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CERTAIN GROUP DYNAMICAL SYSTEMS INDUCED BY HECKE ALGEBRAS

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Abstract. In this paper, we study dynamical systems induced by a certain group $\mathfrak{T}_N^K$ embedded in the Hecke algebra $\mathcal{H}(G_p)$ induced by the generalized linear group $G_p = GL_2(\mathbb{Q}_p)$ over the p-adic number fields $\mathbb{Q}_p$ for a fixed prime p. We study fundamental properties of such dynamical systems and the corresponding crossed product algebras in terms of free probability on the Hecke algebra $\mathcal{H}(G_p)$.

Keywords: free probability, free moments, free cumulants, Hecke algebra, normal Hecke subalgebra, free probability spaces, representations, operators, Hilbert spaces, dynamical systems, crossed product algebras.

Mathematics Subject Classification: 05E15, 11R47, 46L10, 47L30, 47L55.

1. INTRODUCTION

We have considered how *primes* (or *prime numbers*) act on *operator algebras*. The relations between primes and operator algebras have been studied in various different approaches. For instance, in [2], we studied how primes act "on" certain von Neumann algebras generated by p -adic and Adelic measure spaces. Also, the primes as operators in certain von Neumann algebras, have been studied in [3].

Independently, in [5] and [6], we have studied primes as linear functionals acting on *arithmetic functions*, i.e., each prime p induces a free-probabilistic structure $(\mathcal{A}, g_p)$ on the algebra $\mathcal{A}$ of all arithmetic functions. In such a case, one can understand arithmetic functions as *Krein-space operators* (for fixed primes), via certain representations (see [8]).

These studies are all motivated by well-known number-theoretic results (e.g., [1]) under free probability techniques (e.g., [12,13] and [15]).

In modern *number theory* and its applications, *p-adic analysis* provides important tools for studying *geometry at small distance* (e.g., [14]). it is not only interested in various mathematical fields but also in related scientific fields (e.g., [4,8]). The *p-adic* *number fields* $\mathbb{Q}_p$ and the *Adele ring* $\mathbb{A}_{\mathbb{Q}}$ play key roles in modern *number theory*; *analytic number theory*, *L-function theory*, and *algebraic geometry* (e.g., [2, 10] and [11]). Also, analysis on such Adelic structures gives a way of understanding vector analysis under a non-Archimedean metric (e.g., [2, 3, 7], and [8]).

In [7], the author and Gillespie consider free-probabilistic models on the classical *Hecke algebra* $\mathcal{H}(G_p)$, where G_p is the generalized linear group $GL_2(\mathbb{Q}_p)$ over the *p-adic number field* $\mathbb{Q}_p$, for primes p, and characterize the inner freeness of $\mathcal{H}(G_p)$. In [4], the author constructed suitable representations of Hecke algebras under our free-probabilistic settings. Under these representations, each element of $\mathcal{H}(G_p)$ is regarded as an operator on a certain Hilbert space $\mathfrak{H}_p$. The spectral properties; *self-adjointness*, *projection-property*, *normality*, *isometry-property*, *unitarity* and *hyponormality*; of such operators have been considered in [4]. As continuation, in [6], we concentrated on studying *partial isometries* induced by generating elements of the Hecke algebra $\mathcal{H}(G_p)$ under a representation introduced in [4]. In particular, we realize that there are partial isometries having same *initial and final projections* on $\mathfrak{H}_p$, and hence, they generate groups acting on "subspaces" of $\mathfrak{H}_p$, i.e., such partial isometries can be understood as unitaries on the "subspaces." Groups generated by the partial isometries and corresponding C^*-algebras have been considered in [6].

In this paper, we keep study *dynamical systems* induced by groups generated by partial isometries in the sense of [4], and their *crossed product algebras*. In particular, we study free probability, preserving number-theoretic data, on these crossed product algebras.

2. PRELIMINARIES

In this section, we briefly introduce fundamental background concepts we will use in the text.

2.1. HECKE ALGEBRAS AND FREE PROBABILITY

We refer readers [4] and [7] for more detailed information about motivations, definitions and background of our series of study: free probability, representation theory and operator algebra theory on Hecke algebras over p-adic number fields for primes p. We will use the same definitions and notations used in [4] and [7]. Of course, in the following text, we will introduce them precisely.

Also, readers can check fundamental analytic-and-combinatorial *free probability theory* from [13] and [15] (and the cited papers therein). Free probability is understood as the noncommutative operator-algebraic version of classical probability theory. The classical *independence* is replaces by the *freeness*. It has various applications not only in pure mathematics (e.g., [12]) but also in related topics in physics (e.g., [5, 6, 9] and [8]). In particular, we will use the combinatorial free probabilistic approach of *Speicher* (e.g., [13]). *Free moments* and *free cumulants* of operators, or *free random variables*, will be computed in the following text, as free-distributional data of the operators. The precise definitions and computational techniques can be found in [13] and the papers cited therein.

2.2. DYNAMICAL SYSTEMS INDUCED BY ALGEBRAIC STRUCTURES

In this section, we briefly discuss *dynamical systems* induced by algebraic structures equipped with single binary operations; for instance, *semigroups*, or *monoids*, or *groups*, or *groupoids*, etc.

Let $\Gamma = (\Gamma, \cdot)$ be an arbitrary algebraic structure equipped with its single binary operation $(\cdot)$. We will handle Γ arbitrarily in this section, but one may assume now Γ is a group. And let A be an algebra over $\mathbb{C}$. One may / can regard A as a topological algebra. Assume further that there exists an *action* λ of Γ acting on A, i.e., the images $\lambda(w)$, denoted by λ_w of all $w \in \Gamma$ are well-defined functions on A, satisfying

$$\lambda_{w_1 w_2} = \lambda_{w_1} \circ \lambda_{w_2} \quad \text{for all} \quad w_1, w_2 \in \Gamma, \tag{2.1}$$

where $(\circ)$ means the usual functional composition.

Under the setting that A is an algebra over $\mathbb{C}$, one may restrict his / her ideas to the cases where λ_w are linear transformations on A. Also, if A has its topology, then one may assume λ_w's are continuous for the topology of A for all $w \in \Gamma$.

Definition 2.1. The triple (Γ, A, λ) is called the *dynamical system of* Γ *acting on* A *via* λ.

Whenever we have fixed such a dynamical system (Γ, A, λ), if λ induces a *linear transformations on* A, then one can construct the *crossed product algebra*,

$$\mathbb{A}_\Gamma = A \times_\lambda \Gamma \tag{2.2}$$

induced by the dynamical system (Γ, A, λ), as an algebra generated by both A and $\lambda(\Gamma)$ satisfying λ-*relation* (2.3) below:

$$\begin{aligned} &\text{for any } a_1 w_1, a_2 w_2 \in \mathbb{A}_\Gamma, \text{ with } a_1, a_2 \in A \text{ and } w_1, w_2 \in \Gamma, \\ &(a_1 w_1)(a_2 w_2) = (a_1 \lambda_{w_1}(a_2))\, w_1 w_2 \end{aligned} \tag{2.3}$$

for $a_1 \lambda_1(a_2) \in A$ and $w_1 w_2 \in \Gamma$.

If Γ is a group (with group-inverses), or a groupoid (with groupoidal inverses), and if A is a $*$-algebra with its *adjoint*:

$$a^{**} = a \text{ in } A \quad \text{for all} \quad a \in A,$$

and

$$(a_1 + a_2)^* = a_1^* a_2^* \quad \text{and} \quad (a_1 a_2)^* = a_2^* a_1^*$$

in A for all $a_1, a_2 \in A$, then we need an additional condition (2.4) for the λ-relation:

$$\begin{aligned} &\text{for any } aw \in \mathbb{A}_\Gamma \text{ with } a \in A \text{ and } w \in \Gamma \\ &(aw)^* = \lambda_{w^{-1}}(a^*) w^{-1}, \end{aligned} \tag{2.4}$$

where a^* is the adjoint of a in A and w^{-1} is the inverse of w in Γ.

Definition 2.2. Let (Γ, A, λ) be a dynamical system induced by an algebraic structure Γ equipped with a single binary operation, and let $\mathbb{A}_\Gamma = A \times_\lambda \Gamma$ be an algebra in the sense of (2.2), satisfying the λ-relation (2.3). Then it is called the crossed product algebra induced by (Γ, A, λ). If Γ has its invertibility and if A is a $*$-algebra over $\mathbb{C}$, then the λ-relation implies both (2.3) and (2.4).

3. FREE-PROBABILISTIC MODELS ON $\mathcal{H}(G_p)$

In this section, we review free-probabilistic structures obtained in [4]. Moreover, some of the main results of [4] and [7] are introduced for our later results.

3.1. HECKE ALGEBRAS $\mathcal{H}(G_p)$

For a fixed *prime* p, assume we have the corresponding *generalized linear group* $G_p = GL_2(\mathbb{Q}_p)$ over the *p-adic number field* $\mathbb{Q}_p$, and the *Hecke algebra* $\mathcal{H}(G_p)$ is defined by the algebra,

$$\mathcal{H}(G_p) = \mathbb{C}_* \left[\left\{ f = \sum_{j=1}^{N} t_j \ \chi_{x_j K} \ \middle| \ \begin{array}{c} N \in \mathbb{N}, \text{ and } t_j \in \mathbb{C}, \text{ and} \\ K \text{ is a compact subgroup of } G_p, \\ \text{depending on } f, \\ \text{where } x_j \in G_p, \text{ for all } j = 1, \dots, N \end{array} \right\} \right], \tag{3.1}$$

where $\mathbb{C}_*[X]$ mean algebras generated by X under the usual functional addition, and *convolution* $(*)$ in the sense that

$$f_1 * f_2(g) = \int_{G_p} f_1(x) f_2(x^{-1}g) d\mu_p(g) \tag{3.2}$$

for all $f_1, f_2 \in \mathcal{H}(G_p)$ and $g \in G_p$, and where χ_Y mean *characteristic functions of* μ_p*-measurable subsets* Y *of* G_p, where μ_p is the both left-and-right invariant *Haar measure* on G_p (e.g., see [4, 6] and [7]). The subset

$$X_p = \left\{ f = \sum_{j=1}^{N} t_j \ \chi_{x_j K} \ \middle| \ \begin{array}{c} N \in \mathbb{N}, \text{ and } t_j \in \mathbb{C}, \text{ and} \\ K \text{ is a compact subgroup of } G_p, \\ \text{depending on } f, \\ \text{where } x_j \in G_p, \text{ for all } j = 1, \dots, N \end{array} \right\} \tag{3.3}$$

of the Hecke algebra $\mathcal{H}(G_p)$ is said to be the *generating set of* $\mathcal{H}(G_p)$, and we call elements of X_p, *generating elements of* $\mathcal{H}(G_p)$, i.e.,

$$\mathcal{H}(G_p) = \mathbb{C}_*[X_p].$$

Thus, one may re-write

$$\mathcal{H}(G_p) = \left\{ \sum_{j=1}^{N} t_j \ \chi_{x_j K_j} \ \middle| \ \begin{array}{c} N \in \mathbb{N}, \text{ and } t_j \in \mathbb{C}, \text{ and} \\ K_j \text{ are compact subgroups of } G_p, \\ \text{where } x_j \in G_p, \text{ for all } j = 1, \dots, N \end{array} \right\}, \tag{3.4}$$

set-theoretically.

Take now elements $\chi_{x_1K_1}, \chi_{x_2K_2}$ in $\mathcal{H}(G_p)$ for $x_j \in G_p$ and compact-open subgroups K_j of G_p for $j = 1, 2$. Then they satisfy that

$$\begin{aligned}(\chi_{x_1K_1} * \chi_{x_2K_2})(g) &= \int_{G_p} \chi_{x_1K_1}(x)\chi_{x_2K_2}(x^{-1}g)d\mu_p(x) \\ &= \int_{G_p} \chi_{x_1K_1}(x)\chi_{x_2K_2g^{-1}}(x^{-1})d\mu_p(x) \\ &= \int_{G_p} \chi_{x_1K_1}(x)\chi_{gK_2x_2^{-1}}(x)d\mu_p(x)\end{aligned}$$

since $x^{-1} \in x_2K_2g^{-1}$ if and only if $x \in gK_2x_2^{-1}$

$$\begin{aligned}&= \int_{G_p} \chi_{x_1K_1 \cap gK_2x_2^{-1}}(x)d\mu_p(x) \\ &= \mu_p\left(x_1K_1 \ \cap \ gK_2x_2^{-1}\right)\end{aligned}$$

by (3.2), for all $g \in G_p$, i.e.,

$$(\chi_{x_1K_1} * \chi_{x_2K_2})(g) = \mu_p\left(x_1K_1 \cap gK_2x_2^{-1}\right) \tag{3.5}$$

for all $g \in G_p$.

Without loss of generality, one may understand

$$\chi_{xK}(g) = \frac{\mu_p(xK \cap gK)}{\mu_p(xK)} = \frac{\mu_p(xK \cap gK)}{\mu_p(K)}. \tag{3.6}$$

Recall that a subgroup K is *normal* in an arbitrary group Γ if $gK = Kg$ for all $g \in \Gamma$. As usual, we denote this normal subgroup-inclusion by $K \lhd \Gamma$. Define a subset Y_p of the generating set X_p of $\mathcal{H}(G_p)$ by

$$Y_p \overset{def}{=} \left\{ \sum_{j=1}^{N} t_j \chi_{x_jK} \in X_p \,|K \lhd G_p \right\}. \tag{3.7}$$

One may have a subalgebra

$$\mathcal{H}_{Y_p} \overset{def}{=} \mathbb{C}_*[Y_p] \text{ of } \mathcal{H}(G_p). \tag{3.8}$$

Theorem 3.1 (see [4] and [7])**.** *Let $\chi_{x_jK_j}, e_{x_jK_j} \in \mathcal{H}_{Y_p}$, where $x_j \in G_p$ and $K_j \lhd G_p$ compact-open, for $j = 1, 2$. Then*

$$\chi_{x_1K_1} * \chi_{x_2K_2} = \mu_p(K_1 \cap K_2)\chi_{x_1x_2K_1K_2}, \tag{3.9}$$

where $K_1K_2 \lhd G_p$ is the product group of K_1 and K_2.

Definition 3.2. Let Y_p be the subset (3.7) of the generating set X_p, and let $\mathcal{H}_{Y_p} = \mathbb{C}_*[Y_p]$ be the subset (3.8) of the Hecke algebra $\mathcal{H}(G_p)$, for a fixed prime p. Then we call Y_p and $\mathcal{H}_{Y_p}$, the normal sub-generating set of X_p, and the normal Hecke subalgebra of $\mathcal{H}(G_p)$, respectively.

3.2. ON THE NORMAL HECKE SUBALGEBRA $\mathcal{H}_{Y_p}$ OF $\mathcal{H}(G_p)$

In this section, we concentrate on studying the normal Hecke subalgebra $\mathcal{H}_{Y_p}$ of the Hecke algebra $\mathcal{H}(G_p)$, in the sense of (3.8), where Y_p means the normal sub-generating set (3.7) of the generating set X_p of $\mathcal{H}(G_p)$, for a fixed prime p.

