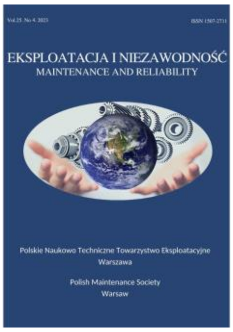
Eksploatacja i Niezawodnosc – Maintenance and Reliability

Volume 25 (2023), Issue 4

journal homepage: http://www.ein.org.pl



Article citation info:
Shangguan A, Feng N, Fei R, Reliability modeling based on multiple wiener degradation-shock competing failure process and dynamic failure threshold, Eksploatacja i Niezawodnosc – Maintenance and Reliability 2023: 25(4) http://doi.org/10.17531/ein/174248

Reliability modeling based on multiple wiener degradation-shock competing failure process and dynamic failure threshold

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Highlights

- The random shock process with a dynamic failure threshold is designed to enhance the performance of the hard failure process.
- The degradation process based on the multiple wiener process with the random effect is constructed.
- The correlation between the multiple degradation processes is analyzed by the copula function and calculated by the Monte Carlo method.

Abstract

Given the presence of multiple degradation failure processes and shock failure processes within the complex system during operation, this paper develops a reliability model that combines the multiple degradation-shock competing failure process and dynamic failure threshold. The Wiener process with random effects is considered as the degradation process model, which includes random effects to account for the heterogeneity among system units. Additionally, the extreme shock model with a dynamic failure threshold is used to depict the random shock. Then, the copula function is carried out to illustrate the correlation between multiple degradation processes, the reliability model is constructed further. To demonstrate the application of this model, a numerical case study and a micro-electro-mechanical system comprising two micro-mechanical resonators are employed. The parameter sensitivity of the proposed model is analyzed. The outcomes of the study highlight that the reliability model, which combines the Wiener process with random effects and the dynamic failure threshold, more accurately reflects the actual operational state of the complex system.

Keywords

degradation process, reliability modeling, shock process, wiener process model, copula function

1. Introduction

Accurately analyzing the failure mechanism is the foundation for enhancing the operational quality of a complex system. However, given the complexity of the operational environment, there are numerous factors that need to be considered in reliability analysis[15]. Therefore, to guarantee the reliability and safety of the complex system, this paper establishes a reliability model that incorporates the relationships between the different failure mechanisms.

In general, the failure mechanism of the complex system in reliability modeling is primarily divided into two aspects: hard failure and soft failure[16]. The former is the external shock loads caused by the complex environment exceed than the hard failure threshold of the system; the soft failure refers to the cumulative degradation of internal factors[1](such as wear[23], corrosion, etc.) and the degradation increment caused by random shock are higher than the soft failure threshold[35,36]. When the system experiences external shocks, the degradation increment occurs, and it becomes easier for the degradation to exceed the soft failure threshold. Since the random shock process will affect the degradation process, and the ability to

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resist random shock will be weakened by the accelerated degradation process. Ultimately, the system will fail once either soft or hard failures occur. Therefore, considering the competitive failure process can enhance the accuracy of the system reliability model [17].

Since the random process method can accurately depict the stochastic characteristics of the system and the randomness of system degradation, it is widely used in degradation process modeling. Some commonly used methods include the Wiener process [9], Gamma process [33,24], inverse Gaussian process [25], linear regression process [18], and more. Among these, the Wiener process is often favored by researchers as the degradation process due to its provision of a clear analytical formula for the distribution of the first arrival time when the degradation amount exceeds the failure threshold. This facilitates reliability calculation and analysis. In [5], the Wiener process is adopted to construct the reliability model and maintenance model. In [19], the multi-phase Wiener process is proposed to determine the remaining useful life (RUL) of the gyroscope. To enhance the effectiveness of degradation process modeling, a Wiener process that combines temporal features and variability is designed to track the degradation path[6]. Liu et al. [10] employ the Wiener process with measurement error and evidence variables to estimate reliability, enabling accurate prediction of the degradation amount. Ref. [20] divides the degradation process into two phases and combines the Wiener process with evidence variables in degradation process modeling. To enhance the nonlinear capability of the reliability model, the nonlinear Wiener process is proposed for reliability estimation in Refs. [2,11,27,21].

However, due to the complexity of the external environment, a single degradation process is insufficient for modeling the degradation process of the system. It is necessary to consider the dependence of multiple degradation processes in reliability analysis. In [28], Li et al. assumed that all failure processes in the system are independent and constructed a reliability model that combines multiple degradation processes and shock processes. However, even a small difference can cause changes in the system's operation, and independent failure processes are likely to result in significant model errors. A reliability model that considers the correlation effects of multiple degradation processes is more aligned with the actual operating state. For example, in [34], a method for modeling mutual influence in failure processes was proposed, and the copula function was used to describe the relationship between multiple degradation processes. In [7], the copula function was adopted to model multivariate correlated accelerated degradation test data. Therefore, considering the interaction relationship of multiple degradation processes can enhance the credibility of the reliability model. Furthermore, the correlation between multiple degradation processes is generally expressed using the copula function. Additionally, the failure of the system is influenced by different failure modes and the magnitude of the failure threshold. The constant threshold is widely used in failure process analysis, typically based on expert experience and experimental results. However, considering the internal dynamic characteristics of complex systems, the ability to resist external shocks will change over time, and a non-fixed threshold can better describe the state of the system. Hence, this paper proposes the concept of a dynamic failure threshold to describe the failure threshold.

