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SOME PROPERTIES OF OPENLY ϱ-CONTINUOUS FUNCTIONS

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Abstract

In the paper we present definition and some properties of openly ϱ-upper continuous functions. Connections with ϱ-upper continuous and porouscontinuous functions are studied.

1. Preliminaries

In the paper we apply standard symbols and notations. By $\mathbb{R}$ we denote the set of all real numbers, by $\mathbb{N}$ we denote the set of all positive integers. The symbol $\lambda(\cdot)$ stands for the Lebesgue measure on $\mathbb{R}$. By $\operatorname{int} A$ we denote the interior of a set A. In the whole paper $I = (a, b)$ is an open interval (not necessarily bounded) and f is a real-valued function defined on I. By $f|A$ we denote the restriction of f to a set $A \subset I$. Symbol $|J|$ stands for length of a interval J.

Let E be a measurable subset of $\mathbb{R}$ and let $x \in \mathbb{R}$. According to [4], the numbers

$$\underline{d}^+(E, x) = \liminf_{t \to 0^+} \frac{\lambda(E \cap [x, x+t])}{t}$$

and

$$\overline{d}^+(E, x) = \limsup_{t \to 0^+} \frac{\lambda(E \cap [x, x+t])}{t}$$

are called the right lower density of E at x and right upper density of E at x, respectively. The left lower and left upper densities of E at x are defined analogously. If

$$\underline{d}^+(E, x) = \overline{d}^+(E, x) \qquad \left(\underline{d}^-(E, x) = \overline{d}^-(E, x)\right)$$

then we call these numbers the right density (left density) of E at x and denote it by $d^+(E, x)$ ($d^-(E, x)$). The numbers

$$\overline{d}(E, x) = \max\{\overline{d}^+(E, x), \overline{d}^-(E, x)\}$$

and

$$\underline{d}(E,x)=\min\{\underline{d}^+(E,x),\underline{d}^-(E,x)\}$$

are called the upper and lower density of E at x, respectively.

If $\overline{d}(E,x)=\underline{d}(E,x)$ then we call this number the density of E at x and denote it by $d(E,x)$. If $d(E,x)=1$ then we say that x is a point of density of E.

First, we recall the notion of ϱ-upper continuity.

Definition 1.1. [6] Let E be a measurable subset of $\mathbb{R}$, $x\in\mathbb{R}$ and $0<\varrho\leq 1$. We say that x is a point of ϱ-type upper density of E if either $\overline{d}(E,x)>\varrho$ if $\varrho<1$ or $\overline{d}(E,x)=1$ if $\varrho=1$.

Definition 1.2. [6] The function $f\colon I\to\mathbb{R}$ is called ϱ-upper continuous at $x\in I$ provided that there is a measurable set $E\subset I$ such that x is a point of ϱ-type upper density of E, $x\in E$ and $f|E$ is continuous at x. If f is ϱ-upper continuous at each point of I then we say that f is ϱ-upper continuous.

By $\mathcal{UC}_\varrho$ we denote the class of all ϱ-upper continuous functions defined on I, whereas the symbol $\mathcal{UC}_\varrho(f)$ denotes the set of all points at which the function f is ϱ-upper continuous.

In an obvious way we define one-sided ϱ-upper continuity. Obviously f is ϱ-upper continuous at x if and only if it is ϱ-upper continuous at x on the right or on the left.

Definition 1.3. [7] Let E be a measurable subset of $\mathbb{R}$. Let $x\in\mathbb{R}$ and $0<\varrho\leq 1$. We say that x is a point of weakly ϱ-type upper density of E if $\overline{d}(E,x)\geq\varrho$.

Definition 1.4. [7] The function $f\colon I\to\mathbb{R}$ is called weakly ϱ-upper continuous at $x\in I$ provided that there is a measurable set $E\subset I$ such that x is a point of weakly ϱ-type upper density of E, $x\in E$ and $f|_E$ is continuous at x. If f is weakly ϱ-upper continuous at each point of I then we say that f is weakly ϱ-upper continuous.

By $w\mathcal{UC}_\varrho$ we denote the class of all weakly ϱ-upper continuous functions defined on I, whereas the symbol $w\mathcal{UC}_\varrho(f)$ denotes the set of all points at which the function f is weakly ϱ-upper continuous.

In an obvious way we define one-sided weakly ϱ-upper continuity. Observe that f is weakly ϱ-upper continuous at x if and only if it is weakly ϱ-upper continuous at x on the right or on the left.

We recall the definition of approximate continuity.

Definition 1.5. [4] The function $f\colon I\to\mathbb{R}$ is called approximately continuous at $x\in I$ provided that there is a measurable set $E\subset I$ such that x is a point of density of E, $x \in E$ and $f|_E$ is continuous at x. If f is approximately continuous at each point of I then we say that f is approximately continuous.

By $\mathcal{A}$ we denote the class of all approximately continuous functions.