For convenience, denote $\prod_{j=1}^{N} x_j$ and $\underset{j=1}{\overset{N}{\times}} K_j$ simply by

$$x_{1,\dots,N} \text{ and } K_{1,\dots,N}, \text{ respectively,}$$

for all $N \in \mathbb{N}$. Also, we will let

$$K^o_{1,\dots,N} = K_{1,\dots,(N-1)} \cap K_N$$

for all $N \in \mathbb{N} \setminus \{1\}$. We will use the same notations throughout this paper.

We obtain the following general computations.

Proposition 3.3 ([7]). *Let $\chi_{x_j K_j}, e_{x_j K_j}$ be generating elements of the normal Hecke subalgebra $\mathcal{H}_{Y_p}$, for $j \in \mathbb{N}$. Then*

$$\underset{j=1}{\overset{N}{*}} \chi_{x_j K_j} = \left(\prod_{l=2}^{N} \mu_p(K^o_{1,\dots,l}) \right) \chi_{x_{1,\dots,N} K_{1,\dots,N}} \tag{3.10}$$

for all $N \in \mathbb{N}$.

Denote the convolution $f * \dots * f$ of n-copies of f simply by $f^{(n)}$ for all $n \in \mathbb{N}$ and $f \in \mathcal{H}(G_p)$.

3.3. FREE-PROBABILISTIC MODELS ON $\mathcal{H}_{Y_p}$

Let $\mathcal{H}(G_p)$ be the Hecke algebra (3.1) generated by the generalized linear group $G_p = GL_2(\mathbb{Q}_p)$ over the p-adic number field $\mathbb{Q}_p$ for a fixed prime p. From Section 3.1 we start to understand this algebra $\mathcal{H}(G_p)$ as an algebra $\mathbb{C}_*[X_p]$ generated by X_p of (3.3), consisting of $\mathbb{C}$-valued functions f formed by

$$f = \sum_{j=1}^{N} t_j \chi_{x_j K} \quad \text{for} \quad t_j \in \mathbb{C}, x_j \in G_p,$$

where K is a compact-open subgroup of G_p for $N \in \mathbb{N}$. So, to consider free-distributional data, it suffices to concentrate on generating elements χ_{xK}'s for $x \in G_p$, and compact-open subgroups K of G_p.

In this section, we further restrict our interests to such elements in the normal Hecke subalgebra $\mathcal{H}_{Y_p}$ (3.8) of $\mathcal{H}(G_p)$.

Let u_p be the group-identity of G_p, i.e.,

$$u_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G_p = GL_2(\mathbb{Q}_p).$$

For the fixed u_p, define now a linear functional φ_p on $\mathcal{H}_{Y_p}$ by

$$\varphi_p(f) \overset{def}{=} f(u_p) \quad \text{for} \quad f \in \mathcal{H}_{Y_p}. \tag{3.11}$$

Clearly, the morphism φ_p is a well-defined point-evaluation linear functional on $\mathcal{H}_{Y_p}$, and hence, the pair $(\mathcal{H}_{Y_p}, \varphi_p)$ forms a *free probability space* in the sense of [13] and [15].

Definition 3.4. We call the free probability space $(\mathcal{H}_{Y_p}, \varphi_p)$, the normal Hecke probability space (for the prime p).

Then we obtain the following fundamental free-moment computations.

Proposition 3.5. *Let $\chi_{x_j K_j}, e_{x_j K_j}$ be generating free random variables in the normal Hecke probability space $(\mathcal{H}_{Y_p}, \varphi_p)$ for all $j \in \mathbb{N}$. Then*

$$\varphi_p\left(\underset{j=1}{\overset{N}{*}} \chi_{x_j K_j}\right) = \frac{\left(\prod_{j=2}^{N} \mu_p(K_{1,\dots,j}^{o})\right)\left(\mu_p(x_{1,\dots,N}K_{1,\dots,N} \cap u_p K_{1,\dots,N})\right)}{\mu_p(K_{1,\dots,N})} \tag{3.12}$$

for all $N \in \mathbb{N}$.

Proof. Indeed, one can get that

$$\varphi_p\Big(\underset{j=1}{\overset{N}{*}} \chi_{x_j K_j}\Big) = \varphi_p\Big(\Big(\prod_{j=2}^{N} \mu_p(K_{1,\dots,j}^{o})\Big)\chi_{x_{1,\dots,N}K_{1,\dots,N}}\Big)$$

by (3.10)

$$= \Big(\prod_{j=2}^{N} \mu_p(K_{1,\dots,j}^{o})\Big)\left(\chi_{x_{1,\dots,N}K_{1,\dots,N}}(u_p)\right)$$

by (3.11)

$$= \Big(\prod_{j=2}^{N} \mu_p(K_{1,\dots,j}^{o})\Big)\left(\frac{\mu_p(x_{1,\dots,N}K_{1,\dots,N} \cap u_p K_{1,\dots,N})}{\mu_p(K_{1,\dots,N})}\right)$$

by (3.6)

$$= \frac{\left(\prod_{j=2}^{N} \mu_p(K_{1,\dots,j}^{o})\right)\left(\mu_p(x_{1,\dots,N}K_{1,\dots,N} \cap u_p K_{1,\dots,N})\right)}{\mu_p(K_{1,\dots,N})}.$$

□

Let $\chi_{x_1K_1}, \ldots, \chi_{x_NK_N} \in (\mathcal{H}_{Y_p}, \varphi_p)$ for $N \in \mathbb{N}$. Then the *joint free cumulants* can be obtained by

$$k_N^p\left(\chi_{x_1K_1}, \ldots, \chi_{x_NK_N}\right) = \sum_{\pi \in NC(N)} \left(\prod_{V \in \pi} \varphi_p\left(\underset{j \in V}{*} \chi_{x_{i_j}K_{i_j}}\right) \mu\left(0_{|V|},\ 1_{|V|}\right)\right)$$

by the free-probabilistic Möbius inversion of [13]

$$= \sum_{\pi \in NC(N)} \left(\prod_{V=(i_1,\ldots,i_{|V|}) \in \pi} \left(\mu_p(V)\right) \mu\left(0_{|V|},\ 1_{|V|}\right)\right), \tag{3.13}$$

by (3.12), where

$$\mu_p(V) = \frac{\mu_p(K^o_{i_1,i_2}) \ldots \mu_p(K^o_{i_1,\ldots,i_{|V|}})\mu_p(x_{i_1,\ldots,i_{|V|}}K_{i_1,\ldots,i_{|V|}} \cap K_{i_1,\ldots,i_{|V|}})}{\mu_p(K_{i_1,\ldots,i_{|V|}})},$$

are the block-depending free moments for all $V \in \pi$ and all $\pi \in NC(N)$, where $k_n^p(\ldots)$ means *free cumulant in terms of* φ_p as in [13].

By the above computation (3.13), we obtain the following freeness condition on the normal Hecke subalgebra $\mathcal{H}_{Y_p}$. And this freeness condition shows that the classical independence guarantees our freeness.

Theorem 3.6 ([7])**.** *Let $f_j = \chi_{K_j}$ be free random variables in the normal Hecke free probability space $(\mathcal{H}_{Y_p}, \varphi_p)$ for $j = 1, 2$. Then*

$$f_1 \text{ and } f_2 \text{ are free in } (\mathcal{H}_{Y_p}, \varphi_p) \Longleftrightarrow \mu_p(K^o_{1,2}) = \mu_p(K_1)\mu_p(K_2). \tag{3.14}$$

The proof of (3.14) is done, by computing "mixed" free cumulants of f_1 and f_2 based on the joint free-cumulant computation (3.13), and the fact

$$\sum_{\pi \in NC(N)} \mu(\pi, 1_N) = 0$$

for all $N \in \mathbb{N}$ (e.g., see [4, 6, 7] and [13]). This freeness characterization (3.14) demonstrates that the freeness on the normal Hecke subalgebra (determined by the linear functional φ_p) is similar to classical independence.

In fact, under normality on the generating set Y_p of $\mathcal{H}_{Y_p}$, the above freeness characterization (3.14) is not so interesting. However, for the extended setting fully on $\mathcal{H}(G_p)$ under a so-called *normal-coring process of* [4], we obtain the similar freeness characterization in [4], by extending (3.14), which is really interesting.

Since we are restricting our interests inside the normal-Hecke-probabilistic frames, we will not mention the full normal-cored free-probabilistic structures of the Hecke algebra $\mathcal{H}(G_p)$, but we need to emphasize at this moment clearly that our future results can be obtained similarly under the normal-coring process of [4] fully on $\mathcal{H}(G_p)$, as in [4] and [6], too.

4. REPRESENTATIONS OF NORMAL HECKE PROBABILITY SPACES

Let p be a fixed prime and let $(\mathcal{H}_{Y_p}, \varphi_p)$ be the normal Hecke probability space in the sense of Section 3. The representations of this section for $(\mathcal{H}_{Y_p}, \varphi_p)$ is understood as a restricted version of those of the Hecke algebra $\mathcal{H}(G_p)$ in the sense of [4] and [6]. Thanks to the properties of our groups (which will be considered in the following text), we (can) restrict our interests to the normal-Hecke-probability cases here.

Define a sesqui-linear form on the normal Hecke subalgebra $\mathcal{H}_{Y_p}$,

$$[\cdot,\cdot]_p : \mathcal{H}_{Y_p} \times \mathcal{H}_{Y_p} \to \mathbb{C}$$

by

$$[f_1, f_2]_p \overset{def}{=} \varphi_p(f_1 * f_2^*) \quad \text{for all} \quad f_1, f_2 \in \mathcal{H}_{Y_p}, \tag{4.1}$$

where

$$f^*(x) \overset{def}{=} \overline{f(x)} \text{ in } \mathbb{C} \quad \text{for all} \quad x \in G_p,$$

where $\overline{z}$ means the *conjugate of* z for all $z \in \mathbb{C}$.

We call the unary operation

$$f \in \mathcal{H}_{Y_p} \longmapsto f^* \in \mathcal{H}_{Y_p} \tag{4.2}$$

in the sense of (4.1), the *adjoint* on $\mathcal{H}_{Y_p}$. And the element f^* is said to be the *adjoint of* f. Since the adjoint (4.2) is well-defined on the normal Hecke subalgebra $\mathcal{H}_{Y_p}$, the algebra $\mathcal{H}_{Y_p}$ is a $*$-*algebra* over $\mathbb{C}$. Indeed, the adjoint (4.2) satisfies that:

$$(f^*)^* = f \quad \text{for all} \quad f \in \mathcal{H}_{Y_p}, \tag{4.3}$$

$$(f_1 + f_2)^* = f_1^* + f_2^* \quad \text{for all} \quad f_1, f_2 \in \mathcal{H}_{Y_p}, \tag{4.4}$$

$$(f_1 f_2)^* = f_2^* f_1^* \quad \text{for all} \quad f_1, f_2 \in \mathcal{H}_{Y_p}. \tag{4.5}$$

So, by (4.3), (4.4) and (4.5), the normal Hecke subalgebra $\mathcal{H}_{Y_p}$ is indeed a $*$-*algebra over* $\mathbb{C}$.

Consider that

$$[f, f]_p \geq 0 \quad \text{for all} \quad f \in \mathcal{H}_{Y_p} \tag{4.6}$$

and

$$[f_1, f_2]_p = \overline{[f_2,\ f_1]_p} \quad \text{for all} \quad f_1, f_2 \in \mathcal{H}_{Y_p}. \tag{4.7}$$

By the sesqui-linearity of $[\cdot,\cdot]_p$, the nonnegativity (4.6), and symmetry (4.7) under conjugate of the form $[\cdot,\cdot]_p$, this form $[\cdot,\cdot]_p$ becomes a *pseudo-inner product on* $\mathcal{H}_{Y_p}$. And hence, the pair $\left(\mathcal{H}_{Y_p},\ [\cdot,\cdot]_p\right)$ forms a *pseudo-inner-product space over* $\mathbb{C}$.

Remark that, by [4], there exists non-zero element $f \in \mathcal{H}_{Y_p}$ such that

$$[f, f]_p = 0. \tag{4.8}$$

So, the pseudo-inner product space $(\mathcal{H}(G_p), [\cdot,\cdot]_p)$ is not an inner product space by the existence of non-zero element f of $\mathcal{H}_{Y_p}$ satisfying (4.8).

When we understand our normal Hecke subalgebra $\mathcal{H}_{Y_p}$ as a pseudo-inner product space $(\mathcal{H}_{Y_p}, [\cdot,\cdot]_p)$we denote it by $\mathcal{H}_p$.

On the pseudo-inner product space $\mathcal{H}_p$, define an equivalence relation $\mathcal{R}_p$ by

$$f_1 \mathcal{R}_p f_2 \overset{def}{\Longleftrightarrow} [f_1, f_1]_p = [f_2, f_2]_p. \tag{4.9}$$

Definition 4.1. Let $\mathcal{H}_p$ be the pseudo-inner product space, and let $\mathcal{R}_p$ be the equivalence relation (4.9) on $\mathcal{H}_p$. Define the quotient space $\mathfrak{H}_p$ by

$$\mathfrak{H}_p = \mathcal{H}_p / \mathcal{R}_p, \tag{4.10}$$

equipped with the inherited pseudo-inner product, also denoted by $[\cdot,\cdot]_p$, on it. Then

$$\mathfrak{H}_p = (\mathfrak{H}_p, [\cdot,\cdot]_p) = (\mathcal{H}_p/\mathcal{R}_p, [\cdot,\cdot]_p)$$

is called the normal Hecke inner product space.

Indeed, our normal Hecke inner product space $\mathfrak{H}_p$ is an inner product space by $\mathcal{R}_p$ of (4.9), i.e., it not only satisfies the sesqui-linearity (under quotient), (4.6) and (4.7), but also satisfies

$$[f, f]_p = 0 \Longleftrightarrow f = 0_{\mathfrak{H}_p} = 0_{\mathcal{H}_p}/\mathcal{R}_p, \tag{4.11}$$

where $0_{\mathcal{H}_p}$ is the zero element of $\mathcal{H}_p$.

By (4.11), we obtain the following proposition immediately.

Proposition 4.2. *The normal Hecke inner product space $\mathfrak{H}_p$ of* (4.10) *is indeed an inner product space over $\mathbb{C}$.*

For the given inner product space $\mathfrak{H}_p$, one can naturally define the corresponding norm $\|\cdot\|_p$ on $\mathfrak{H}_p$ by

$$\|f\|_p \overset{def}{=} \sqrt{[f,\ f]_p} \quad \text{for all} \quad f \in \mathfrak{H}_p, \tag{4.12}$$

and the corresponding metric d_p on $\mathfrak{H}_p$ by

$$d_p(f_1,\ f_2) = \|f_1 - f_2\|_p \quad \text{for all} \quad f_1, f_2 \in \mathfrak{H}_p. \tag{4.13}$$

If there is no confusion, we write the equivalence classes

$$[f]_{\mathcal{R}_p} = \{h \in \mathcal{H}_p : h \mathcal{R}_p f\} \in \mathfrak{H}_p \tag{4.14}$$

simply by f, i.e., we regard it as

$$\sum_{j=1}^{n} t_j \chi_{x_j K_j} = \sum_{j=1}^{n} t_j [\chi_{x_j K_j}]_{\mathcal{R}_p}$$

in $\mathfrak{H}_p$.