Although many reliability studies have established models for multiple degradation processes and competing shock failures, a significant number of them have overlooked the influence of system heterogeneity and competing failure models with non-fixed failure thresholds. For instance, in Ref. [16], only the competitive failure reliability of multivariate Wiener processes and impact processes is considered, without accounting for dynamic failure thresholds. Conversely, in Ref. [32], dynamic failure thresholds are considered, but the degradation process model is overly simplified, and a constant hard failure threshold is assumed, which undermines the credibility of the reliability function results.

Therefore, to address the aforementioned issues, the reliability modeling based on multiple Wiener degradation-shock competing failure process and dynamic failure threshold is proposed, which is based on the competitive failure reliability model. The Copula function is employed to analyze the correlation among the multiple degradation processes within the complex system. The Akaike information criterion and Bayesian information criterion (AIC, BIC) are used to select the optimal Copula function, enabling the derivation of the relationship between the multiple degradation processes of the system. Then, a reliability model of the interaction between multiple degradation processes and the shock process is established in this paper. Numerical cases and real cases are conducted to compare, the experiment results show that the constructed model proposed in this paper aligns more closely with the actual operational state of the system. The contributions of this paper are as follows.

a) This article proposes a competitive failure reliability model that integrates multiple Wiener processes with random effect and random shock processes;
b) In the competitive failure model, the hard failure threshold is associated with the degradation increment, resulting in a dynamic reliability model that accounts for changes in the degradation process;
c) The Copula function is employed to analyze the correlation between multivariate Wiener degradation processes, and the Monte Carlo algorithm is utilized to address the multiple integration problem in the reliability model.

The notations used in this paper are described in the bellow.

Table 1. the different notation used in this paper.

Notation	Description	Notation	Description
$X_i(t)$	The i-th degradation process	$S_i(t)$	The i-th degradation increments
$O_i(t)$	The i-th overall degradation process	L_i	The i-th soft failure threshold
W_i	The j-th shock loads	D_j	The hard failure threshold after the j-th random shock
D_0	The initial value of the hard failure threshold	λ	The intensity of a homogeneous Poisson process
$F_i(t)$	The first arrival time distribution	$R_i(t)$	The reliability function
$C(u,v)$	Bivariate (u and v) copula function	P	The probability that the fatal shock occurs
q	The probability that the nonfatal shock occurs	AIC	Akaike information criterion
BIC	Bayesian information criterion	MLE	Maximum likelihood estimation
HPP	Homogeneous Poisson process		

The rest of the paper is organized as follows. Section 2 represents the operation description of system. Then, the failure process is definite in Section 3, including the degradation process and shock process. Section 4 represent the reliability model based on multiple wiener degradation-shock competing for failure processes and dynamic failure threshold. Section 5 is the experiment results and analysis. The conclusion exists in Section 6.

2. The operation description of system

In general, there are numerous factors contributing to the failure of complex systems, including wear, corrosion, pressure, temperature, and more. Wear and corrosion are considered internal factors, whereas pressure and temperature are external factors. Consequently, the failure types of complex systems can be categorized into two groups: soft failure and hard failure. The former occurs when the performance degradation surpasses the soft failure threshold, while the latter is triggered by random shock loads exceeding the hard failure threshold. Random shocks represent external factors that impact the system. For instance, the battery level and charging speed of a mobile phone fluctuate with usage time, resulting in two soft failure processes for the phone. On the other hand, when the external environment becomes excessively cold, overheated, or when the phone experiences a fall from a height, it may malfunction or experience an increased frequency of failures, representing the hard failure process. Furthermore, multiple studies have demonstrated that external shocks not only accelerate the degradation rate of system components but also weaken the overall resilience of the system. Therefore, adopting a constant failure threshold may undermine the reliability evaluation results of the system, leading to deviations from its actual operating state.

Hence, considering the variability of complex system operation processes, a single degradation process model cannot accurately describe the true state of the system. The multi-degradation process and the external shock process are considered dependent in this paper, and the hard failure threshold will change with the degradation increment. The operation process in the overall life cycle of a complex system is depicted in Fig.1.

In Fig.1, the system will be affected by multi-degradations ($X_i(t)$, i=1,…,m) in its life cycle, and the degradation processes are interdependent during system operation. Since the external shocks of the environment will affect the system operation state, the degradation increment I_{ij} will be generated in the degradation process when the random shock arrived. Then, as the degradation amount increases, the soft failure process will occur when the degradation amount $X_i(t)$ outpaces the soft failure threshold l_i ($X_i(t)$> l_i). Since the degradation increment obtained by the impact of shock process, the resistance ability of system will be weakened.

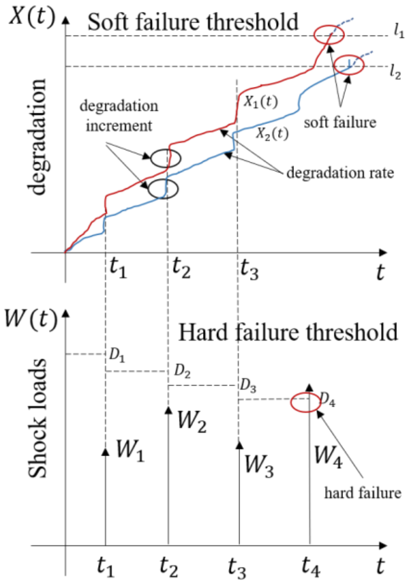

Fig. 1. The operation process of the complex system.