In [1] J. Borsík and J. Holos introduced path continuity connected with the notion of porosity. For a set $A \subset \mathbb{R}$ and an open interval $I \subset \mathbb{R}$ let $\Lambda(A, I)$ denote the length of the largest subinterval of I having an empty intersection with A. Let $x \in \mathbb{R}$. Then, according to [1], [5], the numbers

$$p^+(A, x) = \limsup_{t \to 0^+} \frac{\Lambda(A, (x, x+t))}{t}$$

and

$$p^-(A, x) = \limsup_{t \to 0^+} \frac{\Lambda(A, (x-t, x))}{t}$$

are called the right-porosity of the set A at x and the left-porosity of the set A at x, respectively. The porosity of the set A at x is defined as

$$p(A, x) = \max\{p^-(A, x), p^+(A, x)\}.$$

The set A is called right-porous at a point x if $p^+(A, x) > 0$, left-porous at a point x if $p^-(A, x) > 0$ and porous at a point x if $p(A, x) > 0$. The set A is called porous if A is porous at each point $x \in A$. The set A is called strongly porous at a point x if $p^+(A, x) = 1$ or $p^-(A, x) = 1$.

Definition 1.6. [1] Let $r \in [0, 1)$, $A \subset \mathbb{R}$, $x \in A$. The point x will be called a point of π_r-density of the set A if $p(\mathbb{R} \setminus A, x) > r$.

Let $r \in (0, 1]$, $A \subset \mathbb{R}$, $x \in A$. The point $x \in A$ will be called a point of μ_r-density of the set A if $p(\mathbb{R} \setminus A, x) \geq r$.

Definition 1.7. [1] Let $r \in [0, 1)$, $x \in \mathbb{R}$. The function $f \colon \mathbb{R} \to \mathbb{R}$ will be called

1. $\mathcal{P}_r$-continuous at x if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of π_r-density of A and $f|A$ is continuous at x,
2. $\mathcal{S}_r$-continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of π_r-density of A and $f(A) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.

Let $r \in (0, 1]$, $x \in \mathbb{R}$. The function $f \colon \mathbb{R} \to \mathbb{R}$ will be called

3. $\mathcal{M}_r$-continuous at a point x if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of μ_r-density of A and $f|A$ is continuous at x,
4. $\mathcal{N}_r$-continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of μ_r-density of A and $f(A) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.

All these functions will be called porously continuous. Symbols $\mathcal{P}_r(f)$, $\mathcal{S}_r(f)$, $\mathcal{M}_r(f)$, $\mathcal{N}_r(f)$ will denote the sets of all points at which the function

f is $\mathcal{P}_r$-continuous, $\mathcal{S}_r$-continuous, $\mathcal{M}_r$-continuous, $\mathcal{N}_r$-continuous.

2. Open ϱ-upper continuous functions

We define new classes of functions lying between the class of ϱ-upper continuous and the class of porously continuous functions.

Definition 2.1. Let $\varrho \in [0,1)$, $x \in I$. The function $f\colon I \to \mathbb{R}$ is called

1. $\mathscr{P}_\varrho$-continuous at x if there exists an open set $U \subset \mathbb{R}$ such that $\overline{d}(U,x) > \varrho$ and $f|U \cup \{x\}$ is continuous at x.
2. $\mathscr{S}_\varrho$-continuous at x if for each $\varepsilon > 0$ there exists an open set $U \subset I$ such that $\overline{d}(U,x) > \varrho$ and $f(U) \subset (f(x)-\varepsilon, f(x)+\varepsilon)$.

Let $\varrho \in (0,1]$, $x \in I$. The function $f\colon I \to \mathbb{R}$ is called

1. $\mathscr{M}_\varrho$-continuous at x if there exists an open set $U \subset I$ such that $\overline{d}(U,x) \geq \varrho$ and $f|U \cup \{x\}$ is continuous at x.
2. $\mathscr{N}_\varrho$-continuous at x if for each $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}$ such that $\overline{d}(U,x) \geq \varrho$ and $f(U) \subset (f(x)-\varepsilon, f(x)+\varepsilon)$.

We denote the class of all $\mathscr{P}_\varrho$-continuous, $\mathscr{S}_\varrho$-continuous, $\mathscr{M}_\varrho$-continuous, $\mathscr{N}_\varrho$-continuous by $\mathscr{P}_\varrho$, $\mathscr{S}_\varrho$, $\mathscr{M}_\varrho$, $\mathscr{N}_\varrho$, respectively. Symbols $\mathscr{P}_\varrho(f)$, $\mathscr{S}_\varrho(f)$, $\mathscr{M}_\varrho(f)$, $\mathscr{N}_\varrho(f)$ denotes the sets of all points at which the function f is $\mathscr{P}_\varrho$-continuous, $\mathscr{S}_\varrho$-continuous, $\mathscr{M}_\varrho$-continuous, $\mathscr{N}_\varrho$-continuous, respectively.