Definition 4.3. Construct the d_p-metric topology closure of $\mathfrak{H}_p$ in $\mathcal{H}_p$, also denoted by $\mathfrak{H}_p$. Then this Hilbert space $\mathfrak{H}_p$ is called the normal Hecke Hilbert space.

Then, by the very construction of the normal Hecke Hilbert space $\mathfrak{H}_p$ from the normal Hecke probability space $(\mathcal{H}_{Y_p}, \varphi_p)$, the algebra $\mathcal{H}_{Y_p}$ acts on $\mathfrak{H}_p$ via an algebra-action α^p,

$$\alpha^p(f)(h) = f * h \quad \text{for all} \quad h \in \mathfrak{H}_p, \tag{4.15}$$

for all $f \in \mathcal{H}_{Y_p}$. In fact, by (4.14), one can express (4.15) precisely as follows:

$$\alpha^p(f)(h) = \alpha^p(f)\left([h]_{\mathcal{R}_p}\right) = [f * h]_{\mathcal{R}_p} = f * h \tag{4.16}$$

in $\mathfrak{H}_p$.

For convenience, we denote the image $\alpha^p(f)$ by α^p_f for all $f \in \mathcal{H}_{Y_p}$.

The above morphism α^p of (4.15) (or (4.16)) is indeed a well-defined algebra-action of $\mathcal{H}(G_p)$ acting on $\mathfrak{H}_p$, since

$$\begin{aligned}\alpha^p_{f_1 * f_2}(h) &= f_1 * f_2 * h = f_1 * (f_2 * h) \\ &= f_1 * \left(\alpha^p_{f_2}(h)\right) = \alpha^p_{f_1}\left(\alpha^p_{f_2}(h)\right) = \left(\alpha^p_{f_1}\alpha^p_{f_2}\right)(h)\end{aligned}$$

for all $h \in \mathfrak{H}_p$ and $f_1, f_2 \in \mathcal{H}(G_p)$, i.e.,

$$\alpha^p_{f_1 * f_2} = \alpha^p_{f_1}\alpha^p_{f_2} \text{ on } \mathfrak{H}_p \tag{4.17}$$

for all $f_1, f_2 \in \mathcal{H}(G_p)$.

Also, the algebra-action α^p satisfies that

$$\begin{aligned}\left[\alpha^p_f(h_1),\ h_2\right]_p &= [f * h_1,\ h_2]_p = \varphi_p\left((f * h_1) * h_2^*\right) \\ &= \varphi_p\left(h_1 * f * h_2^*\right) = \varphi_p\left(h_1 * (h_2^* * f)\right) = \varphi_p\left(h_1 * (f^* * h_2)^*\right) \\ &= [h_1, f^* * h_2]_p = \left[h_1,\ \alpha^p_{f^*}(h_2)\right]_p,\end{aligned}$$

since the convolution $(*)$ on $\mathcal{H}_{Y_p}$ is commutative, for all $h_1, h_2 \in \mathfrak{H}_p$ and $f \in \mathcal{H}_{Y_p}$, i.e.,

$$\left(\alpha^p_f\right)^* = \alpha^p_{f^*} \text{ on } \mathfrak{H}_p \quad \text{for all} \quad f \in \mathcal{H}_{Y_p}. \tag{4.18}$$

Since our normal Hecke algebra $\mathcal{H}_{Y_p}$ is a $*$-algebra, the morphism α^p of (4.15) is a $*$-algebra-action of $\mathcal{H}_{Y_p}$ acting on the normal Hecke Hilbert space $\mathfrak{H}_p$, by (4.17) and (4.18).

Theorem 4.4. *The pair $(\mathfrak{H}_p, \alpha^p)$ of the normal Hecke Hilbert space $\mathfrak{H}_p$ and the morphism α^p of* (4.15) *forms a well-determined Hilbert-space representation of the normal Hecke subalgebra $\mathcal{H}_{Y_p}$ acting on $\mathfrak{H}_p$, induced by the normal Hecke probability space $(\mathcal{H}_{Y_p}, \varphi_p)$.*

Proof. By (4.17) and (4.18), indeed, the linear morphism α^p of (4.15) is a $*$-algebra-action of $\mathcal{H}_{Y_p}$ acting on $\mathfrak{H}_p$. Thus, the pair $(\mathfrak{H}_p, \alpha^p)$ forms a Hilbert-space representation of $\mathcal{H}_{Y_p}$. □

Definition 4.5. The representation $(\mathfrak{H}_p, \alpha^p)$ of the normal Hecke Hilbert space $\mathfrak{H}_p$ and the $*$-algebra-action α^p of $\mathcal{H}_{Y_p}$ acting on the normal Hecke Hilbert space $\mathfrak{H}_p$ is called the normal Hecke representation of the normal Hecke probability space $(\mathcal{H}_{Y_p}, \varphi_p)$.

5. GROUPS $\mathfrak{T}_N^K$ INDUCED BY CERTAIN PARTIAL ISOMETRIES ON $\mathfrak{H}_p$

In this section, we construct groups induced by certain operators acting on the normal Hecke Hilbert space $\mathfrak{H}_p$. Such constructions are introduced in [6]. We "re-characterize" and "generalize" the results of [6] to study our dynamical systems in the following text.

Recall that an operator P on an arbitrary Hilbert space H is said to be a *projection*, if it is both self-adjoint and idempotent, i.e.,

$$P^* = P = P^2 \text{ on } H.$$

An operator T on H is called a *partial isometry*, if the operator T^*T is a projection on H. It is well-known that T is a partial isometry if and only if $TT^*T = T$, if and only if T^* is a partial isometry, if and only if TT^* is a projection, if and only if $T^*TT^* = T^*$, on H. The projections T^*T and TT^* induced by a partial isometry T are said to be the *initial projection of* T, and the *final projections of* T, respectively. The (closed) subspaces $(T^*T)H$ and $(TT^*)H$ of the given Hilbert space H are called the *initial subspace of* T, and the *final subspace of* T in H, respectively.

Theorem 5.1. *Let $K \lhd G_p$ be a normal compact-open subgroup with $\mu_p(K) = 1$, and let $T^K = \alpha^p_{\chi_K}$ be an operator on the normal Hecke Hilbert space $\mathfrak{H}_p$. Then it is a projection on $\mathfrak{H}_p$, i.e.,*

$$K \lhd G_p \textit{ compact-open, } \mu_p(K) = 1 \Rightarrow \alpha^p_{\chi_K} \textit{ is a projection on } \mathfrak{H}_p. \tag{5.1}$$

Proof. Suppose K is a compact-open normal subgroup of G_p, and $T^K = \alpha^p_{\chi_K}$, the operator induced by K on $\mathfrak{H}_p$. Then

$$\left(T^K\right)^* = \left(\alpha^p_{\chi_K}\right)^* = \alpha^p_{\chi^*_K} = \alpha^p_{\chi_K} = T^K$$

on $\mathfrak{H}_p$. Thus, the operator T^K is self-adjoint on $\mathfrak{H}_p$.

Observe now that

$$(T^K)^2 = \left(\alpha^p_{\chi_K}\right)^2 = \alpha^p_{\chi_K * \chi_K} = \alpha^p_{\mu_p(K \cap K)\chi_{xKK}} = \alpha^p_{\chi_{\mu_p(K)K}} = \alpha^p_{\chi_K} = T^K$$

on $\mathfrak{H}_p$. So, the operator T^K is idempotent on $\mathfrak{H}_p$.

Therefore, T^K is a projection on $\mathfrak{H}_p$. □

By (5.1), we obtain the following theorem. In fact, the following theorem is proven differently in [6]. However, here we provide a better and generalized proof.

Theorem 5.2. *Let K be a normal compact-open subgroup with $\mu_p(K) = 1$ and let $T^K = \alpha^p_{\chi_K}$ be an operator on $\mathfrak{H}_p$. Suppose $x_1, \ldots, x_N$ are distinct nonzero elements of G_p, satisfying $x_j^2 = u_p$, the group-identity of G_p, and let x_jK be the cosets of K, for $j = 1, \ldots, N$, for $N \in \mathbb{N}$. Then the operators $T_j^K = \alpha^p_{\chi_{x_jK}}$ are partial isometries on $\mathfrak{H}_p$, with both their initial and final projections T^K, for all $j = 1, \ldots, N$, i.e.,*

$$\begin{aligned} &T^K \textit{ is a projection on } \mathfrak{H}_p \textit{ as in } (5.1),\ x_j^2 = u_p \in G_p \Rightarrow \alpha^p_{\chi_{x_jK}} \\ &\textit{are partial isometries on } \mathfrak{H}_p \textit{ for } j = 1, \ldots, N \textit{ and } N \in \mathbb{N}. \end{aligned} \tag{5.2}$$

Proof. By assumption, the operator T^K is a projection on $\mathfrak{H}_p$, by (5.1). Now, let $x_1, \ldots, x_N$ be distinct elements of G_p, and let $x_j K$ be the cosets of K, where x_j are self-invertible in the sense that $x_j^2 = u_p$ in G_p, and let $T_j^K = \alpha^p_{\chi_{x_j K}}$ be corresponding operators on $\mathfrak{H}_p$ for $j = 1, \ldots, N$, for $N \in \mathbb{N}$.

Then we have

$$\left(T_j^K\right)^* \left(T_j^K\right) = \left(\alpha^p_{\chi_{x_j K}}\right)^2 = \alpha^p_{\mu_p(K)\chi_{x_j^2 K}} = \alpha^p_{\chi_K} = T^K = \left(T_j^K\right)\left(T_j^K\right)^*$$

on $\mathfrak{H}_p$, by the conditions that

$$\mu_p(K) = 1 \quad \text{and} \quad x_j^2 = u_p \quad \text{in} \quad G_p$$

for all $j = 1, \ldots, N$.

Since T^K is a projection on $\mathfrak{H}_p$, the operator-product

$$(T_j^K)^*(T_j^K) = (T_j^K)(T_j^K)^*$$

becomes a projection on $\mathfrak{H}_p$ for all $j = 1, \ldots, N$. Thus the self-adjoint operator T_j^K is a partial isometry with its initial-and-final projection, identified with T^K on $\mathfrak{H}_p$ for all $j = 1, \ldots, N$. □

As we have seen in (5.1) and (5.2), if we fix a normal compact-open subgroup K of G_p with $\mu_p(K) = 1$, one can have the projection $T^K = \alpha^p_{\chi_K}$ on $\mathfrak{H}_p$, moreover, if there are $x_j \in G_p$, with $x_j^2 = u_p$, equivalently, $x_j = x_j^{-1}$ in G_p, then we obtain the partial isometries $T_j^K = \alpha^p_{\chi_{x_j K}}$ on $\mathfrak{H}_p$, whose initial-and-final projections are T^K, for $j = 1, \ldots, N$ and $N \in \mathbb{N}$.

Remark 5.3. Note that there are enough self-invertible elements x of the group G_p, such that $x^2 = u_p$. Note that our constructions of T^K and T_j^K are based on the existence of self-invertible elements x in G_p. To confirm there are enough such elements x in G_p, let us consider the following. Suppose that

$$A = \begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix} \in M_2(\mathbb{Q}_p)$$

with

$$a \neq 0 \quad \text{and} \quad b \neq 0 \quad \text{in} \quad \mathbb{Q}_p.$$

Then A^2 is identical to the identity matrix in $M_2(\mathbb{Q}_p)$. It shows that there are enough elements $x \in G_p$ such that $x^2 = u_p$ in G_p.

Construct now a subspace $\mathfrak{H}_p^K$ by $T^K(\mathfrak{H}_p)$ in $\mathfrak{H}_p$. Then, on the subspace $\mathfrak{H}_p^K$, the projection T^K is the identity operator, and each partial isometry T_j^K becomes a unitary for $j = 1, \ldots, N$ and $N \in \mathbb{N}$, i.e.,

$$T^K = 1_{\mathfrak{H}_p^K}, \text{ the identity operator on } \mathfrak{H}_p^K,$$

and

$$T_j^K \text{ are unitaries on } \mathfrak{H}_p^K \quad \text{for all} \quad j = 1, \ldots, N.$$

Recall that an operator U on a Hilbert space H is *unitary* if

$$U^*U = 1_H = UU^*,$$

equivalently, U is invertible on H, moreover, the inverse U^{-1} of U is identical to the adjoint U^* on H.

Thus whenever such operators T_j^K and T^K are given for a fixed normal compact-open subgroup K of G_p, one can construct the group

$$\mathfrak{T}_N^K \overset{def}{=} \left\langle \{T_j^K\}_{j=1}^N \right\rangle, \text{ generated by } \{T_j^K\}_{j=1}^N, \tag{5.3}$$

equipped with its binary operation, the operator multiplication inherited from that on the operator algebra $B(\mathfrak{H}_p)$, consisting of all (bounded linear) operators on $\mathfrak{H}_p$.

And, one can get a C^*-subalgebra $\mathfrak{C}_{K,N}^*$ of the operator algebra $B(\mathfrak{H}_p^K)$ (inside $B(\mathfrak{H}_p)$) generated by the group $\mathfrak{T}_N^K$ of (5.3), i.e., one has

$$\mathfrak{C}_{K,N}^* = C_{\mathfrak{H}_p^K}^* \left(\mathfrak{T}_N^K\right) = \overline{\mathbb{C}\left[\mathfrak{T}_N^K\right]} \text{ in } B(\mathfrak{H}_p^K), \tag{5.4}$$

where $\overline{X}$ here means the operator-norm-closure of sets X in $B(\mathfrak{H}_p^K)$.

Definition 5.4. We call the group $\mathfrak{T}_N^K$ of (5.3), the K(-concentrated)-subgroup (of $B(\mathfrak{H}_p^K)$ in $B(\mathfrak{H}_p)$). And the group C^*-algebra $\mathfrak{C}_{K,N}^*$ of (5.4) is said to be the K(-concentrated)-subgroup C^*-algebra of a compact-open normal subgroup K of G_p.

Then we obtain the following characterizations for the K-subgroups $\mathfrak{T}_N^K$ of (5.3) for a fixed compact-open normal subgroup K of G_p as follows.

Theorem 5.5 ([6])**.** *Let $\mathfrak{T}_N^K$ be the K-group in $B(\mathfrak{H}_p^K)$, and let $\mathfrak{T}_N$ be the finitely presented group,*

$$\mathfrak{T}_N = \left\langle \{w_j\}_{j=1}^N, \ \{w_j = w_j^{-1}\}_{j=1}^N \right\rangle, \tag{5.5}$$

with its generator set $\{w_j\}_{j=1}^N$, consisting of noncommutative indeterminants $w_1, \ldots, w_N$, and its relator set $\{w_j = w_j^{-1}\}_{j=1}^N$. Then the group $\mathfrak{T}_N^K$ and the group $\mathfrak{T}_N$ of (5.5) *are group-homomorphic, i.e.,*

$$\mathfrak{T}_N^K \overset{Group}{=} \mathfrak{T}_N. \tag{5.6}$$

Therefore, the K-subgroup C^*-algebra $\mathfrak{C}_{K,N}^*$ (as a C^*-subalgebra of $B(\mathfrak{H}_p^K)$) is $*$-isomorphic to the group C^*-algebra $\mathfrak{C}_N^*$ generated by $\mathfrak{T}_N$, i.e.,

$$\mathfrak{C}_{K,N}^* \overset{def}{=} C_{\mathfrak{H}_p^K}^* \left(\mathfrak{T}_N^K\right) \overset{*\text{-}iso}{=} C_{l^2(\mathfrak{T}_N)}^* \left(\mathfrak{T}_N\right) \overset{def}{=} \mathfrak{C}_N^*, \tag{5.7}$$

where $l^2(X)$ mean the l^2-Hilbert spaces generated by sets X.