Hence, the dynamic hard failure threshold $D(t)$ is proposed to represent the failure process, as shown in the subgraph below Fig.1. When the degradation increment I_{ij} are generated by the j-th random shock, the hard failure threshold will be decrease with the degradation increment, that is, the initial value D_0 of hard failure threshold is decreased to D_j. Hard failure occurs when the random shock load W_j exceeds the hard failure threshold D_j.

In this paper, the extreme shock model is adopted as the random shock model, which includes non-fatal shocks and fatal shocks. In Fig.1, W_4 represents the fatal shock, directly causing system failure when the shock load W_4 exceeds the threshold D_4 (W_4> D_4). On the other hand, W_1, W_2, W_3 represent non-fatal shocks, which affect the failure threshold and degradation increment but do not result in the hard failure.

3. The definition of the failure process

To enhance the utilization of the complex system, it is necessary to analyze its operational reliability. Due to the complexity of the service environment, a single degradation process cannot accurately depict the system's degradation trend. Therefore, by considering the correlation between multiple degradation processes and the random shock processes, a reliability model based on the dependent competitive failure of degradation - shock processes and dynamic failure threshold is constructed in this paper.

3.1. The random shock modelling

Considering the external shocks of the environment can be classified into fatal shocks and non-fatal shocks. If a fatal shock act on the system, a hard failure will occur immediately. When a non-fatal shock occurs, it leads to a degradation increment and weakens the ability of the system to resist the external shocks. Consequently, the hard failure threshold of the system decreases with each shock occurs [3,8].

Assuming the number of the random shocks $N(t)$ are consistent with the Homogeneous Poisson Process(HPP). The shock affects the system with the constant rate λ. Hence, the probability when the number of the shock being k ($N(t)$=k) can be represented by Eq. (1).

$$p(N(t)=k)=\frac{e^{(-\lambda t)}(\lambda t)^k}{k!}, k=0,1,\ldots \tag{1}$$

Suppose that the probability of the fatal shock occurring is p. Then, the probability of the non-fatal shock occurring is q (where q=1-p). The shock number $N(t)$ includes $N_1(t)$ and $N_2(t)$, *i.e.* $N(t)=N_1(t)+N_2(t)$. Furthermore, the shock strength of the fatal shock and non-fatal shock on the system is $\lambda p, \lambda q$, respectively.

The probability that the fatal shock and the non-fatal shock not occurred is as follows:

$$p(N_1(t)=0)=e^{(-\lambda p)} \tag{2}$$

$$p(N_2(t)=k)=\frac{e^{(-\lambda q t)}(\lambda q t)^k}{k!}, k=0,1,2,\ldots \tag{3}$$

In this paper, W_j is the j-th shock load. Generally, the external shock loads are following the normal distribution, $W_j \sim N(\mu_{W_j}, \sigma^2_{W_j})$. The specific value of p, q and $\mu_{W_j}, \sigma^2_{W_j}$ can be obtained by fitting the shock data.

Considering that the hard failure threshold decreases when the random shock acts on the system, the hard failure threshold $D(t)$ undergoes changes that are associated with the degradation increment and a threshold parameter c, represented as formula (4).

$$D_j(t)=D_0-c\sum_{j=0}^{N_2(t)} I_{ij} \tag{4}$$

where D_0 is the initial value of the hard failure threshold, I_{ij} is the degradation increment, which is caused by the j-th random shock act on the i-th degradation process, c represents the hard failure threshold coefficient, which can adjust the failure threshold value.

Hence, the probability that the hard failure not occurred is same as the probability that the shock loads not exceeds the $D(t)$, the calculation process is described as formula (5).

$$p(W < D(t)) = p(W + c\sum_{j=0}^{N_2(t)} I_{ij} < D_0) = p(E < D_0) \quad (5)$$

where $E \sim N(\mu_W + c\mu_{I_i}, \sigma_W^2 + c\sigma_{I_i}^2)$, the hard failure threshold coefficient c is a constant value and can be obtained by fitting the shock data.

Hence, formula (5) is

$$p(W < D(t)) = p(E < D_0) = \Phi\left(\frac{D_0 - \mu_W - c\mu_{I_i}}{\sqrt{\sigma_W^2 + c\sigma_{I_i}^2}}\right)$$

When the number of fatal shocks is greater than zero ($N_1(t)$>0), the system will fail immediately. When the number of $N_2(t)$ is greater than zero and the number of $N_1(t)$ is zero ($N_1(t)$=0, $N_2(t)$>0), the degradation increment of the system will generate and the hard failure threshold will decrease. Therefore, the calculation process of hard failure not occurred at time t is shown in Eq. (6).

$$p_{NH}(t) = p\left(\bigcap_{j=1}^{N_2(t)} W_j < D_j(t))|N_1(t) = 0, N_2(t) = k\right) \times p(N_1(t) = 0) \times p(N_2(t) = k) \quad = \sum_{k=0}^{\infty} p(W < D(t))^j \times p(N_1(t) = 0) \times p(N_2(t) = k) \quad = \sum_{k=0}^{\infty} \Phi\left(\frac{D_0-\mu_W-c\mu_{I_i}}{\sqrt{\sigma_W^2+c\sigma_{I_i}^2}}\right)^j \times e^{-\lambda pt} \times \frac{e^{-\lambda qt}(\lambda qt)^k}{k!} \quad (6)$$

