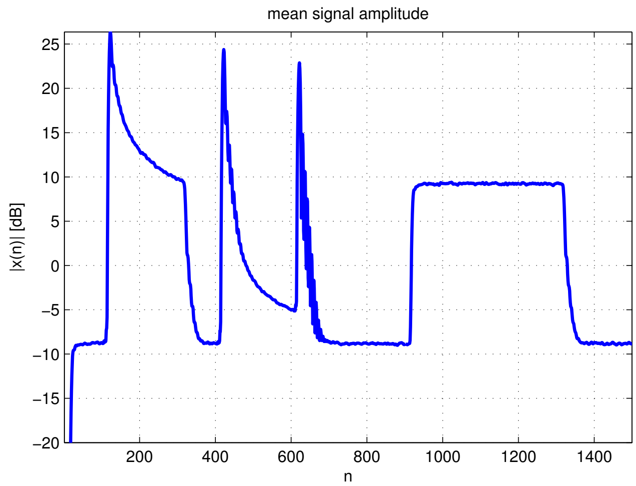
Figure 6 presents the histogram of phases of the output signal $z(n)$ for $n = 620$ which is the time instant where $z(n)$ is strongly noncircular (as indicated in Fig. 5). We see that phases are not distributed uniformly but concentrate around 0 and π, which indicates that the real part is greater than the imaginary part. This indeed represents noncircularity.

2) Simulation 2: input signal correlational properties changing with time: In the second simulation the signal $p(n)$ was nonstationary in a different way. Its instantaneous power was approximately constant, but the spectral content was changing – namely the noise was more narrowband for $n \in [500, 100]$. This can be seen in Fig. 7a, where spectrogram of the output signal $z(n)$ is plotted. In Fig. (7b) the corresponding estimate of the non-circularity coefficient k_1 is plotted. It can be noticed that the coefficient values are small even near to time instants where the spectral contents of the signal changes (is nonstationary) where it does not exceed 0.1.

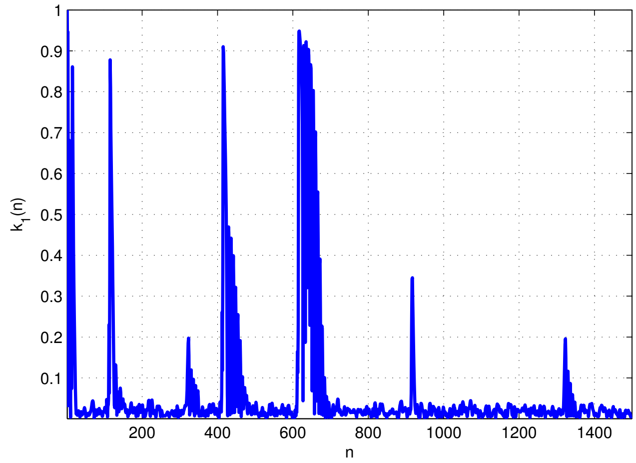
D. Nonstationary Signal with Decreasing Power

We have seen in the examples that particularly high noncircularity coefficient can be observed when the power of the input signal is changing with time. In order to understand why it happens, we will perform the following analysis. Let the input to the demodulator be modelled as a stationary noise damped with a $n^{-\gamma}$ function, that is

$$p(n) = p_0(n) \cdot n^{-\gamma} \mathbf{1}(n) \tag{40}$$



(a)



(b)

Fig. 5. Results of a simulation to give an example of noncircularity of a demodulated nonstationary signal: a) mean amplitude of a demodulated signal as a function of time, b) non-circularity coefficients $k_1(n)$ (0: proper; 1: maximally improper).

where $p_0(n)$ is a stationary real random vector with autocorrelation function $R_{p_0}(m) = \mathrm{E}\{p_0(n)p_0(n+m)\}$, function $\mathbf{1}(n)$ is a Heaviside step function which is equal to 0 if $n < 0$ and equal to 1 if $n \geq 0$, and γ is a positive constant.