6. DYNAMICAL SYSTEMS INDUCED BY $\mathfrak{T}_N^K$

Throughout this section, we fix $N \in \mathbb{N}$, and a normal compact-open subgroup K of G_p with

$$\mu_p(K) = 1.$$

As we have seen above, if K is given as above, then $T^K = \alpha^p_{\chi_K}$ is a projection on $\mathfrak{H}_p$ and $T_j^K = \alpha^p_{\chi_{x_j K}}$ are partial isometries on $\mathfrak{H}_p$, whenever

$$x_j^2 = u_p \text{ in } G_p \quad \text{for} \quad j = 1, \ldots, N,$$

for $N \in \mathbb{N}$, with their initial-and-final projections T^K. And hence, the group $\mathfrak{T}_N^K$ in the sense of (5.3) is well-determined on the subspace $\mathfrak{H}_p^K$ of $\mathfrak{H}_p$.

6.1. ACTING $\mathfrak{T}_N^K$ ON C^*-ALGEBRAS

Let $\mathfrak{T}_N^K$ be the K-subgroup of $B(\mathfrak{H}_p^K)$ in $B(\mathfrak{H}_p)$ for the fixed compact-open subgroup K of G_p, in the sense of (5.3), i.e., it is generated by the partial isometries $T_j^K = \alpha^p_{\chi_{x_j K}}$ having their initial-and-final projections $T^K = \alpha^p_{\chi_K}$ on $\mathfrak{H}_p$, where $x_j^2 = u_p$ in G_p for all $j = 1, \ldots, N$. Since the partial isometries T_j^K on the normal Hecke Hilbert space $\mathfrak{H}_p$ are unitaries on the subspace $\mathfrak{H}_p^K = T^K(\mathfrak{H}_p)$ for $j = 1, \ldots, N$, and since the projection T^K on $\mathfrak{H}_p$ is the identity operator $1_{\mathfrak{H}_p^K}$ of $\mathfrak{H}_p^K$, one can construct the multiplicative subgroup $\mathfrak{T}_N^K$ of $B(\mathfrak{H}_p^K)$ as in Section 5. Moreover, this group $\mathfrak{T}_N^K$ is group-isomorphic to the finitely presented group

$$\mathfrak{T}_N = \left\langle \{w_j\}_{j=1}^N, \ \{w_j = w_j^{-1}\}_{j=1}^N \right\rangle,$$

of (5.5) satisfying (5.6) and (5.7).

Let H be an arbitrary Hilbert space, and $B(H)$, the operator algebra consisting of all operators on H. Construct now the (topological) *tensor product Hilbert space*

$$\mathfrak{H}_p^{K,H} \stackrel{def}{=} H \otimes \mathfrak{H}_p^K, \tag{6.1}$$

where $\mathfrak{H}_p^K = T^K(\mathfrak{H}_p)$ is the subspace of the normal Hecke Hilbert space $\mathfrak{H}_p$ induced by the fixed normal compact-open subgroup K of G_p, where the K-subgroup C^*-algebra $\mathfrak{C}^*_{K,N}$ is acting. Then the group generators T_j^K of $\mathfrak{T}_N^K$ are understood again as self-adjoint unitary operators

$$T_j^{K,H} = 1_H \otimes T_j^K, \tag{6.2}$$

in the tensor product Hilbert space $\mathfrak{H}_{K,H}$ of (6.1), for all $j = 1, \ldots, N$, where 1_H means the identity operator on H.

Now, let A be a C^*-subalgebra of $B(H)$, i.e., all elements a of A are operators acting on H (under the embedding action λ_H in the sense that $\lambda_H(a) = a$ on H). Also, let $Aut(A)$ be the collection of all $*$-isomorphisms (or ($*$-)automorphisms) on A.

Then the group-dynamical system $(\mathfrak{T}_N^K, A, \lambda)$ is well-determined, where λ is in the sense of (6.3) below extended on A under linearity, i.e., it is an action of $\mathfrak{T}_N^K$ acting on A satisfying

$$\lambda\left(T_j^K\right)\left(\sum_{i=1}^{n} t_i a_i\right) = \sum_{i=1}^{n} t_i \lambda_j(a_i) \tag{6.3}$$

for all $\sum_{i=1}^{n} t_i a_i \in A$ with $t_i \in \mathbb{C}$, $a_i \in A$, for $n \in \mathbb{N} \cup \{\infty\}$, for all $j = 1, \ldots, n$, where $\lambda_j = \lambda\left(T_j^K\right)$ are in the sense that

$$\lambda_j(a_i) = T_j^{K,H} \otimes \left(a_i \otimes 1_{\mathfrak{H}_p^K}\right) = a_i \otimes T_j^K,$$

where $T_j^{K,H}$ are in the sense of (6.2).

Definition 6.1. Let $\mathfrak{T}_N^K$ be the K-subgroup of $B(\mathfrak{H}_p^K)$, for a compact-open normal subgroup K of G_p, and let λ be a group-action (6.3) of $\mathfrak{T}_N^K$ acting on a C^*-algebra A (in $B(H)$). Then the group-dynamical system $(\mathfrak{T}_N^K, A, \lambda)$ is called the K-subgroup dynamical system of $\mathfrak{T}_N^K$ (acting) on A (via λ).

For convenience, we denote $\lambda(T)$ simply by λ_T for all $T \in \mathfrak{T}_N^K$.

Let $(\mathfrak{T}_N^K, A, \lambda)$ be a K-subgroup dynamical system. As in Section 2.2, one can construct the corresponding crossed product C^*-algebra

$$\mathbb{A}_N^K = A \times_\lambda \mathfrak{T}_N^K \tag{6.4}$$

induced by the dynamical system $(\mathfrak{T}_N^K, A, \lambda)$, satisfying the λ-relation, expressed by (6.5) and (6.6), below:

$$(a_1 T_1)(a_2 T_2) = a_1 \lambda_{T_1}(a_2) T_1 T_2, \tag{6.5}$$

and

$$(aT)^* = \lambda_{T^{-1}}(a^*) T^{-1} \tag{6.6}$$

for all $aT, a_1T_1, a_2T_2 \in \mathbb{A}_N^K$ with $a, a_1, a_2 \in A$ and $T, T_1, T_2 \in \mathfrak{T}_N^K$, i.e., the C^*-algebra $\mathbb{A}_N^K$ is the C^*-subalgebra of

$$B(\mathfrak{H}_{p,H}^K) = B(H \otimes \mathfrak{H}_p^K) = B(H) \otimes_{\mathbb{C}} B(\mathfrak{H}_p^K),$$

where $\mathfrak{H}_{p,H}^K$ is the tensor product Hilbert space $H \otimes \mathfrak{H}_p^K$ in the sense of (6.1), and where $(\otimes_{\mathbb{C}})$ means the tensor product on algebras over $\mathbb{C}$, generated by A and $\lambda(\mathfrak{T}_N^K)$ satisfying the above λ-relation, (6.5) and (6.6).

Definition 6.2. We call the crossed product C^*-algebra $\mathbb{A}_N^K = A \times_\lambda \mathfrak{T}_N^K$ of (6.4) induced by a K-subgroup dynamical system $(\mathfrak{T}_N^K, A, \lambda)$, the K-subgroup dynamical C^*-algebra over A (in $B(\mathfrak{H}_{p,H}^K)$).

For convenience, if there is no confusion, let us denote $\lambda_T(a)$ simply by a^T, for all $T \in \mathfrak{T}_N^K$, and $a \in A$. With this new notation one can re-write (6.5) and (6.6) as follows:

$$(a_1T_1)(a_2T_2) = a_1 a_2^{T_1} T_1 T_2$$

and

$$(aT)^{*} = (a^{*})^{T^{-1}} T^{-1} = (a^{*})^{T^{-1}} T^{*},$$

respectively, where T^{-1} means the group-inverse of T in $\mathfrak{T}_N^K$ and T^* means the adjoint of T in $\mathbb{A}_N^K$.

Since $\mathfrak{T}_N^K$ and $\mathfrak{T}_N$ are group-isomorphic, one can establish an "equivalent" group-dynamical system $(\mathfrak{T}_N, A, \lambda \circ \Omega)$. From below, we denote a group-isomorphism of $\mathfrak{T}_N$ and $\mathfrak{T}_N^K$ by

$$\Omega : \mathfrak{T}_N \to \mathfrak{T}_N^K,$$

which is a generator-preserving isomorphism from $\mathfrak{T}_N$ onto $\mathfrak{T}_N^K$. Clearly, such an isomorphism Ω exists by (5.6).

Also, this group-isomorphism Ω is nicely extended to a $*$-isomorphism Ω_o from the corresponding C^*-algebras $\mathfrak{C}_N^*$ onto $\mathfrak{C}_{K,N}^*$, under linearity, by (5.7), i.e.,

$$\Omega(w_j) = T_j^K \quad \text{for} \quad j = 1, \ldots, N \tag{6.7}$$

(with possible re-arrangements), where

$$\mathfrak{T}_N^K = \left\langle \{T_j^K\}_{j=1}^N \right\rangle \text{ in } B(\mathfrak{H}_p^K)$$

and

$$\mathfrak{T}_N = \left\langle \{w_j\}_{j=1}^N, \ \{w_j^{-1} = w_j\}_{j=1}^N \right\rangle.$$

So, indeed, one can establish an equivalent dynamical system $(\mathfrak{T}_N, A, \lambda \circ \Omega)$, whenever we have the K-subgroup dynamical system $(\mathfrak{T}_N^K, A, \lambda)$.

Theorem 6.3. *Let $\mathfrak{T}_N^K$ be the K-subgroup in $B(\mathfrak{H}_p^K)$ and let $\mathfrak{T}_N$ be the finitely presented group $\Omega^{-1}\left(\mathfrak{T}_N^K\right)$ with group-isomorphism Ω of (6.7). Then the K-subgroup-dynamical systems $(\mathfrak{T}_N^K, A, \lambda)$ and $(\mathfrak{T}_N, A, \lambda \circ \Omega)$ are equivalent, i.e.,*

$$\left(\mathfrak{T}_N^K, \ A, \ \lambda\right) \overset{equi}{=} \left(\mathfrak{T}_N, \ A, \ \lambda \circ \Omega\right). \tag{6.8}$$

Therefore, the crossed product C^*-algebras $\mathbb{A}_N^K = A \times_\lambda \mathfrak{T}_N^K$ and $\mathbb{A}_N = A \times_{\lambda \circ \Omega} \mathfrak{T}_N$ are $*$-isomorphic, i.e.,

$$\mathbb{A}_N^K \overset{*\text{-}iso}{=} \mathbb{A}_N. \tag{6.9}$$

Proof. The proof of (6.8) is by the definition of equivalence on dynamical systems, and by (6.7). By (6.8), the $*$-isomorphic relation (6.9) holds. □

By the equivalence (6.8), one can understand two group-dynamical systems $(\mathfrak{T}_N^K, A, \lambda)$ and $(\mathfrak{T}_N, A, \lambda \circ \Omega)$, alternatively. Similarly, we understand two C^*-algebras $\mathbb{A}_N^K$ and $\mathbb{A}_N$, alternatively, by (6.9). In the rest of this paper, we let

$$\lambda_o \overset{denote}{=} \lambda \circ \Omega.$$

Let $\mathbb{A}_N^K$ be the K-subgroup dynamical C^*-algebra (6.4) induced by a K-subgroup dynamical system $(\mathfrak{T}_N^K, A, \lambda)$, and let

$$a_j T_j^K \in \mathbb{A}_N^K \quad \text{for} \quad j = 1, \ldots, N,$$

for any $a_j \in A$, where $T_j^K = \alpha^p_{\chi_{x_j K}}$ are generating elements of the K-subgroup $\mathfrak{T}_N^K$ for $j = 1, \ldots, N$. Then one has

$$\begin{aligned}\left(a_{j_1} T_{j_1}^K\right)\left(a_{j_2} T_{j_2}^K\right) &= a_{j_1} a_{j_2}^{T_{j_1}^K} T_{j_1}^K T_{j_2}^K = a_{j_1} a_{j_2}^{T_{j_1}^K} \alpha^p_{\chi_{x_{j_1} K}} \alpha^p_{\chi_{x_{j_2} K}} \\ &= a_{j_1} a_{j_2}^{T_{j_1}^K} \alpha^p_{\chi_{\mu_p(K)} \chi_{x_{j_1} x_{j_2} KK}} = a_{j_1} a_{j_2}^{T_{j_1}^K} \alpha^p_{\chi_{x_{j_1} x_{j_2} K}}.\end{aligned} \tag{6.10}$$

So, in $\mathbb{A}_N = A \times_{\lambda_o} \mathfrak{T}_N$, we have the equivalent formula of (6.10):

$$(a_{j_1} w_{j_1})(a_{j_1} w_{j_1}) = a_{j_1} a_{j_2}^{w_{j_1}} w_{j_1} w_{j_2}. \tag{6.11}$$

By regarding our K-subgroup dynamical C^*-algebra $\mathbb{A}_N^K$ as its $*$-isomorphic C^*-algebra $\mathbb{A}_N$, we obtain the following isomorphism theorem.