3.2. The degradation process modelling

Generally, the first arrival time at which the degradation amount exceeds the failure threshold follows an inverse Gaussian distribution. Incorporating this distribution significantly enhances the theoretical foundation for reliability analysis. In this paper, the Wiener process with the random effects [29,12] is used to model the degradation process. The specific description of the Wiener process is presented in Eq. (7).

$$X_i(t) = \mu_i t + \sigma_i B(t) \quad (7)$$

Where $X_i(t)$ is the i-th degradation process in the time t, μ_i,σ_i is the drift parameters and the diffusion parameter of the i-th degradation process, respectively. $B(t)$ is the standard Brownian motion.

Due to the various factors such as operational errors and complex environmental conditions, the degradation process of subsystems or key components within a complex system exhibits variations. In this paper, the Wiener process with the stochastic parameters is used as the degradation process model. The individual differences among subsystems are primarily described by the parameters $\delta = (\mu, \sigma)$. In the degradation process model, μ represents the random variable, which follows a normal distribution $\mu \sim N(\eta, \sigma_\eta^2)$ and is utilized to capture the heterogeneity of the subsystem, σ is a constant parameter that represents the common attributes of the system. Therefore, considering the multiple degradation processes within the system, the degradation model is presented in Eq. (8).

$$\begin{cases} X_i(t) = \mu_i t + \sigma_i B(t) \\ \mu_i \sim N(\eta_i, \sigma_{\eta_i}^2) \end{cases} \quad (8)$$

where $X_i(t)$ is the degradation process of the i-th subsystem in the time t, the parameters $\delta = (\mu, \sigma)$ can be obtained by the fitting of the degradation data.

The degradation increment generated by the external shock, the shock load is proportional to the degradation increment. The relationship between the shock and degradation increment is shown in(9).

$$I_{ij} = a_i W_j, j = 1, \ldots, k, \quad (9)$$

where a is the correlation coefficient between degradation increment and shock loads, I_{ij} is the i-th degradation increment of the j-th shock, k is the sum number of the shock.

The coefficient a_i can be calculated by the experiment [22]. Hence, when the shock number of the system is k ($N_2(t)$=k) in the time t, the sum of degradation increment is $S_i(t) = \sum_{j=0}^{k} I_{ij}$.

Generally, the degradation process of the system includes the self-degradation and degradation increment caused by the random shock, which is described as formula (10).

$$O_i(t) = X_i(t) + S_i(t) = \mu_i t + \sigma_i B(t) + \sum_{j=0}^{k} I_{ij} \quad (10)$$

Under the assumption of independence, since the shock load follows the normal distribution, when the $N_2(t)$=k, the sum of the increment is also following the normal distribution. $S_i(t) = \sum_{j=0}^{N_2(t)} I_{ij} \sim N(k\mu_I, k\sigma_I^2)$

where $\mu_I = a_i\mu_W$,$\sigma_I^2 = a_i^2\sigma_W^2$

Hence, the probability distribution function is

$$F_{S_i}(s) = \Phi\left(\frac{s-ka_i\mu_W}{\sqrt{k}a_i\sigma_W}\right) \quad (11)$$

The accurate calculation of reliability involves determining the probability of neither hard nor soft failure processes occurring throughout the entire life cycle of the system. Consequently, determining a reasonable failure threshold and the interaction relationship between the two failure processes are the focus of research. By incorporating the degradation process and the shock process, the reliability model can be formulated.

4. Reliability modeling considering the competing failure process

The external shock accelerates the degradation process and the hard failure threshold, thereby weakening the system's ability to resist external shocks. Consequently, it is crucial to consider the correlation between the degradation process and the shock process when modelling system's reliability.

4.1. The reliability model without considering correlation

According to the two types of random shock: fatal shock and non-fatal shock, the reliability modeling based on multiple wiener degradation-shock competing failure process should be discussed by the different situations.

a) When $N(t)=0$, the system is not affected by the external shock but the degradation process. Hence, the probability P_{SF1_i} that the system will not failure is

$$P_{SF1_i}(t|N(t)=0)=p(X_i(t)<l_i)=p(T>t)=1-\int_0^t f(t|u_i)dt$$

$$=1-\left(\phi\left(\frac{\eta_i t-l_i}{\sqrt{\sigma_{\eta_i}^2 t^2+\sigma_i^2 t}}\right)+exp\left(\frac{2\eta l_i}{\sigma_{\eta_i}^2}+\frac{2\sigma_i^2 l_i}{\sigma_i^2}\right)\cdot\phi\left(\frac{2\sigma_{\eta_i}^2 t l_i+\sigma_i^2(\eta_i t+l_i)}{\sigma_i^2\sqrt{\sigma_{\eta_i}^2 t^2+\sigma_i^2 t}}\right)\right) \quad (12)$$

where $\phi(\cdot)$ represent the standard normal distribution, (η_i,σ_{η_i}) is the parameters of the i-th degradation process. The specific derivation process is shown in Ref.[32].

Then the reliability model of the system based on the competing failure process is

$$R_i^1(t)=p(X_i(t)<l_i|N_1(t)=0,N_2(t)=0)\times p(N_1(t)=0,N_2(t)=0)=P_{SF1\_i}(t)\times e^{-\lambda pt}\times e^{-\lambda qt} \quad (13)$$

b) When the number of the external shock greater than zeros($N(t)>0$), the system is influenced by the degradation process and the random shock process.