For the input signal (40) and perfectly balanced demodulator, the complementary covariance function is given by the equation

$$\tilde{R}_z(n_1, n_2) = (n_1 n_2)^{-\gamma} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} S_p(e^{j(\theta+\theta_c)}) \cdot H_1(e^{j\theta}) H_2(e^{j(-\theta-2\theta_c)}) e^{-jn_1 2\theta_c} e^{j(n_2-n_1)\theta} d\theta. \tag{41}$$

where

$$H_1(e^{j\theta}) = \sum_{n=0}^{n_1} n^{-\gamma} h(n) e^{-j\theta n} \tag{42}$$

$$H_2(e^{j\theta}) = \sum_{n=0}^{n_2} n^{-\gamma} h(n) e^{-j\theta n} \tag{43}$$

are frequency characteristics of the filters with impulse responses $n^{-\gamma} h(n)$ cropped to $n_1 + 1$ or $n_2 + 1$ samples.

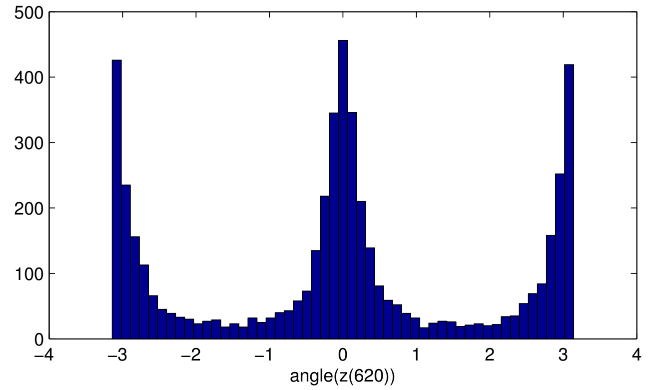


Fig. 6. Histogram of angles of $z(n)$ for n=620, showing strong noncircularity.

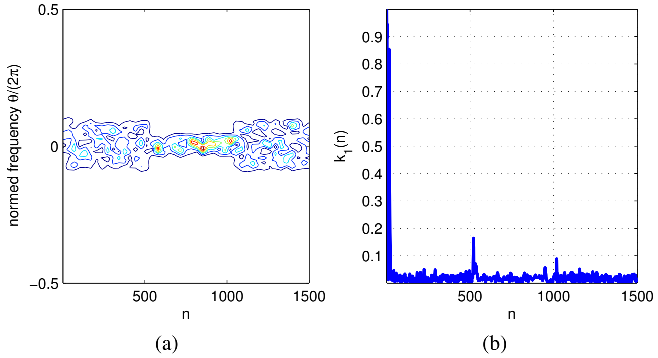


Fig. 7. Results of the simulation 2: a) spectrogram of the nonstationary input signal $p(n)$, b) corresponding non-circularity coefficients as a function of the discrete time n.

Complementary correlation is not necessarily equal to zero because $H_1(e^{j\theta})H_2(e^{j(-\theta-2\theta_c)})$ is not necessarily equal to zero. This is true because multiplying the filter impulse response by $n^{-\gamma}$ destroys frequency characteristics and causes them to overlap, what can be seen in Fig. 8. Another reason of

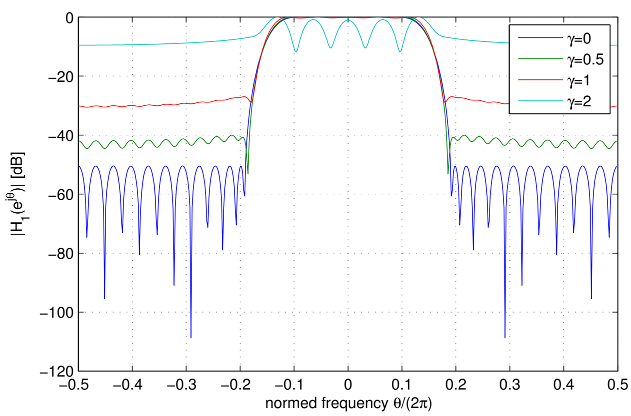


Fig. 8. Effective characteristics of a demodulator filter "destroyed" by nonstationarity of the input signal.