Remark 2.1. *In [3] similar functions are considered. But in the definitions $A_r(f)$ and $B_r(f)$ in [2] symmetric density is used. And there is connections between $A_r(f)$, $B_r(f)$, ϱ-upper continuity and porouscontinuity.*

Some obvious relations between sets of open ϱ-continuity of f will be described in the following propositions.

Proposition 2.1. *Let $f\colon I \to \mathbb{R}$. Then*

1. $\mathscr{P}_{\varrho_2}(f) \subset \mathscr{P}_{\varrho_1}(f)$ and $\mathscr{S}_{\varrho_2}(f) \subset \mathscr{S}_{\varrho_1}(f)$ for $0 \leq \varrho_1 < \varrho_2 < 1$,
2. $\mathscr{M}_{\varrho_2}(f) \subset \mathscr{M}_{\varrho_1}(f)$ and $\mathscr{N}_{\varrho_2}(f) \subset \mathscr{N}_{\varrho_1}(f)$ for $0 < \varrho_1 < \varrho_2 \leq 1$,
3. $\mathscr{P}_\varrho(f) \subset \mathscr{M}_\varrho(f)$ and $\mathscr{S}_\varrho(f) \subset \mathscr{N}_\varrho(f)$ for $0 < \varrho < 1$,
4. $\mathscr{M}_{\varrho_2}(f) \subset \mathscr{P}_{\varrho_1}(f)$ and $\mathscr{N}_{\varrho_2}(f) \subset \mathscr{S}_{\varrho_1}(f)$ for $0 \leq \varrho_1 < \varrho_2 \leq 1$,
5. $\mathscr{P}_\varrho(f) \subset \mathscr{S}_\varrho(f)$ for $0 \leq \varrho < 1$,
6. $\mathscr{M}_\varrho(f) \subset \mathscr{N}_\varrho(f)$ for $0 < \varrho \leq 1$.

Proposition 2.2. *Let $f\colon I \to \mathbb{R}$, $\varrho \in [0,1)$. Then $\mathscr{P}_\varrho(f) \subset \mathcal{UC}_\varrho(f)$.*

Proposition 2.3. *Let $f\colon I \to \mathbb{R}$, $\varrho \in (0,1]$. Then $\mathscr{M}_\varrho(f) \subset u\mathcal{UC}_\varrho(f)$.*

The following two propositions follow directly from the definitions.

Proposition 2.4. *Let $\varrho \in [0,1)$, $x \in I$. If $f\colon I \to \mathbb{R}$ is continuous at x from the left or from the right then $x \in \mathscr{S}_\varrho(f) \cap \mathscr{P}_\varrho(f)$.*

Proposition 2.5. *Let $\varrho \in (0,1]$, $x \in I$. If $f\colon I \to \mathbb{R}$ is continuous at x from the left or from the right then $x \in \mathscr{N}_\varrho(f) \cap \mathscr{M}_\varrho(f)$.*

We will show that approximate continuity does not imply any open ϱ-upper continuity. To this end we need well known theorem of Zahorski.

Theorem 2.1. [4] *Let E be a set of F_σ type such that $d(E,x) = 1$ for all $x \in E$. There exists an approximately continuous function $f\colon E \to \mathbb{R}$ such that $0 < f(x) \leq 1$ for all $x \in E$ and $f(x) = 0$ for all $x \notin E$. Then the function f is also upper semi-continuous.*

Example 2.1. We will give an example of approximately continuous function which does not belong to $\mathscr{S}_0$.

Let $E \subset \mathbb{R}$ be nowhere dense closed set with positive Lebesgue measure. Let $L(E)$ be a set of density points of E. Then $\lambda(L(E)) = \lambda(E)$, by Lebesgue Density Theorem [4]. Let $F \subset L(E)$ be a set of F_σ type such that $\lambda(F) = \lambda(L(E))$. Then $F \subset L(F)$. By Theorem 2.1, there exists an approximately continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) \in (0,1]$ for all $x \in F$ and $f(x) = 0$ for all $x \in \mathbb{R} \setminus F$. Let $x_0 \in F$, so $f(x_0) > 0$. For all $0 < \varepsilon < f(x_0)$ we have

$$\{x\colon |f(x) - f(x_0)| < \varepsilon\} \subset F.$$

The set F is nowhere dense, so $\operatorname{int}\{x\colon |f(x) - f(x_0)| < \varepsilon\} = \emptyset$. Hence f is not $\mathscr{S}_0$-continuous at x_0.

The class of all weakly ϱ-upper continuous functions consists the class of all Lebesgue measurable functions [7], so all considered classes of functions $\mathscr{P}_\varrho$,$\mathscr{S}_\varrho$,$\mathscr{M}_\varrho$, $\mathscr{N}_\varrho$ consist the class of all Lebesgue measurable functions.