Theorem 6.4. *Let $\mathbb{A}_N^K = A \times_\lambda \mathfrak{T}_N^K$ be our K-subgroup dynamical C^*-algebra* (6.4) *induced by the K-subgroup dynamical system $(\mathfrak{T}_N^K, A, \lambda)$. Then this C^*-algebra $\mathbb{A}_N^K$ is $*$-isomorphic to the conditional tensor product C^*-algebra $\mathfrak{A}_N^K$,*

$$\mathfrak{A}_N^K \overset{def}{=} A \otimes_\lambda \mathfrak{C}^*_{K,N},$$

where $\mathfrak{C}^*_{K,N} = C^*_{\mathfrak{H}_p^K}\left(\mathfrak{T}_N^K\right)$ is the C^*-subalgebra in the sense of (5.7) *in $B(\mathfrak{H}_p^K)$, where the conditional tensor product $\otimes_\lambda$ satisfies the λ-relations:*

$$\left(a_1 \otimes T_{j_1}^K\right)\left(a_2 \otimes T_{j_2}^K\right) = a_1 a_2^{T_{j_1}^K} T_{j_1}^K T_{j_2}^K$$

and

$$\left(a \otimes T_j^K\right)^* = (a^*)^{T_j^K} \otimes T_j^K$$

for all j, $j_1, j_2 = 1, \ldots, N$, under linearity, i.e.,

$$\mathbb{A}_N^K = A \times_\lambda \mathfrak{T}_N^K \overset{*\text{-}iso}{=} A \otimes_\lambda \mathfrak{C}^*_{K,N} \overset{*\text{-}iso}{=} A \otimes_{\lambda_o} \mathfrak{C}^*_N \overset{*\text{-}iso}{=} \mathbb{A}_N. \tag{6.12}$$

Proof. Let us understand the K-subgroup dynamical C^*-algebra $\mathbb{A}_N^K$ induced by a K-subgroup dynamical system $(\mathfrak{T}_N^K, A, \lambda)$ as its $*$-isomorphic crossed product C^*-algebra $\mathbb{A}_N$ induced by an equivalent group-dynamical system $(\mathfrak{T}_N, A, \lambda_o)$, where

$$\mathfrak{T}_N = \left\langle \{w_j\}_{j=1}^N,\ \{w_j^{-1} = w_j\}_{j=1}^N \right\rangle. \tag{6.13}$$

First construct a conditional tensor product C^*-algebra $\mathfrak{A}_N$ by a C^*-subalgebra of the usual tensor product C^*-algebra $A \otimes_{\mathbb{C}} \mathfrak{C}^*_N$, as the conditional tensor product C^*-algebra,

$$\mathfrak{A}_N = A \otimes_{\lambda_o} \mathfrak{C}^*_N,$$

where $\mathfrak{C}^*_N = C^*_{l^2(\mathfrak{T}_N)}(\mathfrak{T}_N) \overset{*\text{-iso}}{=} \mathfrak{C}^*_{K,N}$, satisfying the λ_o-relations (6.14) and (6.15) below:

$$(a_1 \otimes w_{j_1})(a_2 \otimes w_{j_2}) = a_1 a_2^{w_{j_1}} \otimes w_{j_1} w_{j_2}, \tag{6.14}$$

where $a_2^{w_{j_1}} = \lambda_{o,w_{j_1}}(a_2)$ in A, for all $a_1, a_2 \in A$, and w_{j_1}, w_{j_2} are the generating elements of $\mathfrak{T}_N$ of (6.13), under linearity, and

$$(a \otimes w_j)^* = (a^*)^{w_j^{-1}} \otimes w_j^{-1} = (a^*)^{w_j} \otimes w_j, \tag{6.15}$$

since $w_j^{-1} = w_j$, for all $j = 1, \ldots, N$, under linearity, for all $a \in A$, and w_j are the generators of $\mathfrak{T}_N$ of (6.13), where $(a^*)^w = \lambda_{o,w}(a^*)$ in A.

Define now a morphism $\Phi : \mathfrak{A}_N \to \mathbb{A}_N$ by a linear transformation satisfying

$$\Phi\left(\sum_{i=1}^{n} t_i(a \otimes w_{j_i})\right) = \sum_{i=1}^{n} t_i\left(a w_{j_i}\right) \tag{6.16}$$

for all $\sum_{i=1}^{n} t_i(a \otimes w_{j_i}) \in \mathfrak{A}_N$, with $t_i \in \mathbb{C}, a \in A, w_{j_i}$ are the generating elements of $\mathfrak{T}_N$, for $n \in \mathbb{N} \cup \{\infty\}$. Then, as a generator-preserving morphism, Φ is bijective. Also, it satisfies that

$$\Phi\left((a_1 \otimes w_{j_1})(a_2 \otimes w_{j_2})\right) = \Phi\left(a_1 a_2^{w_{j_1}} \otimes w_{j_1} w_{j_2}\right)$$

by (6.14)

$$= a_1 a_2^{w_{j_1}} w_{j_1} w_{j_2} = \left(a_1 w_{j_1}\right)\left(a_2 w_{j_2}\right)$$

in $\mathbb{A}_N$

$$= \Phi\left(a_1 \otimes w_{j_1}\right) \Phi\left(a_2 \otimes w_{j_2}\right)$$

for all $a_1, a_2 \in A$ and the generators $w_{j_1}, w_{j_2} \in \mathfrak{T}_N$. Thus, this linear morphism Φ of (6.16) is multiplicative, i.e.,

$$\Phi\left(x_1 x_2\right) = \Phi(x_1)\Phi(x_2) \text{ in } \mathbb{A}_N \quad \text{for all} \quad x_1, x_2 \in \mathfrak{A}_N. \tag{6.17}$$

Furthermore, this multiplicative bijective linear transformation Φ satisfies that

$$\Phi\left((a \otimes w_j)^*\right) = \Phi\left((a^*)^{w_j} \otimes w_j\right)$$

by (6.15)

$$= (a^*)^{w_j} w_j = (a w_j)^*$$

in $\mathbb{A}_N$, by (6.6) under $\mathfrak{T}_N \overset{\text{Group}}{=} \mathfrak{T}_N^K$. Thus, we have

$$\Phi(x^*) = \Phi(x)^* \text{ in } \mathbb{A}_N \quad \text{for all} \quad x \in \mathfrak{A}_N. \tag{6.18}$$

Therefore, by (6.17) and (6.18), the bijective linear transformation Φ of (6.16) is both multiplicative and adjoint-preserving, equivalently, it is a $*$-isomorphism. So, two C^*-algebras $\mathfrak{A}_N$ and $\mathbb{A}_N$ are $*$-isomorphic, i.e.,

$$\mathfrak{A}_N = A \otimes_{\lambda_o} \mathfrak{C}_N^* \overset{*\text{-iso}}{=} A \times_{\lambda_o} \mathfrak{T}_N = \mathbb{A}_N. \tag{6.19}$$

By (6.8), (6.9) and (6.19), we obtain that

$$\mathbb{A}_N \overset{*\text{-iso}}{=} \mathfrak{A}_N \overset{*\text{-iso}}{=} A \otimes_{\lambda} \mathfrak{T}_N^K \overset{*\text{-iso}}{=} \mathbb{A}_N^K.$$

Therefore, the $*$-isomorphic relation (6.12) holds. □

The above characterization (6.12) shows that our K-subgroup dynamical C^*-algebra $\mathbb{A}_N^K$ induced by a K-subgroup dynamical system $(\mathfrak{T}_N^K, A, \lambda)$ is $*$-isomorphic to the conditional tensor product C^*-algebra

$$\mathfrak{A}_N^K \overset{denote}{=} A \otimes_\lambda \mathfrak{C}_{K,N}^*,$$

having λ-relations (6.5) and (6.6).

Now, we understand these $*$-isomorphic C^*-algebras $\mathfrak{A}_N, \mathfrak{A}_N^K, \mathbb{A}_N$ and $\mathbb{A}_N^K$, as the same C^*-algebra. Case-by-case, we will use suitable settings.

6.2. FREE PROBABILITY ON $\mathbb{A}_N^K$

In this section, we establish free probability on the K-subgroup dynamical C^*-algebra $\mathbb{A}_N^K$ induced by the K-subgroup dynamical system $(\mathfrak{T}_N^K, A, \lambda)$ As we discussed at the end of Section 6.1, we use the C^*-algebras $\mathbb{A}_N^K, \mathbb{A}_N, \mathfrak{A}_N^K$ and $\mathfrak{A}_N$ as the same C^*-algebra, here.

Recall that the C^*-algebra $\mathbb{A}_N^K$ is acting on the tensor product Hilbert space $\mathfrak{H}_{p,H}^K = H \otimes \mathfrak{H}_p^K$ of (6.1), whenever A is a C^*-(sub)algebra in the operator algebra $B(H)$.

The inner product $[\cdot,\cdot]$ on $\mathfrak{H}_{p,H}^K$ is naturally determined by

$$[h \otimes w, h' \otimes w'] = [h, h']_H [w, w']_p, \tag{6.20}$$

where $h, h' \in H$, and $w, w' \in \mathfrak{H}_p^K$, where $[\cdot,\cdot]_H$ means the *inner product on* H, and $[\cdot,\cdot]_p$ is the inner product (4.1) on $\mathfrak{H}_p^K$, inherited from that on $\mathfrak{H}_p$.

Now, let us fix an element h_0 in the Hilbert space H. And take the identity element

$$h_K \overset{denote}{=} 1_{\mathfrak{H}_p^K} = T^K = \alpha_{\chi_K}^p \tag{6.21}$$

in the subspace $\mathfrak{H}_p^K$ of the normal Hecke Hilbert space $\mathfrak{H}_p$. Fix now a Hilbert-space element $h_{p,H}^K \in \mathfrak{H}_{p,H}^K$,

$$h_{p,H}^K \overset{def}{=} h_0 \otimes h_K \in \mathfrak{H}_{p,H}^K, \tag{6.22}$$

where h_0 is arbitrarily fixed in H, and h_K is in the sense of (6.21) in $\mathfrak{H}_p^K$.

For the fixed Hilbert-space element $h_{p,H}^K$in $\mathfrak{H}_{p,H}^K$, define a morphism

$$\varphi_{p,H} : \mathbb{A}_N^K \to \mathbb{C}$$

by a linear functional satisfying

$$\varphi_{p:H}\,(aw) = \left[(a \otimes w)(h_{p,H}^K),\ h_{p,H}^K\right], \tag{6.23}$$

for all $aw = a \otimes w \in \mathbb{A}_N^K$, with $a \in A$ and $w \in \mathfrak{C}_{K,N}^*$ (by understanding $aw \in \mathbb{A}_N^K$ as $a \otimes w \in \mathfrak{A}_N^K$), under linearity, where $[\cdot,\cdot]$ is in the sense of (6.20).

Now, let $T = aT_j^K \in \mathbb{A}_N^K$, where $a \in A$ and T_j^K are the generators of the K-subgroup $\mathfrak{T}_N^K$ for $j = 1, \ldots, N$. Then

$$\begin{aligned}\varphi_{p:H}\left(aT_j^K\right) &= \left[(a \otimes T_j^K)(h_{p,H}^K),\ h_{p,H}^K\right] = \left[(a \otimes T_j^K)(h_0 \otimes h_p),\ h_0 \otimes h_p\right] \\ &= [a(h_0),\ h_0]_H \left[T_j^K h_p,\ h_p\right]_p = [a(h_0),\ h_0]_H \left[\alpha^p_{\chi_{x_jK}},\ \alpha^p_{\chi_K}\right]_p \\ &= [a(h_0),\ h_0]_H\, \varphi_p\left(\chi_{x_jK} * \chi_K\right)\end{aligned}$$

by (4.1)

$$\begin{aligned}&= \varphi_p\left(\chi_{x_jK}\right)[a(h_0),\ h_0]_H = \frac{\mu_p(x_jK \cap K)}{\mu_p(K)}[a(h_0),\ h_0]_H \\ &= \left(\mu_p\left(x_jK \cap K\right)\right)[a(h_0),\ h_0]_H\end{aligned} \tag{6.24}$$

by (3.8), (3.9).

The formula (6.24) shows that if we define a linear functional $\psi_0 : A \to \mathbb{C}$ on A by

$$\psi_0(a) \overset{def}{=} [a(h_0),\ h_0]_H \quad \text{for all} \quad a \in A, \tag{6.25}$$

then the linear functional $\varphi_{p,H}$ of (6.23) on $\mathbb{A}_N^K$ can be understood by

$$\varphi_{p:H} = \psi_0 \otimes \varphi_p \text{ on } \mathbb{A}_N^K, \tag{6.26}$$

by (6.24) and (6.25), in the sense that

$$\varphi_{p:H}(aw) = (\psi_0 \otimes \varphi_p)(a \otimes w) = (\psi_0(a))(\varphi_p(w))$$

for all $aw \in \mathbb{A}_N^K$ with $a \in A, w \in \mathfrak{T}_N^K \subset \mathfrak{C}_{K,N}^*$, under linearity, where ψ_0 is in the sense of (6.25) and φ_p is in the sense of (4.1).

By definition (6.23), and by (6.24) and (6.26), we get

$$\varphi_{p,H}\left(\sum_{i=1}^{n} t_i\, a_i w_i\right) = \sum_{i=1}^{n} t_i \varphi_{p,H}\left(a_i w_i\right) = \sum_{i=1}^{n} t_i \psi_0(a_i)\varphi_p(w_i)$$

for all $t_i \in \mathbb{C}, a_i \in A, w_i \in \mathfrak{T}_N^K \subset \mathfrak{C}_{K,N}^*$, for $i = 1, \ldots, n$, and $n \in \mathbb{N} \cup \{\infty\}$.

Proposition 6.5. *Let $\varphi_{p,H}$ be the linear functional* (6.23) *on the given K-subgroup dynamical C^*-algebra $\mathbb{A}_N^K$, and let ψ_0 and φ_p be the linear functionals in the sense of* (6.25) *and* (4.1)*, respectively. Then*

$$\varphi_{p,H} = \psi_0 \otimes \varphi_p.$$

Proof. The proof is done by (6.26). □

As we have discussed above, the linear functional $\varphi_{p,H}$ is well-determined on $\mathbb{A}_N^K$. So, the pair $\left(\mathbb{A}_N^K,\ \varphi_{p,H}\right)$ forms a C^*-probability space in the sense of [13] and [15].

Definition 6.6. The C^*-probability space $\left(\mathbb{A}_N^K,\ \varphi_{p,H}\right)$ is called the K-(subgroup-)dynamical C^*-probability space induced by a K-subgroup dynamical system $(\mathfrak{T}_N^K, A, \lambda)$.

6.3. FREE-DISTRIBUTIONAL DATA ON $(\mathbb{A}_N^K, \varphi_{p,H})$

In this section, we fix a K-subgroup dynamical system $(\mathfrak{T}_N^K,\, A, \lambda)$, and its corresponding crossed product C^*-algebra $\mathbb{A}_N^K = A \times_\lambda \mathfrak{T}_N^K$, understood also by its $*$-isomorphic C^*-algebras $\mathbb{A}_N, \mathfrak{A}_N, \mathfrak{A}_N^K$, where the C^*-algebra A is acting on a Hilbert space H in $B(H)$. Also, let $\left(\mathbb{A}_N^K,\ \varphi_{p:H}\right)$ be the K-dynamical C^*-probability space in the sense of Section 6.2. Here, we are interested in free-distributional data of certain operators of $\mathbb{A}_N^K$ in terms of $\varphi_{p,H}$, represented by free-moments or free-cumulats.

Recall and note that

$$\varphi_{p:H} = \psi_0 \otimes \varphi_p \text{ on } \mathfrak{A}_N^K = \mathbb{A}_N^K, \tag{6.27}$$

as in (6.26).

We concentrate on studying free distributions of the operators formed by $aT_j^K \in (\mathbb{A}_N^K, \varphi_{p,H})$ with $a \in A$ and $T_j^K = \alpha^p_{\chi_{x_j K}}$, which are the generators of $\mathfrak{T}_N^K \subset \mathfrak{C}^*_{K,N}$, for $j = 1, \ldots, N$.

For convenience, we use the terms aw_j for aT_j^K for $j = 1, \ldots, N$, in this section.