The overall degradation is combined with the self-degradation and degradation increment, the probability P_{SF2_i} that the system will not have the soft failure is

$$P_{SF2\_i}(t|N(t)>0)=p(O_i(t)<l_i|N_2(t)=k)=p(X_i(t)+S_i(t)<l_i|N_2(t)=k) \quad (14)$$

When the number of fatal shock $N_l(t)>0$, the hard failure will occur immediately. To analyze the hard failure process, the non-fatal shock is mainly discussed in this section. The reliability model of the system subjected to the competing failure process is

$$R_i^2(t)=\sum_{k=1}^{\infty}p\left(O_i(t)<l_i,\bigcap_{j=1}^{N_2(t)}W_j<D(t)\,|N_2(t)=k\right)\times p(N_2(t)=k)$$
$$=\sum_{k=1}^{\infty}\frac{1}{\sqrt{\left(2\pi\left(\sigma_i^2+\sigma_{\eta_i}^2 t\right)t^3\right)^{\frac{1}{2}}}}\cdot\int_0^{l_i}\phi\left(\frac{l_i-x-ka_i\mu_w}{\sqrt{k}a_i\sigma_w}\right)\cdot x\cdot exp\left[-\frac{(x-\eta_i t)^2}{2\left(\sigma_{\eta_i}^2 t^2+\sigma_i^2 t\right)}\right]dx-$$
$$\phi\left(\frac{-ka_i\mu_w}{\sqrt{k}a_i\sigma_w}\right)\left[1-\left(\phi\left(\frac{\eta_i t-l_i}{\sqrt{\sigma_{\eta_i}^2 t^2+\sigma_i^2 t}}\right)+exp\left(\frac{2\eta l_i}{\sigma_{\eta_i}^2}+\frac{2\sigma_i^2 l_i}{\sigma_i^2}\right)\cdot\right.\right.$$
$$\left.\left.\phi\left(\frac{2\sigma_{\eta_i}^2 t l_i+\sigma_i^2(\eta_i t+l_i)}{\sigma_i^2\sqrt{\sigma_{\eta_i}^2 t^2+\sigma_i^2 t}}\right)\right)\right]\times\phi\left(\frac{D_0-\mu_w-c\mu_{I_i}}{\sqrt{\sigma_w^2+c\sigma_{I_i}^2}}\right)^k\times\frac{\exp(-\lambda qt)\cdot(\lambda qt)^k}{k!} \quad (15)$$

The specific derivation process is shown in Appendix A.

Based on the mutually exclusive event, the reliability model of the i-th subsystem at time t is shown in formula (16).

$$R_i(t)=R_i^1(t)+R_i^2(t) \quad (16)$$

Since the analytical solution of the definite integral in formula(16) is difficult to obtain. In this paper, the Monte Carlo method is adopted to calculate the area of the integrated function between the zero and soft failure threshold (*i.e.* $[0,l_i]$). The specific calculation process is shown in Fig. 2.

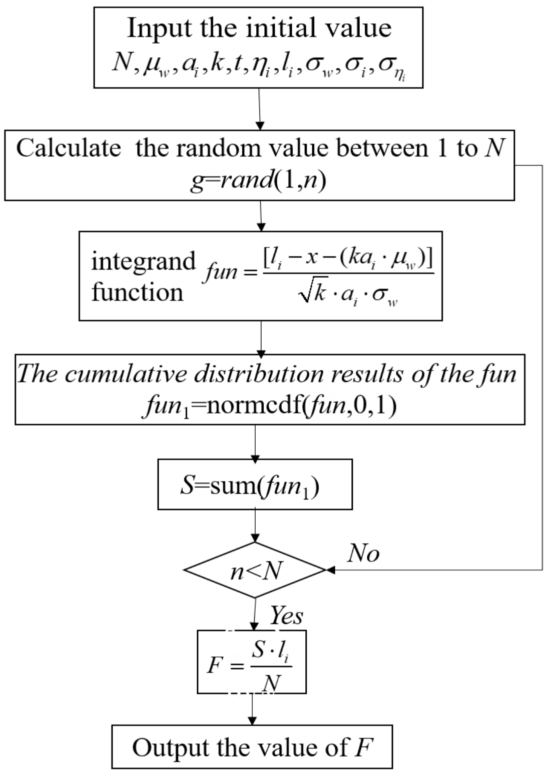

Fig. 2. The flow chart of the integrated function calculation.

4.2. The reliability model construction considering the correlation

When the degradation amount of any subsystem is greater than the soft failure threshold l_i, the system will fail immediately. To describe the relationship among the different degradation processes, the copula function[26] is used in the reliability modeling.