Lemma 2.1. *Let $U \subset \mathbb{R}$ be open set, $x_0 \in \mathbb{R}$. Then*

$$\overline{d}(U, x_0) \geq p(\mathbb{R} \setminus U, x_0).$$

Proof. Let $p(\mathbb{R} \setminus U, x_0) = c$. Then $p^+(\mathbb{R} \setminus U, x_0) = c$ or $p^-(\mathbb{R} \setminus U, x_0) = c$. Without loss of generality we may assume that $p^+(\mathbb{R} \setminus U, x_0) = c$. Therefore there is decreasing sequence $\{h_n\}_{n\geq 1}$ of positive numbers such that $\lim\limits_{n\to\infty} h_n = 0$ and

$$p^+(\mathbb{R} \setminus U, x_0) = \lim_{n\to\infty} \frac{\Lambda\left(\mathbb{R} \setminus U, (x_0, x_0 + h_n)\right)}{h_n}.$$

Therefore there is a sequence of open intervals $\{I_n\}_{n\geq 1}$ such that $I_n \cap (\mathbb{R} \setminus U) = \emptyset$ and $|I_n| = \Lambda\left(\mathbb{R} \setminus U, (x_0, x_0 + h_n)\right)$. Then $I_n \subset U$ for each $n \geq 1$

and

$$\overline{d}(U,x_0) \geq \overline{d}^+(U,x_0) \geq \overline{d}\left(\bigcup_{k\geq 1} I_k, x_0\right) \geq$$

$$\geq \limsup_{n\to\infty} \frac{\lambda\left(\bigcup_{k\geq 1} I_k \cap [x_0, x_0+h_n]\right)}{h_n} =$$

$$= \limsup_{n\to\infty} \frac{\lambda\left(\bigcup_{k\geq n} I_k\right)}{h_n} \geq \limsup_{n\to\infty} \frac{\lambda(I_n)}{h_n} =$$

$$= \limsup_{n\to\infty} \frac{\Lambda\left(\mathbb{R}\setminus U, (x_0, x_0+h_n)\right)}{h_n} = p^+(\mathbb{R}\setminus U, x_0) = p(\mathbb{R}\setminus U, x_0).$$

□

The next theorem follows immediately from Lemma 2.1

Theorem 2.2. *Let $f\colon I \to \mathbb{R}$. Then*

1. $\mathcal{P}_\varrho(f) \subset \mathscr{P}_\varrho(f)$ *for* $\varrho \in [0,1)$,
2. $\mathcal{S}_\varrho(f) \subset \mathscr{S}_\varrho(f)$ *for* $\varrho \in [0,1)$,
3. $\mathcal{M}_\varrho(f) \subset \mathscr{M}_\varrho(f)$ *for* $\varrho \in (0,1]$,
4. $\mathcal{N}_\varrho(f) \subset \mathscr{N}_\varrho(f)$ *for* $\varrho \in (0,1]$.

We will show, in the next example, that all inclusions in Theorem 2.2 are proper.

Example 2.2. We will construct $f \in \mathscr{M}_1$ such that $0 \notin \mathcal{S}_0(f)$, e.g. $\mathscr{M}_1(f)\setminus \mathcal{S}_0(f) \neq \emptyset$.

Let $\{x_n\}_{n\geq 1}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} x_n = 0$, $x_n - x_{n+1} \geq x_{n+1} - x_{n+2}$ and $\lim_{n\to\infty} \frac{x_n - x_{n+1}}{x_{n+1}} = 0$ (for example, $x_n = \frac{1}{n}$). Let $y_n, z_n \in (x_{n+1}, x_n)$ be such that $x_n - z_n = \frac{1}{n+5}(x_n - x_{n+1})$, $y_n - x_{n+1} = \frac{1}{n+5}(x_n - x_{n+1})$. Thus $x_{n+1} < y_n < z_n < x_n$ for each $n \geq 1$. Notice that $z_n - y_n = \frac{n+3}{n+5}(x_n - x_{n+1})$ for each $n \geq 1$.. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \bigcup_{n=1}^\infty [y_n, z_n] \cup \{0\}, \\ 1 & \text{if } x \in (0,\infty) \setminus \bigcup_{n=1}^\infty [y_n, z_n], \\ f(-x) & \text{if } x \in (\infty, 0). \end{cases}$$

Obviously, at each $x \neq 0$ the function f is continuous from the right or from the left, and therefore $\mathbb{R}\setminus\{0\} \subset \mathscr{M}_1(f)$. Let $U = \bigcup_{n=1}^\infty (y_n, z_n)$. Then for each $n \geq 1$ we have

$$\frac{\lambda(U \cap [0, x_n])}{x_n} = \frac{\sum_{k=n}^{\infty} \lambda([y_k, z_k])}{x_n} \geq \frac{\sum_{k=n}^{\infty} \frac{k+5}{k+7}(x_k - x_{k+1})}{y_n} \geq$$

$$\geq \frac{n+5}{n+7} \frac{\sum_{k=n}^{\infty} (x_k - x_{k+1})}{y_n} = \frac{n+5}{n+7} \frac{x_n}{x_n} = \frac{n+5}{n+7}.$$

Therefore

$$d(U, 0) = d^+(U, 0) \geq \liminf_{n\to\infty} \frac{\lambda(U \cap [0, y_n])}{y_n} = \liminf_{n\to\infty} \frac{\lambda(U \cap [0, x_n])}{y_n} \geq$$

$$\geq \liminf_{n\to\infty} \frac{n+5}{n+7} = 1.$$

Hence $d(U, 0) = 1$ and f is approximately continuous at 0. Moreover, U is open, so $0 \in \mathscr{M}_1(f)$.