Let $a_i w_{j_i}$ be such free random variables $a_i T_{j_i}^K$ in the K-dynamical C^*-probability space $(\mathbb{A}_N^K, \varphi_{p,H})$ for $i = 1, \ldots, n$, where $a_1, \ldots, a_n \in A$ and $w_{j_i} = T_{j_i}^K$ are generators of $\mathfrak{T}_N^K \subset \mathfrak{C}^*_{K,N}$ for $i = 1, \ldots, n$, for some $n \in \mathbb{N}$. Then

$$\prod_{i=1}^{n} a_i w_{j_i} = a_1 a_2^{w_{j_1}} a_3^{w_{j_1} w_{j_2}} \ldots a_n^{w_{j_1} w_{j_2} \ldots w_{j_{n-1}}} w_{j_1} w_{j_2} \ldots w_{j_i}, \tag{6.28}$$

where $a^w = \lambda_w(a)$ in A.

So, by (6.26) and (6.28), we obtain the following free-momental information.

Proposition 6.7. *Let $a_i w_{j_i}$ be free random variables of the given K-dynamical C^*-probability space $\left(\mathbb{A}_N^K,\ \varphi_{p,H}\right)$, with $a_i \in A$, and $w_{j_i} = \alpha^p_{\chi_{x_j K}}$ are the generators of $\mathfrak{T}_N^K \subset \mathfrak{C}^*_{K,N}$, for $i = 1, \ldots, n$, for $n \in \mathbb{N}$. Then*

$$\varphi_{p,H}\left(\prod_{i=1}^{n} a_i w_{j_i}\right) = \left(\mu_p\left(\left(\prod_{i=1}^{n} x_{j_i}\right) K \cap K\right)\right) \left[\left(\prod_{i=1}^{n} a_i^{\prod_{k=1}^{i-1} w_{j_i}}\right)(h_0),\ h_0\right]_H, \tag{6.29}$$

with axiomatization $a_1^{\prod_{k=1}^{0} w_{j_i}} = a_1$ in A.

Proof. Observe that

$$\varphi_{p,H}\left((a_1 w_{j_1})(a_2 w_{j_2})\dots(a_n w_{j_n})\right)$$
$$= \varphi_{p,H}\left(a_1 a_2^{w_{j_1}} a_3^{w_{j_1}w_{j_2}} \dots a_n^{w_{j_1}\dots w_{j_{n-1}}} w_{j_1} w_{j_2} \dots w_{j_n}\right)$$

by (6.28)

$$= \psi_0\left(a_1 a_2^{w_{j_1}} \dots a_n^{w_{j_1}\dots w_{j_{n-1}}}\right) \varphi_p\left(T_{j_1}^K \dots T_{j_n}^K\right)$$

by (6.26)

$$= \psi_0\left(a_1 a_2^{w_{j_1}} \dots a_n^{w_{j_1}\dots w_{j_{n-1}}}\right) \varphi_p\left(\alpha^p_{\chi_{x_{j_1}\dots x_{j_n}K}}\right)$$
$$= \left(\psi_0\left(a_1 a_2^{w_{j_1}} \dots a_n^{w_{j_1}\dots w_{j_{n-1}}}\right)\right)\left(\mu_p\left(x_{j_1}\dots x_{j_n}K \cap K\right)\right),$$

for all $(j_1, \dots, j_n) \in \{1, \dots, N\}^n$ and $(a_1, \dots, a_n) \in A^n$ and $n \in \mathbb{N}$. □

The above formula (6.29) provides general joint free-momental free-distributional data of $a_i w_{j_i}$ for $i = 1, \dots, n$ and $n \in \mathbb{N}$.

By the Möbius inversion of Section 2.2, one can get the following equivalent free-distributional data.

Proposition 6.8. *Let $a_i w_{j_i}$ be free random variables in the K-dynamical C^*-probability space $\left(\mathbb{A}_N^K, \varphi_{p,H}\right)$ with $a_i \in A$, and $w_{j_i} = \alpha^p_{\chi_{x_{j_i}K}}$ are generators of $\mathfrak{T}_N^K \subset \mathfrak{C}^*_{K,N}$, for $i = 1, \dots, n$ and $n \in \mathbb{N}$. Then we have*

$$k_n^{p,H}\left(a_1 w_{j_1}, \dots, a_n w_{j_n}\right) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \varphi_{p,H}(V)\right) \mu(\pi, 1_n), \tag{6.30}$$

where

$$\varphi_{p,H}(V) = \mu_p\left(\prod_{k \in V}^{o} x_{j_{i_k}} K \cap K\right)\left[\left(\prod_{k \in V}^{o} a_{i_k}\right)(h_0), h_0\right]_H,$$

where $\prod^o$ means the product under order in the sense that if

$$V = (i_1 < i_2 < \dots < i_k) \text{ in } \pi \in NC(n),$$

then

$$\prod_{k \in V}^{o} x_{j_{i_k}} = x_{j_{i_1}} x_{j_{i_2}} \dots x_{j_{i_k}} \text{ in } G_p,$$

and

$$\prod_{k \in V}^{o} a_{i_k} = a_{i_1} a_{i_2} \dots a_{i_k} \text{ in } A,$$

where $k_n^{p,H}(\dots)$ means the free cumulant determined by the linear functional $\varphi_{p,H}$.

Proof. The proof of (6.31) is done by the free-probabilistic Möbius inversion of [15] and by (6.29). □

Now, both by (6.29) and by (6.30), we obtain the following necessary freeness condition.

Theorem 6.9. *Let $a_i w_{j_i} \in \left(\mathbb{A}_N^K,\ \varphi_{p,H}\right)$ be given by above propositions for $i = 1, 2$ with assumption $j_1 < j_2$. If a_1 and a_2 are free in (A, ψ_0) and if*

$$\mu_p\left(\left(\prod_{i=1}^{n} x_{j_i}\right) K \cap K\right) = 1 \tag{6.31}$$

for all $(j_1, \ldots, j_n) \in \{1, 2\}^n$ and $n \in \mathbb{N}$, then $a_1 w_{j_1}$ and $a_2 w_{j_2}$ are free in $\left(\mathbb{A}_N^K,\ \varphi_{p,H}\right)$.

Proof. Suppose the condition (6.31) holds. Then

$$k_n^{p,H}\left(a_{i_1} w_{j_{i_1}}, \ldots, a_{i_n} w_{j_{i_n}}\right)$$
$$= \sum_{\pi \in NC(n)} \left(\mu_p\left(\prod_{k \in V}^{o} x_{j_{i_k}} K \cap K\right)\left[\left(\prod_{k \in V}^{o} a_{i_k}\right)(h_0), h_0\right]_H\right) \mu(\pi, 1_n)$$

by [13]

$$= \sum_{\pi \in NC(n)} \left(\left[\left(\prod_{k \in V}^{o} a_{i_k}\right)(h_0), h_0\right]_H\right) \mu(\pi, 1_n)$$

by (6.31)

$$= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \psi_0\left(\prod_{k \in V}^{o} a_{i_k}\right)\right) \mu(\pi, 1_n)$$
$$= \sum_{\pi \in NC(n)} \psi_{0:\pi}\left(a_{i_1}, \ldots, a_{i_n}\right) \mu(\pi, 1_n)$$
$$= k_n^{\psi_0}\left(a_{i_1}, \ldots, a_{i_n}\right)$$

where $k_n^{\psi_0}(\ldots)$ means the free cumulant on (A, ψ_0) determined by the linear functional ψ_0

$$= 0, \tag{6.32}$$

by the assumption that a_1 and a_2 are free in (A, ψ_0).

It shows that the mixed free cumulants of $a_1 w_{j_1}$ and $a_2 w_{j_2}$ vanish under the condition (6.32), and hence, they are free in $\left(\mathbb{A}_N^K,\ \varphi_{p,H}\right)$. □

7. APPLICATIONS

In this section, we keep studying our K-dynamical C^*-probability space $\left(\mathbb{A}_N^K,\ \varphi_{p,H}\right)$. In particular, we will take some specific C^*-algebras A for a given K-subgroup dynamical C^*-algebra $\mathbb{A}_N^K = A \times_\lambda \mathfrak{T}_N^K$. As in Section 6, we understand $*$-isomorphic C^*-algebras

$$\mathfrak{A}_N^K = A \otimes_\lambda \mathfrak{C}_{K,N}^*, \quad \mathfrak{A}_N = A \otimes_{\lambda_o} \mathfrak{C}_N^*,$$
$$\mathbb{A}_N = A \times_{\lambda_o} \mathfrak{T}_N,$$

and $\mathbb{A}_N^K$ as the same C^*-algebra.

7.1. A IS A GROUP C^*-ALGEBRA IN $(\mathfrak{T}_N^K, A, \lambda)$

As a first case, let us consider a K-subgroup dynamical system $(\mathfrak{T}_N^K, A_G, \lambda)$ is given where A_G is a *group C^*-algebra*.

Let G be a discrete countable group. Then one can construct the corresponding *group-Hilbert space* $H_G = l^2(G)$ as a l^2*-space* with its *orthonormal basis*

$$\mathcal{B}_G = \{\xi_g : g \in G\} \subset H_G,$$

i.e., for the l^2*-inner product* $[\cdot,\cdot]_2$ on H_G, we have

$$[\xi_{g_1}, \xi_{g_2}]_2 = \delta_{g_1,g_2} \quad \text{for all} \quad g_1, g_2 \in G,$$

where δ means the *Kronecker delta*. Remark that, among orthonormal-basis elements in H_G, we have multiplication

$$\xi_{g_1}\xi_{g_2} = \xi_{g_1 g_2} \text{ in } H_G$$

for all $g_1, g_2 \in G$.

Thus, one can construct the *left-regular unitary representation* λ_G as a group-action of G acting on H_G by

$$\lambda_G(g) = u_g \in B(H_G), \text{ a unitary on } H_G,$$

such that

$$u_g(\xi_{g'}) = \xi_g \xi_{g'} = \xi_{gg'} \quad \text{for all} \quad g' \in G$$

and

$$u_g^* = u_g^{-1} \text{ on } H_G$$

for all $g \in G$.

Then the subset $\lambda_G(G)$ of $B(H_G)$ generates the C^*-algebra

$$A_G \overset{def}{=} C_{H_G}^*\left(\lambda_G(G)\right) = \overline{\mathbb{C}\left[\lambda_G(G)\right]} \text{ in } B(H_G), \tag{7.1}$$

where $\overline{X}$ means the topological closure in $B(H_G)$ under operator-norm topology.

Definition 7.1. We call the C^*-subalgebra A_G of $B(H_G)$ generated by $G = \lambda_G(G)$, the *group C^*-algebra of G*.

All elements u of a group C^*-algebra A_G of G are expressed by

$$u = \sum_{g\in G} t_g \lambda_G(g) = \sum_{g\in G} t_g u_g \quad \text{with} \quad t_g \in \mathbb{C}.$$

In fact, our K-subgroup C^*-subalgebra $\mathfrak{C}^*_{K,N} = C^*_{\mathfrak{H}_p^K}\left(\mathfrak{T}_N^K\right)$ is in fact under a similar setting in $B(\mathfrak{H}_p^K)$, since the generators $T_j^K = \alpha^p_{\chi_{x_j K}}$ are unitaries on $\mathfrak{H}_p^K$ for $j = 1, \ldots, N$.

Let us fix a discrete countable group G, and its corresponding group C^*-algebra A_G acting on the group Hilbert space H_G, and assume we have a K-subgroup dynamical system $(\mathfrak{T}_N^K, A_G, \lambda)$ generating the K-subgroup dynamical C^*-algebra

$$\mathbb{A}_N^K = A_G \times_\lambda \mathfrak{T}_N^K \text{ in } B(\mathfrak{H}^K_{p,H_G}),$$

where

$$\mathfrak{H}^K_{p,H_G} = H_G \otimes \mathfrak{H}_p^K.$$

Remark that it is $*$-isomorphic to the conditional tensor product C^*-algebra

$$\mathfrak{A}_N^K = A_G \otimes_\lambda \mathfrak{C}^*_{K,N}.$$

We consider the more detailed isomorphism theorem in this special case.

Theorem 7.2. *Let $\mathbb{A}^K_{N,G} = A_G \times_\lambda \mathfrak{T}_N^K$ be the K-subgroup dynamical C^*-algebra induced by the K-subgroup dynamical system $(\mathfrak{T}_N^K, A_G, \lambda)$, where A_G is a group C^*-algebra in the sense of* (7.1)*. Then*

$$\mathbb{A}^K_{N,G} \overset{*\text{-iso}}{=} C^*_{\mathfrak{H}^K_{p,H_G}}\left(G \times^\lambda \mathfrak{T}_N^K\right), \tag{7.2}$$

where $G \times^\lambda \mathfrak{T}_N^K$ is the semi-product group of G and $\mathfrak{T}_N^K$ with its operation

$$(g_1, w_1)(g_2, w_2) = (g_1 g_2^{w_1}, w_1 w_2)$$

for $g_1, g_2 \in G$ and $w_1, w_2 \in \mathfrak{T}_N^K$, where $g_1 g_2^{w_1}$ is under operation on G and $w_1 w_2$ is under operation on $\mathfrak{T}_N^K$. Here

$$g_2^{w_1} = \lambda_G^{-1}\left(\lambda_{w_1}(u_{g_2})\right),$$

where λ_G is the left-regular representation of G as in (7.1)*, and λ is the group-action of $\mathfrak{T}_N^K$ acting on $\mathfrak{H}^K_{p,H_G}$ in the sense of* (6.3)*.*

Proof. First, define the semi-product group

$$G_N^K = G \times^\lambda \mathfrak{T}_N^K \text{ of groups } G \text{ and } \mathfrak{T}_N^K \tag{7.3}$$

by the subgroup of the usual product group $G \times \mathfrak{T}_N^K$ satisfying the operation

$$(g_1, w_1)(g_2, w_2) = (g_1 g_2^{w_1},\ w_1 w_2),$$

where

$$g_2^{w_1} = \lambda_G^{-1}(\lambda_{w_1}(u_{g_2})) \text{ in } G,$$

where $\lambda_{w_1} \in Aut(A_G)$ and $u_{g_2} = \lambda_G(g_2) \in A_G$, for $g_2 \in G$ and $w_1 \in \mathfrak{T}_N^K$. Then the operation is closed and associative. And it acts on the Hilbert space

$$\mathfrak{H}_{p,H_G}^K = H_G \otimes \mathfrak{H}_p^K$$

via a group-action $\lambda_G \otimes \lambda$. So, one can have the representation $(\mathfrak{H}_{p,H_G}^K, \lambda_G \otimes \lambda)$ of the group G_N^K in (7.3). Consider now the group C^*-algebra

$$\mathcal{A}_{N,G}^K \overset{def}{=} C^*_{\mathfrak{H}_{p,H_G}^K}\left(G_N^K\right) \tag{7.4}$$

as a C^*-subalgebra of $B\left(\mathfrak{H}_{p,H_G}^K\right)$.