Assuming that the system is composed of m subsystems, that is the m degradation processes. For the i-th degradation process(i=1,…,m), the cumulative distribution function of the failure time is

$$p(T_i \le t_i) = p(X_i > l_i)F_i(t) = 1 - R_i(t) \qquad (17)$$

Then, the joint cumulative distribution function of the failure time for the m degradation processes is shown in formula (18).

$$f(X_1,\dots,X_m) = p(X_1 > l_1, X_2 > l_2, \dots, X_m > l_m) = c(F_1(X_1), F_2(X_2), \dots, F_m(X_m); \alpha) \prod_{i=1}^{m} f_i(X_i;\theta) = \frac{\partial^m C(F_1(X_1),F_2(X_2),\dots,F_m(X_m);\alpha)}{\partial F_1(X_1)\,\partial F_2(X_2)\dots\partial F_m(X_m)} \prod_{i=1}^{m} f_i(X_i;\theta) \qquad (18)$$

where $c(F_1(X_1), F_2(X_2), \dots, F_m(X_m); \alpha)$ is the probability density function of the copula function, α is the parameters of the function, $f_i(X_i;\theta)$ is the failure time probability density function of the i-th degradation process, θ is the parameters of the $f_i(.)$.

In the process of parameters estimating (α,θ), the maximum likelihood estimation is adopted in this paper. The log-likelihood function of the copula function is

$$l(\Theta) = \sum_{t=1}^{T} ln\, c\,(F_1(X_{1t}), F_2(X_{2t}), \dots, F_m(X_{mt}); \alpha) + \sum_{t=1}^{T}\sum_{i=1}^{m} ln\, f_i\,(X_{it};\theta) \qquad (19)$$

where Θ is the overall parameters $\Theta = \{\alpha, \theta\}$, the maximum likelihood function of parameters α,θ is shown in formula (20).

$$\begin{cases} \hat{\alpha}_{MLE} = argmax_{\alpha} \times \sum_{t=1}^{T} ln\, c\,(F_1(X_1), F_2(X_2), \dots, F_m(X_m); \alpha) \\ \hat{\theta}^i{}_{MLE} = argmax_{\theta^i}\, ln\, f_i\,(X_{it};\theta^i) \end{cases} \qquad (20)$$

The specific deduced process of the copula method is described in Ref. [27].

Based on the formula (16), considering the correlation of multiple degradation processes in the failure process, the probability that the system does not fail in the time t is

$$p(T_1 > t_1, T_2 > t_2, \dots, T_m > t_m) = C(R_1(t_1), R_2(t_2), \dots, R_m(t_m)) = 1 - \sum_{i=1}^{m} F_i(t_i) + \sum_{1\le i<h\le m} C(F_i(t_i), F_h(t_h), \dots) - \sum_{1\le i<h<k\le m} C(F_i(t_i), F_h(t_h), F_k(t_k), \dots) + (-1)^m C(F_1(t_1), F_2(t_2), \dots, F_m(t_m)) \qquad (21)$$

Assume that there are only two degradation process (O_1,O_2) existed in the complex system, the reliability calculation process of the system is denoted as(22).

$$R(t) = C(R_1(t_1), R_2(t_2)) = 1 - F_1(t_1) - F_2(t_2) + C(F_1(t_1), F_2(t_2)) = R_1(t_1) + R_2(t_2) - 1 + C(u,v) \qquad (22)$$

where $u = F_1(t_1)$=1-$R_1(t_1)$, $v = F_2(t_2)$=1-$R_2(t_2)$.

The widely used Copula functions of two degradation process are described in Table 2.

Table 2. The copula function of two degradation process.

Copula functions	$C(u,v)$	The range of parameter
Gumbel copula	$C(u,v;\alpha) = exp\left\{-[(-\ln u)^{\alpha} + (-\ln v)^{\alpha}]^{\frac{1}{\alpha}}\right\}$	$\alpha \in [1,+\infty)$
Clayton copula	$C(u,v;\alpha) = (u^{-\alpha} + v^{-\alpha} - 1)^{\frac{1}{\alpha}}$	$\alpha \in (0,\infty)$
Frank copula	$C(u,v;\alpha) = -\frac{1}{\alpha} ln\left(1 + \frac{(e^{-\alpha u}-1)(e^{-\alpha v}-1)}{(e^{-\alpha}-1)}\right)^{-\frac{1}{\alpha}}$	$\alpha \neq 0$
Norm copula	$C(u_1,u_2;\alpha) = \int_{-\infty}^{\phi^{-1}(u_1)}\int_{-\infty}^{\phi^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} exp\left\{-\frac{s^2 - 2\rho st + t^2}{s(1-\rho^2)}\right\} dsdt$	$\phi^{-1}$is inverse function of standard normal distribution
t-copula	$C(u_1,u_2;\rho,v) = \int_{-\infty}^{{t_1}^{-1}(u_1)}\int_{-\infty}^{{t_2}^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} exp\left\{1 + \frac{{\zeta_1}^2 - 2\rho st + {\zeta_2}^2}{s(1-\rho^2)}\right\}^{-\frac{v+2}{2}} d\zeta_1 d\zeta_2$	$t_v{}^{-1}$is the inverse function of t-distribution with degrees of freedom v

Hence, reliability considering the correlation between the multiple degradation process and the shock process can be obtained based on the different copula functions.

Furthermore, in order to select an optimal copula function, AIC, BIC are used in this paper, thereby enhancing the accuracy of the reliability results. The lower the criteria model, the better the copula function. The calculation formulas of AIC and BIC are shown in (23).

$$\begin{cases} AIC = 2\,max[l(\Theta)] + 2n_p \\ BIC = -2\,max[l(\Theta)] + 2n_p\,ln(n) \end{cases} \tag{23}$$

where $max[l(\Theta)]$ is the max value of the log-likelihood function (formula(19)), n_p is the number of parameters that need to be estimated, n is the number of the samples.