For each $\varepsilon \in (0, 1)$, $\mathbb{R} \setminus \{x \colon |f(x) - f(0)| < \varepsilon\} \subset \mathbb{R} \setminus \bigcup_{n=1}^{\infty} \{x_n\}$. Let $h \in [x_{n+1}, x_n]$. Since $\frac{\Lambda\left(\mathbb{R}\setminus \bigcup_{n=1}^{\infty} \{x_n\}, (0,h)\right)}{h} \leq \frac{x_n - x_{n+1}}{x_{n+1}}$ and $\lim_{n\to\infty} \frac{x_n - x_{n+1}}{x_{n+1}} = 0$, we deduce

$$p(\mathbb{R} \setminus \{x \colon |f(x) - f(0)| < \varepsilon\}, 0) = \lim_{h\to 0^+} \frac{\Lambda\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty} \{x_n\}, (0, h)\right)}{h} = 0.$$

Thus $0 \notin \mathcal{S}_0(f)$.

Lemma 2.2. *Let $\varrho \in [0, 1]$ and $x \in \mathbb{R}$. Let $\{E_n \colon n \in \mathbb{N}\}$ be a descending family of open sets such that $x \in \bigcap_{n=1}^{\infty} E_n$, $\overline{d}(E_n, x) \geq \varrho$ for $n \geq 1$. Then there exists an open set E such that $\overline{d}(E, x) \geq \varrho$ and for every positive integer n there exists $\delta_n > 0$ such that $E \cap (x - \delta_n, x + \delta_n) \subset E_n$.*

Proof. By assumptions, $\overline{d}(E_n, x) \geq \varrho$ for $n \geq 1$. Therefore $\overline{d}^+(E_n, x) \geq \varrho$ or $\overline{d}^-(E_n, x) \geq \varrho$ for each n. Hence there exists an infinite family $\{E_{n_k} \colon k \in \mathbb{N}\}$ such that $\overline{d}^+(E_{n_k}, x) \geq \varrho$ for all $k \geq 1$ or $\overline{d}^-(E_{n_k}, x) \geq \varrho$ for all $k \geq 1$. Without loss of generality we may assume that the first possibility occurs. Then $\overline{d}^+(E_n, x) \geq \varrho$ for all $n \geq 1$, because $\{E_n \colon n \in \mathbb{N}\}$ is a descending family.

We shall construct inductively a decreasing sequence $\{x_n\}_{n\geq 1}$ converging to x such that

$$\frac{\lambda(E_n \cap [x_{n+1}, x_n])}{x_n - x} > \varrho\Big(1 - \frac{1}{2^n}\Big) \quad \text{for } n \geq 1. \tag{1}$$

Let $x_1 > x$ be any point for which $\frac{\lambda(E_1 \cap [x, x_1])}{x_1 - x} > \varrho\left(1 - \frac{1}{2}\right)$ and $x_1 - x < 1$. Next, we can find $x_2 \in (x, x_1)$ such that $\frac{\lambda(E_1 \cap [x_2, x_1])}{x_1 - x} > \varrho\left(1 - \frac{1}{2}\right)$, $\frac{\lambda(E_2 \cap [x, x_2])}{x_2 - x} > \varrho\left(1 - \frac{1}{4}\right)$ and $x_2 - x < \frac{1}{2}$. There exists $x < x_3 < x_2$ for which $\frac{\lambda(E_2 \cap [x_3, x_2])}{x_2 - x} > \varrho\left(1 - \frac{1}{4}\right)$, $\frac{\lambda(E_3 \cap [x, x_3])}{x_3 - x} > \varrho\left(1 - \frac{1}{8}\right)$ and $x_3 - x < \frac{1}{3}$.

Assume that points $x_1, x_2, \ldots, x_n$ with properties $x < x_n < \ldots < x_1$, $\frac{\lambda(E_{i-1} \cap [x_i, x_{i-1}])}{x_{i-1} - x} > \varrho\left(1 - \frac{1}{2^{i-1}}\right)$ for $i \in \{2, \ldots, n\}$, $\frac{\lambda(E_i \cap [x, x_i])}{x_i - x} > \varrho\left(1 - \frac{1}{2^i}\right)$ and $x_i - x < \frac{1}{i}$ for $i \in \{1, 2, \ldots, n\}$ are chosen. Then there exists $x < x_{n+1} < x_n$ such that $\frac{\lambda(E_n \cap [x_{n+1}, x_n])}{x_n - x} > \varrho\left(1 - \frac{1}{2^n}\right)$, $\frac{\lambda(E_{n+1} \cap [x, x_{n+1}])}{x_{n+1} - x} > \varrho\left(1 - \frac{1}{2^{n+1}}\right)$ and $x_{n+1} - x < \frac{1}{n+1}$.
Thus we have constructed inductively the sequence $\{x_n\}_{n\geq 1}$ satisfying condition (1).