Define a linear transformation

$$\Psi : \mathfrak{A}_{N,G}^K = A_G \otimes_\lambda \mathfrak{C}_{K,N}^* \to \mathcal{A}_{N,G}^K$$

by the morphism satisfying

$$\Psi(u_g \otimes w_j) = (\lambda_G \otimes \lambda)(g,\ w_j) \tag{7.5}$$

for all $g \in G$ and the generators w_j of $\mathfrak{T}_N^K$ for $j = 1, \ldots, N$. It is not difficult to check this linear transformation Ψ is bijective, by the very construction (7.5). Then it satisfies that

$$\begin{aligned}\Psi((u_{g_1} \otimes w_{j_1})(u_{g_2} \otimes w_{j_2})) &= \Psi\left(u_{g_1 g_2^{w_{j_1}}} \otimes w_{j_1} w_{j_2}\right)\\ &= (\lambda_G \otimes \lambda)\left(g_1 g_2^{w_{j_1}},\ w_{j_1} w_{j_2}\right)\\ &= ((\lambda_G \otimes \lambda)(g_1, w_{j_1}))((\lambda_G \otimes \lambda)(g_2, w_{j_2}))\\ &= \Psi(u_{g_1} \otimes w_{j_1})\,\Psi(u_{g_2} \otimes w_{j_2})\end{aligned}$$

for all $g_1, g_2 \in G$ and generators w_{j_1} and w_{j_2} of $\mathfrak{T}_N^K \subset \mathfrak{C}_{K,N}^*$. So, for any $y_1, y_2 \in \mathfrak{A}_{N,G}^K$, we have

$$\Psi(y_1 y_2) = \Psi(y_1)\Psi(y_2) \text{ in } \mathcal{A}_{N,G}^K,$$

i.e., this bijective linear transformation Ψ of (7.5) is multiplicative.

Observe also that

$$\Psi((u_g \otimes w_j)^*) = \Psi\left(u_{g^{-1}} \otimes w_j\right)$$

since $u_g^* = u_{g^{-1}} = u_g^{-1}$ on H_G, under the left regular unitary representation

$$= (\lambda_G \otimes \lambda)(g^{-1}, w_j) = (\lambda_G \otimes \lambda)(g^{-1}, w_j^{-1})$$

since w_j are self-invertible for all $j = 1, \ldots, N$

$$= (\lambda_G \otimes \lambda)\left((g,\ w_j)^{-1}\right) = \Psi\left(u_g \otimes w_j\right)^*$$

for all $g \in G$ and generators w_j of $\mathfrak{T}_N^K \subset \mathfrak{C}_{K,N}^*$ for all $j = 1, \ldots, N$. Therefore, under linearity of Ψ, it satisfies

$$\Psi(y^*) = \Psi(y)^* \text{ in } \mathcal{A}_{N,G}^K \quad \text{for all} \quad y \in \mathfrak{A}_{N,G}^K.$$

So, the multiplicative bijective linear transformation Ψ of (7.5) is in fact a $*$-isomorphism from $\mathfrak{A}_{N,G}^K$ onto $\mathcal{A}_{N,G}^K$. It shows that two C^*-algebras $\mathfrak{A}_{N,G}^K$ and $\mathcal{A}_{N,G}^K$ are $*$-isomorphic. Since $\mathfrak{A}_{N,G}^K$ and $\mathbb{A}_{N,G}^K$ are $*$-isomorphic by (6.12), we can conclude that

$$\mathfrak{A}_{N,G}^K \overset{*\text{-iso}}{=} \mathcal{A}_{N,G}^K \overset{*\text{-iso}}{=} \mathbb{A}_{N,G}^K.$$

Therefore, our K-subgroup dynamical C^*-algebra $\mathbb{A}_{N,G}^K = A_G \times_\lambda \mathfrak{T}_N^K$ is $*$-isomorphic to the group C^*-algebra $\mathcal{A}_{N,G}^K = C^*_{\mathfrak{H}_{p,H_G}^K}\left(G_N^K\right)$, where G_N^K is in the sense of (7.3). □

The above characterization (7.2) shows that if our K-subgroup C^*-algebra $\mathbb{A}_{N,G}^K = A_G \times_\lambda \mathfrak{T}_N^K$ is induced by a group C^*-algebra A_G, then it is understood as a new group C^*-algebra $\mathcal{A}_N^K = C^*_{\mathfrak{H}_{p,H_G}^K}\left(G_N^K\right)$ generated by the semi-product group

$$G_N^K = G \times^\lambda \mathfrak{T}_N^K.$$

Now, let us concentrate on the K-dynamical C^*-probability space

$$\left(\mathbb{A}_{N,G}^K,\ \varphi_{p,H_G}\right),$$

where $\mathbb{A}_{N,G}^K = A_G \times_\lambda \mathfrak{T}_N^K$, for a group G.

Define canonically the linear functional ψ_G on the group C^*-algebra A_G by

$$\psi_G\left(\sum_{g\in G} t_g u_g\right) \overset{def}{=} t_{e_G} = \left[\left(\sum_{g\in G} t_g u_g\right)(\xi_{e_G}),\ \xi_{e_G}\right]_2, \tag{7.6}$$

where e_G is the *group-identity of* G. Then this linear functional ψ_G is not only a well-defined linear functional but also it is a *trace* in the sense that

$$\psi_G\left(y_1 y_2\right) = \psi_G\left(y_2 y_1\right) \quad \text{for all} \quad y_1, y_2 \in A_G$$

(e.g., [12]), i.e., even though $y_1 y_2 \neq y_2 y_1$ in A_G, one has the same tracial (or linear-functional) values for them under ψ_G in $\mathbb{C}$.

Then, by (6.26), (6.27) and (7.6), we have a well-defined linear functional $\varphi_{p,H_G} = \psi_G \otimes \varphi_p$ on the K-subgroup dynamical C^*-algebra

$$\mathfrak{A}_{N,G}^K = A_G \otimes_\lambda \mathfrak{C}_{K,N}^* = \mathbb{A}_{N,G}^K,$$

as in Sections 6.2 and 6.3. And it forms our K-dynamical C^*-probability space $\left(\mathbb{A}_{N,G}^K,\ \varphi_{p,H_G}\right)$.

Proposition 7.3. *Let $u_{g_i}w_{j_i}$ be free random variables in a K-dynamical C^*-probability space $\left(\mathbb{A}_{N,G}^K,\ \varphi_{p,H_G}\right)$ for $g_i \in G$ and generators $w_{j_i} = \alpha^p_{\chi_{x_j K}}$ of the K-subgroup $\mathfrak{T}_N^K$ for $i = 1, \ldots, n$ and $n \in \mathbb{N}$. Then*

$$\varphi_{p,H_G}\left(\prod_{i=1}^{n} u_{g_i}w_{j_i}\right) = \delta_{\prod\limits_{i=1}^{n} g_i,\ e_G}\ \mu_p\left(\prod_{i=1}^{n} x_{j_i}K \cap K\right), \tag{7.7}$$

where δ means the Kronecker delta.

Proof. Observe that

$$\begin{aligned}&\varphi_{p,H_G}\left((u_{g_1}w_{j_1})(u_{g_2}w_{j_2})\ldots(u_{g_n}w_{j_n})\right)\\ &\quad= \psi_G\left(u_{g_1}u_{g_2}\ldots u_{g_n}\right)\varphi_p\left(w_{j_1}w_{j_2}\ldots w_{j_n}\right)\end{aligned}$$

by (6.27)

$$= \psi_G\left(u_{g_1g_2\ldots g_n}\right)\left(\mu_p\left((x_{j_1}x_{j_2}\ldots x_{j_n})K \cap K\right)\right)$$

by (6.29)

$$= \begin{cases}\mu_p\left((x_{j_1}\ldots x_{j_n})K \cap K\right) & \text{if } g_1g_2\ldots g_n = e_G,\\ 0 & \text{otherwise,}\end{cases}$$

by (7.6), for all $g_1, \ldots, g_n \in G$ and generators $w_{j_1} = \alpha^p_{\chi_{x_{j_1}K}}, \ldots, w_{j_n} = \alpha^p_{\chi_{x_{j_n}K}}$ of $\mathfrak{T}_N^K$ for all $n \in \mathbb{N}$. □

By (7.7), one has the following equivalent free-distributional data on $\left(\mathbb{A}_{N,G}^K,\ \varphi_{p,H_G}\right)$ via the Möbius inversion of Section 2.2.

Proposition 7.4. *Let $u_{g_i}w_{j_i}$ be free random variables of $\left(\mathbb{A}_{N,G}^K,\ \varphi_{p,H_G}\right)$, where $g_i \in G - \{e_G\}$, and $w_{j_i} = \alpha^p_{\chi_{x_{j_i}K}}$ are generators of $\mathfrak{T}_N^K \subset \mathfrak{C}_{K,N}^*$, for $i = 1, \ldots, n$, for $n \in \mathbb{N}$. Then*

$$\begin{aligned}&k_n^{p,H_G}\left(u_{g_1}w_{j_1}, \ldots, u_{g_n}w_{j_n}\right)\\ &= \begin{cases}\sum\limits_{\pi \in NC_e^U(n)}\left(\prod\limits_{V\in\pi}\mu_p\left(\left(\prod\limits_{k\in V}^{o} x_{j_k}\right)K \cap K\right)\right)\mu(\pi, 1_n) & \textit{if } n \in 2\mathbb{N},\\ 0 & \textit{otherwise,}\end{cases}\end{aligned} \tag{7.8}$$

where $NC_e^U(n)$ is the subset of the noncrossing partition lattice $NC(n)$ introduced in (7.11) and (7.12) below.

Proof. Under the notations used in (6.31), we have that

$$k_n^{p,H_G}\left(u_{g_1}w_{j_1}, \ldots, u_{g_n}w_{j_n}\right) = \sum_{\pi\in NC(n)}\left(\prod_{V\in\pi}\varphi_{p,H_G}(V)\right)\mu(\pi, 1_n) \tag{7.9}$$

where

$$\varphi_{p,H_G}(V) = \delta_{\prod\limits_{k \in V}^{o} g_k,\, e_G} \cdot \mu_p \left(\prod_{k \in V}^{o} x_{j_k} K \cap K \right),$$

by (6.31) and (7.7), for all $V \in \pi$, $\pi \in NC(n)$ and $n \in \mathbb{N}$.

Remark first that if n is odd in $\mathbb{N}$, then each noncrossing partition $\pi \in NC(n)$ contains at least one odd block V in the sense that V has an odd cardinality, i.e., $|V|$ is odd. If V is odd, then

$$\delta_{\prod\limits_{k \in V}^{o} g_k,\, e_G} = 0, \tag{7.10}$$

since each g_k are assumed to be a non-identity in G. Therefore, the corresponding block-depending free moment $\varphi_{p,H_G}(V) = 0$, and hence, the corresponding partition-depending free moment vanish, too;

$$\varphi_{p,H_G}(\pi) = \prod_{B \in \pi} \varphi_{p,H_G}(B) = (\varphi_{p,H_G}(V)) \left(\prod_{B \neq V \in \pi} \varphi_{p,H_G}(B) \right) = 0,$$

by (7.10). Therefore, whenever n is odd in $\mathbb{N}$, the free cumulant computation (7.9) vanish, i.e.,

$$k_n^{p,H_G}(u_{g_1} w_{j_1}, \ldots, u_{g_n} w_{j_n}) = 0, \text{ whenever } n \text{ is odd.} \tag{7.11}$$

Assume now that n is even in $\mathbb{N}$. Define now a subset $NC_e(n)$ of $NC(n)$ by

$$NC_e(n) = \{\pi \in NC(n) : \pi \text{ has only even blocks}\},$$

i.e., if $\theta \in NC_e(n)$ and if $V \in \theta$ is a block, then $|V|$ is even, and vice versa.

Now, for the n-tuple of our given free random variables (under order)

$$U = (u_{g_1} w_{j_1},\ u_{g_2} w_{j_2},\ \ldots,\ u_{g_n} w_{j_n}),$$

define a subset $NC_e^U(n)$ of $NC_e(n)$ by

$$NC_e^U(n) = \left\{ \theta \in NC_e(n) \,\middle|\, \begin{array}{c} \text{for all } V = (i_1, \ldots, i_n) \in \theta, \\ g_{i_1} g_{i_2} \ldots g_{i_n} = e_G \text{ in } G \end{array} \right\}. \tag{7.12}$$

By the construction of the subset $NC_e^U(n)$ of (7.12) for an arbitrarily fixed even number $n \in \mathbb{N}$, the formula (7.9) becomes

$$\begin{aligned} k_n^{p,H_G}(u_{g_1} w_{j_1}, \ldots, u_{g_n} w_{j_n}) &= \sum_{\pi \in NC_e^U(n)} \varphi_{p,H_G}(\pi) \mu(\pi, 1_n) \\ &= \sum_{\pi \in NC_e^U(n)} \left(\prod_{V \in \pi} \varphi_{p,H_G}(V) \right) \mu(\pi, 1_n) \\ &= \sum_{\pi \in NC_e^U(n)} \left(\prod_{V \in \pi} \mu_p \left(\left(\prod_{k \in V}^{o} x_{j_k} \right) K \cap K \right) \right) \mu(\pi, 1_n). \quad \square \end{aligned}$$

Recall that, by (6.32), if a_1 and a_2 are free in (A, ψ_0) and if

$$\mu_p(x_{j_i} K \cap K) = 1 = \mu_p(x_{j_1} x_{j_2} K \cap K)$$

for $x_{j_1}, x_{j_2} \in G_p$, inducing the generator $w_{j_i} = \alpha^p_{\chi_{x_{j_i}K}}$ of our K-subgroup $\mathfrak{T}^K_N$ for $i = 1, 2$, then two free random variables $a_1 w_{j_1}$ and $a_2 w_{j_2}$ are free in the K-dynamical C^*-probability space $\left(\mathbb{A}^K_N,\ \varphi_{p,H}\right)$ in general, for given C^*-probability space (A, ψ_0).

Proposition 7.5. *Suppose a group G is the free group F_n with n-generators $\{g_1, \ldots, g_n\}$, for $n \in \mathbb{N}$ and assume further that $w_j = \alpha^p_{\chi_{x_j K}}$ are the generators of our K-subgroup $\mathfrak{T}^K_N$, satisfying*

$$\mu_p\left(\prod_{k=1}^{n} x_{j_k} K \cap K\right) = 1$$

for all $(j_1, \ldots, j_n) \in \{1, \ldots, N\}^n$ and $n \in \mathbb{N}$. Then the K-subgroup C^*-algebra $\mathbb{A}^K_{N,G}$ is $*$-isomorphic to

$$\mathbb{A}^K_{N,G} \overset{*\text{-}iso}{=} \underset{i=1}{\overset{n}{*_{\mathbb{C}}}} \left(C^*(\mathbb{Z}) \times_\lambda \mathfrak{T}^K_N\right) \overset{*\text{-}iso}{=} \underset{i=1}{\overset{n}{*_{\mathbb{C}}}} \left(C^*(\mathbb{Z}) \otimes_\lambda \mathfrak{C}^*_{K,N}\right), \tag{7.13}$$

where $(*_{\mathbb{C}})$ means the topological free product algebra, and where $C^*(\mathbb{Z})$ is the group C^*-algebra generated by the abelian infinite cyclic group $\mathbb{Z}$ of the integers.