broadening the functions $|H_1(e^{j\theta})|$ and $|H_2(e^{j\theta})|$ is that the corresponding impulse responses are shortened. This effects especially abrupt short changes in signal power where mainly transients are observed at the output of the demodulator filter.

VI. Conclusions

In the paper, we reviewed notions of complex random variables and signals and properties of circularity and properness which is a second-order circularity. Then we have presented several methods of obtaining the complex baseband signal. It was shown that all of the presented methods the obtained signal is *proper* if the real input signal is wide-sense stationary and the quadrature mixer is perfectly balanced. But if the input signal is nonstationary or the mixer is not well balanced, then the demodulator output can be improper. The reason of this can be interpreted as transients of the demodulator filters, which produces improper output if the input changes its power with time; particularly if the signal power changes much within time corresponding to the length of the filter impulse response.

The result is important, because in the pulse Doppler radar the unwanted component of the input signal can be modelled as the sum of a stationary receiver noise, nonstationary clutter echoes, and sometimes active interferences (which can also be nonstationary). After quadrature demodulator, only the thermal noise component is exactly circular, while clutter echoes and nonstationary interferences will be *improper*.

It was also shown that impropriety of the demodulated signal means that it is sometimes "almost real". It seems that this could be used to improve detection and estimation algorithms (which for example could pay greater attention to the imaginary part of the useful signal which is less noisy), but it is not yet clear if this would give any real improvement in practice.

References

[1] F. Barbaresco and P. Chevalier, "Noncircularity exploitation in signal processing overview and application to radar," in *Waveform Diversity Digital Radar Conference - Day 1: Waveform Diversity Design, 2008 IET*, 2008, pp. 1–6.

[2] P. Chevalier, A. Blin, F. Pipon, and F. Delaveau, "GLRT-based array receivers to detect a known signal corrupted by noncircular interferences," in *Proc. EUSIPCO*, Poznań, Poland, Sept. 2007.

[3] F. D. Neeser and J. L. Massey, "Proper complex random processes with applications to information theory," *IEEE Transactions on Information Theory*, vol. 39, no. 4, pp. 1293–1302, 1993.

[4] P. Schreier and L. Scharf, *Statistical Signal Processing of Complex-Valued Data: The Theory of Improper and Noncircular Signals*. Cambridge University Press, 2010.

[5] B. Picinbono, "Second-order complex random vectors and normal distributions," *IEEE Transactions on Signal Processing*, vol. 44, no. 10, pp. 2637–2640, 1996.

[6] ——, "On circularity," *IEEE Transactions on Signal Processing*, vol. 42, no. 12, pp. 3473–3482, 1994.

[7] P. Schreier, L. Scharf, and A. Hanssen, "A generalized likelihood ratio test for impropriety of complex signals," *Signal Processing Letters, IEEE*, vol. 13, no. 7, pp. 433–436, 2006.

[8] J.-P. Delmas and H. Abeida, "On the degree of second-order noncircularity of complex random variables," in *Acoustics, Speech and Signal Processing, 2008. ICASSP 2008. IEEE International Conference on*, 2008, pp. 3905–3908.

[9] J. Eriksson, E. Ollila, and V. Koivunen, "Essential statistics and tools for complex random variables," *IEEE Transactions on Signal Processing*, vol. 58, no. 10, pp. 5400–5408, 2010.

[10] M. Zakai, "The representation of narrow-band processes," *Information Theory, IRE Transactions on*, vol. 8, no. 4, pp. 323–325, 1962.