Let $E = \bigcup_{n=1}^{\infty} \big(E_n \cap (x_{n+1}, x_n)\big)$. Obviously, E is open. Since

$$\limsup_{n\to\infty} \frac{\lambda(E \cap [x, x_n])}{x_n - x} \geq \limsup_{n\to\infty} \frac{\lambda(E_n \cap [x_{n+1}, x_n])}{x_n - x} \geq \geq \lim_{n\to\infty} \varrho\Big(1 - \frac{1}{2^n}\Big) = \varrho,$$

we obtain $\overline{d}(E, x) \geq \varrho$.

By the definition of the set E, for each n there exists $\delta_n = x_n - x > 0$ such that $E \cap (x - \delta_n, x + \delta_n) = E \cap [x, x_n) \subset E_n$. The proof is completed. $\square$

Theorem 2.3. *Let $f\colon I \to \mathbb{R}$ and $\varrho \in (0, 1]$. Then $\mathscr{M}_\varrho(f) = \mathscr{N}_\varrho(f)$.*

Proof. From Proposition 2.1 it is clear that it is sufficient to show $\mathscr{N}_\varrho(f) \subset \mathscr{M}_\varrho(f)$. Let $x_0 \in \mathscr{N}_\varrho(f)$. Then for each positive integer n there is an open set E_n such that $\overline{d}(E_n, x_0) \geq \varrho$ and $f(E_n) \subset \left(f(x_0) - \frac{1}{n}, f(x_0) + \frac{1}{n}\right)$. By Lemma 2.2 for sets E_n, we can construct an open set E such that $\overline{d}(E, x_0) \geq \varrho$ and for each n there exists $\delta_n > 0$ for which $E \cap (x_0 - \delta_n, x_0 + \delta_n) \subset E_n$. The last condition implies that $f|E \cup \{x_0\}$ is continuous at x_0. Thus $x_0 \in \mathscr{M}_\varrho(f)$. $\square$

Theorem 2.4. *Let $\varrho \in [0, 1)$, $f\colon I \to \mathbb{R}$, $x_0 \in I$. Then $x_0 \in \mathscr{P}_\varrho(f)$ if and only if*

$$\lim_{\varepsilon\to 0^+} \overline{d}\left(\operatorname{int}\{x\colon |f(x) - f(x_0)| < \varepsilon\}, x_0\right) > \varrho.$$

Proof. Assume that f is $\mathscr{P}_\varrho$-continuous at x_0. Let $U \subset I$ be an open set such that $\overline{d}(U, x_0) > \varrho$ and $f|U \cup \{x_0\}$ is continuous at x_0. Let $\varepsilon > 0$. Since $f|U \cup \{x_0\}$ is continuous at x_0, we can find $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \cap U \subset \{x \colon |f(x_0) - f(x)| < \varepsilon\}$. Hence

$$\overline{d}\big(\{x \in I \colon |f(x_0) - f(x)| < \varepsilon\}, x_0\big) \geq \\ \geq \overline{d}\big(\operatorname{int}\{x \in U \colon |f(x_0) - f(x)| < \varepsilon\}, x_0\big) = \overline{d}(U, x_0)$$

for each $\varepsilon > 0$. Therefore

$$\lim_{\varepsilon \to 0^+} \overline{d}\big(\operatorname{int}\{x \in I \colon |f(x_0) - f(x)| < \varepsilon\}, x_0\big) \geq \overline{d}(U, x_0) > \varrho.$$

Finally, assume that

$$\varrho_1 = \lim_{\varepsilon \to 0^+} \overline{d}\big(\operatorname{int}\{x \in I \colon |f(x_0) - f(x)| < \varepsilon\}, x_0\big) > \varrho.$$

Using Lemma 2.2 for sets $E_n = \{x \in I \colon |f(x_0) - f(x)| < \frac{1}{n}\}$ we can construct an open set U such that $\overline{d}(U, x_0) \geq \varrho_1 > \varrho$ and for each n there exists $\delta_n > 0$ for which $U \cap (x_0 - \delta_n, x_0 + \delta_n) \subset E_n$. The last condition implies that $f|_{U \cup \{x_0\}}$ is continuous at x_0. It follows that f is $\mathscr{P}_\varrho$-continuous at x_0, what was to be shown. □

Theorem 2.5. *Let* $0 < \varrho_1 < \varrho_2 < 1$ *and* $f \colon I \to \mathbb{R}$. *Then*

$$\mathscr{M}_1(f) = \mathscr{N}_1(f) \subset \mathscr{P}_{\varrho_2}(f) \subset \mathscr{S}_{\varrho_2}(f) \subset \mathscr{M}_{\varrho_2}(f) = \\ = \mathscr{N}_{\varrho_2}(f) \subset \mathscr{P}_{\varrho_1}(f) \subset \mathscr{P}_0(f) \subset \mathscr{S}_0(f).$$