Proof. By (6.32), the free random variables $a_i w_j$ are free from each other in the K-dynamical C^*-probability space $\left(\mathbb{A}^K_{N,G},\ \varphi_{p,H_G}\right)$, where G is the free group F_n with n-generators $\{g_1, \ldots, g_n\}$ for $n \in \mathbb{N}$. Recall also that under the canonical trace ψ_G, the group C^*-algebra A_G generated by the free group G is $*$-isomorphic to

$$A_G = A_{F_n} \overset{*\text{-iso}}{=} A_{F_{n_1}} *_{\mathbb{C}} A_{F_{n_2}} \overset{*\text{-iso}}{=} \underset{i=1}{\overset{n}{*_{\mathbb{C}}}} C^*(\mathbb{Z}), \tag{7.14}$$

whenever $n_1 + n_2 = n$ for $n_1, n_2 \in \mathbb{N}$ (e.g., [12] and [15]). Remark again that the above $*$-isomorphic relation (7.14) is determined by the trace ψ_G on A_G. So, one can get that

$$\begin{aligned}\mathbb{A}^K_{N,G} &\overset{*\text{-iso}}{=} \mathfrak{A}^K_{N,G} = A_G \otimes_\lambda \mathfrak{C}^*_{K,N} \\ &\overset{*\text{-iso}}{=} \left(\underset{i=1}{\overset{n}{*_{\mathbb{C}}}}\ C^*(\mathbb{Z})\right) \otimes_\lambda \mathfrak{C}^*_{K,N}\end{aligned}$$

by (7.14)

$$\overset{*\text{-iso}}{=} \underset{i=1}{\overset{n}{*_{\mathbb{C}}}} \left(C^*(\mathbb{Z}) \otimes_\lambda\ \mathfrak{C}^*_{K,N}\right)$$

by (6.32). $\square$

7.2. A IS A CERTAIN QUOTIENT ALGEBRA OF $M_n(\mathbb{C})$ IN $(\mathfrak{T}_N^K, A, \lambda)$

In this section, we fix $n \in \mathbb{N} \setminus \{1\}$, and a K-subgroup dynamical system $(\mathfrak{T}_N^K, A_n, \lambda)$ induced by the K-subgroup

$$\mathfrak{T}_N^K = \left\langle \{w_j = \alpha^p_{\chi_{x_j K}}\}_{j=1}^N \right\rangle$$

generating the corresponding K-subgroup dynamical C^*-algebra

$$\mathbb{A}_{N,n}^K = A_n \times_\lambda \mathfrak{T}_N^K.$$

Here A_n is a certain *quotient algebra* of the *matricial algebra* $M_n(\mathbb{C})$ for $n \in \mathbb{N}$. Throughout this section, we fix $n \in \mathbb{N} \setminus \{1\}$.

Now, let tr_n be the usual *trace on* $M_n(\mathbb{C})$,

$$tr_n\left([t_{ij}]_{n\times n}\right) = \sum_{k=1}^{n} t_{kk} \tag{7.15}$$

for all $(n \times n)$-matrices $[t_{ij}]_{n\times n} \in M_n(\mathbb{C})$ with $t_{ij} \in \mathbb{C}$ for all $i, j = 1, \ldots, n$.

The algebra A_n is defined by the quotient algebra of $M_n(\mathbb{C})$ by an equivalence relation $\mathcal{R}$,

$$A_n = M_n(\mathbb{C})/\mathcal{R}, \tag{7.16}$$

where

$$a_1 \mathcal{R} a_2 \overset{def}{\Longleftrightarrow} spec(a_1) = spec(a_2)$$

for all $a_1, a_2 \in M_n(\mathbb{C})$, where $spec(a)$ means the *spectrum of* a for all $a \in M_n(\mathbb{C})$.

Recall that the spectrum $spec(a)$ of a matrix a is the collection of all *eigenvalues of* $a \in M_n(\mathbb{C})$. Recall also that, two matrices a_1 and a_2 are *unitarily equivalent* in $M_n(\mathbb{C})$ if and only if

$$spec(a_1) = spec(a_2)$$

as subsets of $\mathbb{C}$. So, the above equivalence relation $\mathcal{R}$ of (7.16) again means that

$$a_1 \mathcal{R} a_2 \iff a_1 \text{ and } a_2 \text{ are unitarily equivalent in } M_n(\mathbb{C}).$$

Note that whenever a is given in $M_n(\mathbb{C})$ with its spectrum

$$spec(a) = \{t_1, t_2, \ldots, t_n\}$$

(without considering multiplicities of eigenvalues), one can find so-called the *spectral form* a_o *of* a,

$$a_o = \begin{pmatrix} t_1 & & & * \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_n \end{pmatrix}, \tag{7.17}$$

in $M_n(\mathbb{C})$ with

$$spec(a_o) = \{t_1, t_2, \ldots, t_n\} = spec(a).$$

Therefore, without loss of generality, one can understand the quotient algebra A_n of (7.16) as the collection of all spectral forms (7.17) of matrices in $M_n(\mathbb{C})$. So, one can naturally define a linear functional ψ_n on A_n by

$$\psi_n(x) \overset{def}{=} tr_n(x) \quad \text{for all} \quad x \in A_n. \tag{7.18}$$

In fact, the element x of A_n is an equivalence class $[x_o]_{\mathcal{R}}$ of the spectral form x_o of x in the sense of (7.17) in A_n, by the above discussion. So, one can simply let

$$x = x_o \text{ in } A_n.$$

From now on, all elements $x = [x]_{\mathcal{R}}$ of A_n are regarded as the spectral forms x_o of x in A_n. Thus, one can get that

$$\psi_n(x) = \psi_n([x]_{\mathcal{R}}) = tr_n(x_o) = \sum_{t \in spec(x_o)} m_t t, \tag{7.19}$$

by (7.17) and (7.18), where x_o is the spectral forms of x_o, for all $x \in A_n$, and m_t mean the *multiplicities of* $t \in spec(x_o)$.

Thus, indeed, one can get a well-determined C^*-probability space (A_n, ψ_n), and hence we have the K-dynamical C^*-probability space

$$\left(\mathbb{A}_{N,n}^K,\ \varphi_{p,H_n}\right),$$

induced by the K-subgroup dynamical system $(\mathfrak{T}_N^K, A_n, \lambda)$. In particular,

$$\varphi_{p,H_n} = \psi_n \otimes \varphi_p \quad \text{on} \quad \mathfrak{A}_{N,n}^K = A_n \otimes_\lambda \mathfrak{C}_{K,N}^* = \mathbb{A}_{N,n}^K,$$

where ψ_n is in the sense of (7.18), satisfying (7.19).

Proposition 7.6. *Let $a_1, \ldots, a_n \in A_n$, and $w_{j_i} = \alpha^p_{\chi_{x_j}K}$, generators of $\mathfrak{T}_N^K \subset \mathfrak{C}_{K,N}^*$, for $n \in \mathbb{N}$, and let $T_i = a_i w_{j_i}$ be free random variables of the K-dynamical C^*-probability space $\left(\mathbb{A}_{N,n}^K,\ \varphi_{p,H_n}\right)$. We naturally assume each $a_j = [a_j]_{\mathcal{R}}$ of A_n as its spectral form for all $j = 1, \ldots, n$. Let*

$$spec(a_i) = \{t_{i1}, \ldots, t_{in}\} \quad \textit{for all} \quad i = 1, \ldots, n, \tag{7.20}$$

without considering multiplicities of eigenvalues. Then

$$\varphi_{p,H_n}\left(\prod_{i=1}^n T_i\right) = \mu_p\left(\left(\prod_{i=1}^n x_{j_i}\right) K \cap K\right)\left(\sum_{i=1}^n\left(\prod_{k=1}^n t_{ik}\right)\right). \tag{7.21}$$

Proof. By (6.29), the linear functional $\varphi_{p,H_n} = \psi_n \otimes \varphi_p$ satisfies

$$\begin{aligned}
&\varphi_{p,H_n}\left((a_1 w_{j_1})(a_2 w_{j_2}) \ldots (a_n w_{j_n})\right) \\
&\quad = \psi_n\left(a_1 a_2^{w_{j_1}} a_3^{w_{j_1} w_{j_2}} \ldots a_n^{w_{j_1} \ldots w_{j_{n-1}}}\right) \varphi_p\left(w_{j_1} w_{j_2} \ldots w_{j_n}\right) \\
&\quad = \psi_n\left(a_1 a_2 \ldots a_n\right) \mu_p\left(x_{j_1} x_{j_2} \ldots x_{j_n} K \cap K\right)
\end{aligned}$$

since each $a_i^{w_{j_1} w_{j_2} \ldots w_{j_{i-1}}}$ is the isomorphic image of a_i, which is unitarily equivalent to a_i, sharing the identical spectral forms, for all $i = 2, \ldots, n$

$$= tr_n \left(a_1 a_2 \ldots a_n \right) \mu_p \left(x_{j_1} x_{j_2} \ldots x_{j_n} K \cap K \right)$$

$$= tr_n \left(\begin{pmatrix} \prod_{k=1}^{n} t_{1k} & & & * \\ & \prod_{k=1}^{n} t_{2k} & & \\ & & \ddots & \\ 0 & & & \prod_{k=1}^{n} t_{nk} \end{pmatrix} \right) \cdot \mu_p \left(x_{j_1} x_{j_2} \ldots x_{j_n} K \cap K \right)$$

by (7.20)

$$= \left(\sum_{i=1}^{n} \left(\prod_{k=1}^{n} t_{ik} \right) \right) \left(\mu_p \left(\left(\prod_{i=1}^{n} x_{j_i} \right) K \cap K \right) \right).$$

□

Now, let $n_1, n_2 \in \mathbb{N} \setminus \{1\}$, and let A_{n_1} and A_{n_2} be the corresponding quotient algebras in the sense of (7.16). Construct now a direct product algebra $A_{n_1+n_2}$,

$$A_{n_1,n_2} = A_{n_1} \oplus A_{n_2}. \tag{7.22}$$

Note that it is understood as the quotient algebra of the direct product algebra $M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C})$ under an equivalence relation $\mathcal{R}_{n_1+n_2}$, where

$$(a_1 \oplus b_1) \, \mathcal{R}_{n_1+n_2} \, (a_2 \oplus b_2) \overset{def}{\Longleftrightarrow}$$

$$spec(a_1) = spec(a_2) \quad \text{for} \quad a_1, a_2 \in M_{n_1}(\mathbb{C})$$

and

$$spec(b_1) = spec(b_2) \quad \text{for} \quad b_1, b_2 \in M_{n_2}(\mathbb{C}).$$

It is not difficult to check that A_{n_1,n_2} has an inherited trace ψ_{n_1,n_2} from the trace $tr_{n_1+n_2}$ on the matricial algebra $M_{n_1+n_2}(\mathbb{C})$. In fact, one has

$$\psi_{n_1,n_2} = \psi_{n_1} \oplus \psi_{n_2} \text{ on } A_{n_1,n_2} \tag{7.23}$$

in the sense that

$$\psi_n \left(a_1 \oplus a_2 \right) = \psi_{n_1}(a_1) + \psi_{n_2}(a_2)$$

for all $a_1 \oplus a_2 \in M_n$ with $a_k \in A_{n_k}$ for $k = 1, 2$, under linearity.

As we discussed above, if $a_1 \oplus a_2 \in A_{n_1,n_2}$, then a_1 and a_2 are spectral forms in A_{n_1}, respectively, in A_{n_2}. So,

$$\psi_{n_1,n_2} \left(a_1 \oplus a_2 \right) = \psi_{n_1} \left(a_1 \right) + \psi_{n_2} \left(a_2 \right) = \left(\sum_{t \in spec(a_1)} m_t t \right) + \left(\sum_{s \in spec(a_2)} m_s s \right) \tag{7.24}$$

for all $a_1 \oplus a_2 \in A_{n_1,n_2}$, by (7.19) and (7.23), where m_t and m_s are multiplicities of t and s, respectively. However, recall that the direct sum $a_1 \oplus a_2$ of two matrices a_1 and a_2 satisfies

$$spec\,(a_1 \oplus a_2) = spec(a_1) \cup spec(a_2) \text{ in } \mathbb{C}. \tag{7.25}$$

So, the formula (7.24) satisfies

$$\psi_{n_1,n_2}\,(a_1 \oplus a_2) = \sum_{t \in spec(a_1) \cup spec(a_2)} m_t t, \tag{7.26}$$

where m_t means the multiplicities of t "in $a_1 \oplus a_2$", by (7.25).

It is clear that two algebras A_{n_1} and A_{n_2} are free in the C^*-probability space $(A_{n_1,n_2}, \psi_{n_1,n_2})$. Moreover,

$$\begin{aligned}(A_{n_1,n_2}, \psi_{n_1,n_2}) &= (A_{n_1}, \psi_{n_1}) *_{\mathbb{C}} (A_{n_2}, \psi_{n_2}) \\ &= (A_{n_1}, \psi_{n_1}) \oplus (A_{n_2}, \psi_{n_2}) = (A_{n_1} \oplus A_{n_2},\ \psi_{n_1} \oplus \psi_{n_2})\,.\end{aligned} \tag{7.27}$$

Proposition 7.7. *Let $(A_{n_1,n_2}, \psi_{n_1,n_2})$ be a free probability space, where A_{n_1,n_2} and ψ_{n_1,n_2} are in the sense of* (7.22) *and* (7.26), *respectively. Assume that the generators $w_j = \alpha^p_{\chi_{x_j K}}$ of $\mathfrak{T}^K_N \subset \mathfrak{C}^*_{K,N}$ satisfy*

$$\mu_p\left(\left(\prod_{l=1}^{n} x_{j_l}\right) K \cap K\right) = 1 \tag{7.28}$$

for all $(j_1, \ldots, j_n) \in \{1, 2\}^n$ and $n \in \mathbb{N}$. Then

$$\mathbb{M}^K_{N,n} \overset{def}{=} M_n \times_\lambda \mathfrak{T}^K_N \overset{*\text{-}iso}{=} \mathbb{A}^K_{N,n_1} *_{\mathbb{C}} \mathbb{A}^K_{N,n_2} \overset{*\text{-}iso}{=} \mathbb{A}^K_{N,n_1} \oplus \mathbb{A}^K_{N,n_2}. \tag{7.29}$$

Proof. Two C^*-subalgebras

$$\mathbb{A}_1 = A_{n_1} \otimes_\lambda \mathfrak{C}^*_{K,N} \quad \text{and} \quad \mathbb{A}_2 = A_{n_2} \otimes_\lambda \mathfrak{C}^*_{K,N}$$

are free in $\left(\mathbb{M}^K_{N,n},\ \varphi_{p,H_n}\right)$, since condition (7.28) satisfies the general case (6.32), by (7.27). So,

$$\mathbb{M}^K_{N,n} \overset{*\text{-iso}}{=} \mathbb{A}_1 *_{\mathbb{C}} \mathbb{A}_2.$$

However, again by (7.27), we obtain

$$\mathbb{A}_1 *_{\mathbb{C}} \mathbb{A}_2 = \mathbb{A}_1 \oplus \mathbb{A}_2.$$ □

The above structure theorem (7.29) can be proved by computing free cumulants directly. Such free cumulants can be computed with help of (7.24), (7.25) and (7.26). However, we provide the above alternative proof.

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