In the parameter estimation process of copula function, it can be calculated by the '*Matlab toolkit*', such as the 'copulafit' function. The results of likelihood function can be solved by the '*matlab copula patton toolbox*'[4]. The specific optimal copula function selection process is described in Section 5.

5. Experiment results and analysis

To validate the effectiveness of the model proposed in this paper, both a numerical case and a real case are examined. Considering the mutual influence between the degradation process and the random shock impacts on the complex system during operation, both cases involve two sub-degradation processes. By utilizing the reliability model proposed in this study, the reliability results throughout the system's life cycle are obtained. These results serve as an analytical foundation for intelligent decision-making in the later stage of the system.

5.1. The numerical example

In this paper, given that the system experiences two degradation processes and an external shock process, the multiple wiener degradation-shock competition failure process is proposed to construct the reliability model. Both degradation processes are modeled as Wiener process with random effect, while the shock process is presented by the extreme shock model. It should be noted that the variables of numerical simulation have no physical meaning, mainly reflecting the changing trend of the reliability curve in different situations. The specific value of the parameters in the model is presented in Table 3.

Table 3. The specific value of the parameters.

Parameter	Description	Value
σ_1, σ_2	The diffusion parameter	σ_1=0.2,$\sigma_2 = 0.4$
$\begin{cases} \mu_1 \sim N(\eta_1, \sigma_{\eta_1}) \\ \mu_2 \sim N(\eta_2, \sigma_{\eta_2}) \end{cases}$	The drift parameters	$\begin{cases} \mu_1 \sim N(10.15,\ 0.2) \\ \mu_2 \sim N(15.3, 0.15) \end{cases}$
l_1, l_2	The soft threshold of degradation process	l_1=40,$l_2 = 42$
λ	The shock rate	3×10^{-3}
k	The number of the shock	3
P	The probability of fatal shock occurring	1/3
$W_j \sim N(\mu_W, \sigma_W)$	The shock loads	$\mu_W = 0.6, \sigma_W = 0.01$
c	the coefficient of the degradation increment	0.75
D_0	The initial value of the hard failure threshold	15

Based on the parameters of Table 3 and the calculation process of *Eq*.(8), two degradation curve of the numerical case is shown in Fig. 3.

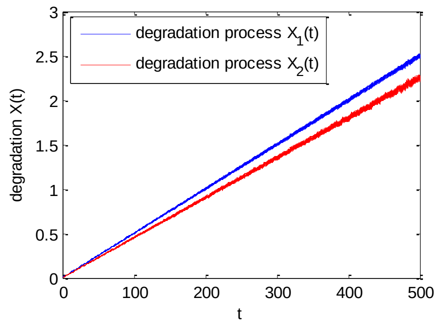

Fig. 3. The Wiener-degradation processes with random effect in the numerical case.

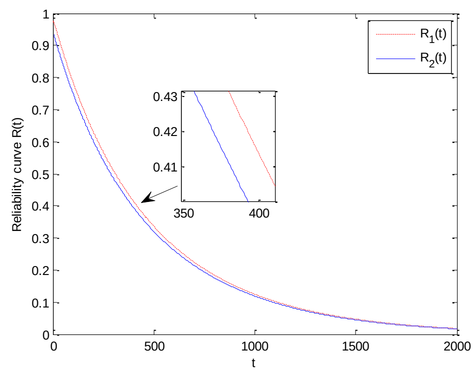

Fig. 4. Reliability results without considering correlation.

The two-degradation process is similar to each other in Fig.3. Then, combining the formulas (16) and (22), the reliability based on the two different degradation and external shock is obtained, which is shown in Fig.4.

In Fig.4, the reliability curve is declined with the variable t (no physical meaning) change, which is similar to each other by the similar degradation process and also conforms to the real operation situation.

Considering the reliability results corresponding to different degradation processes, the correlation between two reliability R_1, R_2 is calculated by the Copula function in this paper. To select the optimal Copula function in reliability analysis, based on the formulas (18) and (22), the log-likelihood results and AIC, BIC of the two reliability R_1, R_2 are calculated. The Copula function corresponding to the minimum results of AIC and BIC is selected as the optimal function[34]. Hence, the Copula calculation results of numerical case are shown in Table 4.

Table 4. the results of different copula function.

Copula function	LL[a]	AIC[b]	BIC[c]	parameter	Rank
Norm	8.4988e5	1699762	1699767.6	1	5
Clyton	-3.7226e3	-7443.2	-7437.6	29.1726	4
Frank	-4.6974e3	-9392.80	-9387.2	122.0997	2
Gumbel	**-7.8710e3**	**-15740**	**-15734.4**	**25.5476**	**1**
T	-4.0309e3	-8057.8	-8046.6	[0.9,2.1]	3

[a] Log-likelihood; [b] Akaike information criterion; [c] Bayes information criterion

In Table 4, five copula functions are compared to obtain the optimal Copula function. It is clear that the AIC/BIC results of Gumbel function is the smallest, AIC=-15740, BIC=-15734.4. The Copula function that can describe the joint distribution of reliability corresponding to two degradation processes is Gumbel function, followed by the Frank function, and the result of normal function is the worst. Therefore, this paper chooses the Gumbel function to analyze the correlation of the multiple degradation process in the numerical case.

Based on the optimal Copula function, combining the formulas (16) and (22) in section 4, the reliability of the independent degradation process is used to compare with the model proposed in this paper, the comparison results are shown in Fig.5.