Proof. The proof follows immediately from Proposition 2.1 and Theorem 2.3. □

Theorem 2.6. *Let* $0 < \varrho_1 < \varrho_2 < 1$. *Then*

$$\mathscr{M}_1 = \mathscr{N}_1 \subset \mathscr{P}_{\varrho_2} \subset \mathscr{S}_{\varrho_2} \subset \mathscr{M}_{\varrho_2} = \mathscr{N}_{\varrho_2} \subset \mathscr{P}_{\varrho_1} \subset \mathscr{P}_0 \subset \mathscr{S}_0$$

and all incusions are proper.

Proof. All inclusions follow from the previous theorem. We will only show (in Examples 2.3-2.5) that they are proper. □

Example 2.3. Let $0 \leq \varrho_1 < \varrho_2 \leq 1$. We will construct $f \colon \mathbb{R} \to \mathbb{R}$ such that $f \in \mathscr{P}_{\varrho_1} \setminus \mathscr{M}_{\varrho_2}$.

We can find a sequence $\{[a_n, b_n]\}_{n \geq 1}$ of pairwise disjoint closed intervals such that $0 < b_{n+1} < a_n < b_n$ for each n and $\overline{d}^+ \left(\bigcup_{n=1}^{\infty} [a_n, b_n], 0\right) = \frac{\varrho_1 + \varrho_2}{2}$. Denote $I_n = [a_n, b_n]$ for every $n \geq 1$. Define a function $f \colon \mathbb{R} \to \mathbb{R}$ letting

$$f(x)=\begin{cases}0 & \text{if } x\in\{0\}\cup\bigcup_{n=1}^{\infty} I_n,\\ 1 & \text{if } x\in(-\infty,0)\cup\bigcup_{n=1}^{\infty}(b_{n+1},a_n)\cup(b_1,\infty).\end{cases}$$

The function f is continuous from the left or from the right at every point except 0. Hence $\mathbb{R}\setminus\{0\}\subset\mathscr{P}_{\varrho_1}(f)$. If $E=\bigcup_{n=1}^{\infty}(a_n,b_n)$ then E is open and the function f restricted to $E\cup\{0\}$ is constant, so in particular, it is continuous at zero. Moreover,

$$\overline{d}(E,0)=\overline{d}^{+}\left(\bigcup_{n=1}^{\infty}(a_n,b_n),0\right)=\overline{d}^{+}\left(\bigcup_{n=1}^{\infty}I_n,0\right)=\frac{\varrho_1+\varrho_2}{2}>\varrho_1.$$

Hence $0\in\mathscr{P}_{\varrho_1}(f)$ and $f\in\mathscr{P}_{\varrho_1}$.

But

$$\overline{d}^{+}(\{x\colon f(x)<1\},0)=\overline{d}^{+}\left(\bigcup_{n=1}^{\infty}I_n,0\right)=\frac{\varrho_1+\varrho_2}{2}<\varrho_2.$$

Moreover $\overline{d}^{-}(\{x\colon f(x)<1\},0)=0$. Hence $\overline{d}(\{x\colon f(x)<1\},0)<\varrho_2$ and f is not $\mathscr{M}_{\varrho_2}$-continuous at 0. Therefore $0\notin\mathscr{M}_{\varrho_2}(f)$ and $f\notin\mathscr{M}_{\varrho_2}$.

Example 2.4. Let $\varrho\in(0,1)$. We will construct $f\colon\mathbb{R}\to\mathbb{R}$ such that $f\in\mathscr{M}_{\varrho}\setminus\mathscr{S}_{\varrho}$.

We can find a sequence $\{[a_n,b_n]\}_{n\geq 1}$ of pairwise disjoint closed intervals such that $0<b_{n+1}<a_n<b_n$ for each n and $\overline{d}^{+}(\bigcup_{n=1}^{\infty}[a_n,b_n],0)=\varrho$. Define a function $f\colon\mathbb{R}\to\mathbb{R}$ letting

$$f(x)=\begin{cases}0 & \text{if } x\in\{0\}\cup\bigcup_{n=1}^{\infty}[a_n,b_n],\\ 1 & \text{if } x\in(-\infty,0)\cup\bigcup_{n=1}^{\infty}(b_{n+1},a_n)\cup[b_1,\infty).\end{cases}$$

Observe that the function f is continuous from the left or from the right at every point except 0. Hence $\mathbb{R}\setminus\{0\}\subset\mathscr{M}_{\varrho}(f)$.Denote $E=\bigcup_{n=1}^{\infty}(a_n,b_n)$. Then the function $f|E\cup\{0\}$ is constant, so in particular, it is continuous at zero. Moreover,

$$\overline{d}(E,0)\geq\overline{d}^{+}(E,0)=\overline{d}^{+}\left(\bigcup_{n=1}^{\infty}[a_n,b_n],0\right)=\varrho.$$

Hence $0\in\mathscr{M}_{\varrho}(f)$ and $f\in\mathscr{M}_{\varrho}$.

Let $\varepsilon \in (0,1)$. Since

$$\overline{d}\left(\{x\colon |f(x)-f(0)|<\varepsilon\},0\right)=\overline{d}^{+}\left(\bigcup_{n=1}^{\infty}[a_n,b_n],0\right)=\varrho$$

we conclude that $0 \notin \mathscr{S}_\varrho(f)$ and $f \notin \mathscr{S}_\varrho$.

Example 2.5. Let $\varrho \in [0,1)$. We will construct $f\colon \mathbb{R}\to\mathbb{R}$ such that $f \in \mathscr{S}_\varrho \setminus \mathscr{P}_\varrho$.

We can find a sequence $\{[a_n,b_n]\}_{n\geq 1}$ of pairwise disjoint closed intervals such that $0<b_{n+1}<a_n<b_n$ for each n and $\overline{d}^{+}\left(\bigcup_{n=1}^{\infty}[a_n,b_n],0\right)=\varrho$. Define a function $f\colon\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\begin{cases} 0 & \text{if } x\in\{0\}\cup(b_1,\infty)\cup\bigcup\limits_{n=1}^{\infty}[a_n,b_n),\\ 1 & \text{if } x\in(-\infty,0)\cup\bigcup\limits_{n=2}^{\infty}\{b_n\},\\ \frac{a_n-x}{a_n-b_{n+1}} & \text{if } x\in(b_{n+1},a_n),\, n\geq 1.\end{cases}$$

The function f is continuous from the right at every point except 0. Hence $\mathbb{R}\setminus\{0\}\subset\mathscr{S}_\varrho(f)$. Let $U_\varepsilon=\{x\colon |f(x)-f(0)|<\varepsilon\}\setminus\{0\}$ for each $\varepsilon>0$. Then $U_\varepsilon=\bigcup_{n=1}^{\infty}(a_n-\varepsilon(a_n-b_{n+1}),b_n)$. Hence U_ε is open. Moreover,

$$\begin{aligned}\overline{d}(U_\varepsilon,0)&=\overline{d}^{+}\left(\bigcup_{n=1}^{\infty}(a_n-\varepsilon(a_n-b_{n+1}),b_n),0\right)=\\ &=\limsup_{n\to\infty}\frac{\sum_{k=n}^{\infty}(b_k-a_k+\varepsilon(a_k-b_{k+1}))}{b_n}=\\ &=\limsup_{n\to\infty}\frac{\sum_{k=n}^{\infty}\big((1-\varepsilon)(b_k-a_k)+\varepsilon(b_k-b_{k+1})\big)}{b_n}=\\ &=\limsup_{n\to\infty}\left((1-\varepsilon)\frac{\sum_{k=n}^{\infty}(b_k-a_k)}{b_n}+\varepsilon\frac{b_n}{b_n}\right)=\\ &=(1-\varepsilon)\varrho+\varepsilon>\varrho.\end{aligned}$$

Therefore $0\in\mathscr{S}_\varrho(f)$ and $f\in\mathscr{S}_\varrho$.

On the other hand,

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\overline{d}(\operatorname{int}\{x\colon |f(x)-f(0)|>\varepsilon\},0)=\\ &=\lim_{\varepsilon\to 0^+}\overline{d}^{+}\left(\bigcup_{n=1}^{\infty}(a_n-\varepsilon(a_n-b_{n+1}),b_n),0\right)=\lim_{\varepsilon\to 0^+}\big((1-\varepsilon)\varrho+\varepsilon\big)=\varrho.\end{aligned}$$

Hence $0\notin\mathscr{P}_\varrho(f)$ and $f\notin\mathscr{P}_\varrho$, by Theorem 2.4.

References

[1] J. Borsík, J. Holos, *Some properties of porouscontinuous fucntions*, Math. Slovaca **64** (2014), No. 3, 741-750.

[2] J. Borsík, *Some classes of strongly quasicontinuous functions*, Real Anal. Exchange **30** (2004-2005) 689-702.

[3] J. Borsík,*Quasicontinuous functions with small set of discontinuity points*, Modern Real Analysis, Lódź University Press, Lodz 2015, 11-26.

[4] A. M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Mathematics, Vol. **659**, Springer-Verlag Berlin Heidelberg New York, 1978.

[5] E. P. Dolženko, *Boundary properties of arbitrary functions*, Math. USSR Izv. **31** (1967), 3-14 (in Russian).

[6] S. Kowalczyk, K. Nowakowska, *A note on ϱ-upper continuous functions*, Tatra Mt. Math. Publ. **44** (2009), 153-158.

[7] K. Nowakowska, M. Turowska, *A note on weakly ϱ-upper continuous functions*, Jan Długosz University of Częstochowa, Scientific Issuse, Mathematics XIX (2014,) 195-206.

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