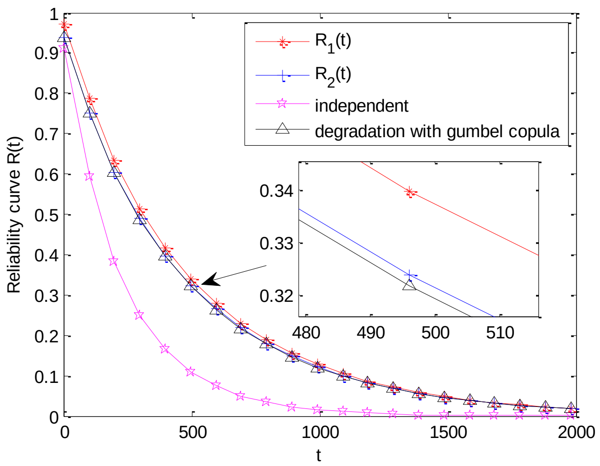

Fig.5. The reliability results comparison based on different method.

In Fig.5, the X-axis represent the variable t (no physical meaning), the Y-axis is the reliability results. The reliability corresponding to the two degradation processes are similar to the reliability considering the correlation of degradation. The reliability results calculated by the Gumbel Copula function can better show the similarity between the different degradation processes. Hence, the model established in this paper can better evaluate the reliability of the system. Moreover, it is obvious that the reliability with the independent degradation process is far lower than the real reliability, which not only does not reflect the similarity between the two subsystems but underestimates the reliability of the system.

To analyze the impact of random shock rate λ on the reliability, the different λ are calculated for the reliability calculation. The comparison results are shown in Fig.6.

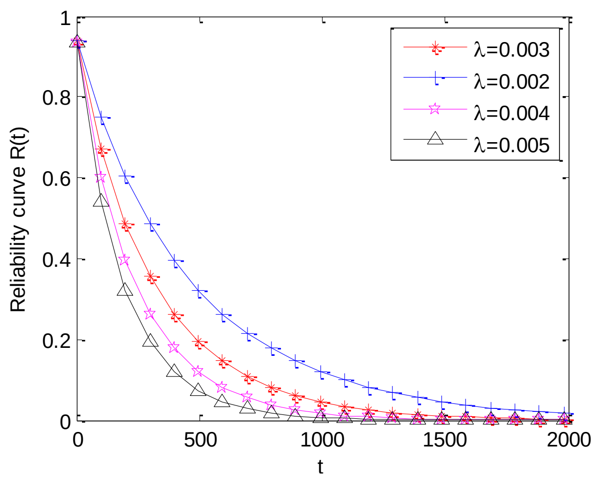

Fig. 6. The reliability results under different λ.

In Fig.6, the higher the value of the λ, the lower the reliability results, which is satisfied with the real situation. When the shock rate λ is higher, the ability that resists the random shock of the system will be decreased, and the reliability will be further reduced. Therefore, external environmental factors need to be considered in the assessment of reliability.

5.2. Case application

To further validate the advantages of the proposed method, a widely used case-Micro-electro-mechanical systems (MEMS) is adopted in this paper[13]. MEMS is extensively used in the industry due to its cost-effectiveness, light and easy integration. But its small size makes it susceptible to external impacts, resulting in competitive failure processes. On the other hand, the MEMS system consist of multiple micro-mechanical resonators (MMR) operating in series and parallel modes, and the sampling frequency of these resonators is extremely affected by the external environment[36]. Each resonator will undergo a degradation failure process during operation, and the MEMS composed of different resonators will experience multiple degradation failure processes within the whole lifecycle, which is meeting the multi-degradation process model established in this paper. Moreover, wear degradation and external random shock are the main reasons for MEMS failure. With the extension of working time, the wear debris caused by friction will further affect the MEMS operation, and gradually reducing its ability to resist external impacts. This is equivalent to the dynamic failure threshold caused by the MEMS being affected by external shocks during operation.

Hence, MEMS is a typical system that is influenced by multiple degradation processes and a shock process with a dynamic failure threshold, which is according to the reliability model constructed in this paper. It is worth noting that the MEMS system is often composed of multiple resonators in series, and if any resonator fails, the entire system will fail. In this paper, the system that two resonators in a series are selected as the verification object. Therefore, the MEMS system used in this study involves two degradation processes.

Since the parameters of the MEMS system have been estimated in references [13,30], and obtaining the original experiment details is challenging, the values of the estimated parameters are cited directly in this paper. The other parameters involved in the reliability model (such as the threshold coefficient *c*, shock number *k*, etc.) are mainly determined by the user usage of the system. For example, the threshold coefficient c is related to the size of the failure threshold $D(t)$. when the coefficient c increases, the failure threshold D is decreased faster, which indicates the system is greatly affected by external shocks; The smaller the shock number k, the less external shock act on the system, and the higher the reliability of the system. Different these parameters will not affect the analysis results of experiment, and can be continuously adjusted to gradually meet the safety requirements of the product. Hence, some parameters are set by a priori assumption. The purpose of it is to obtain the changing trend of the reliability model under the different parameters. The specific parameters of MEMS reliability are shown in Table 5.

Table 5. The parameters of two degradation processes and shock process.

Parameter	Description	Value	Sources
σ_1, σ_2	The diffusion parameter	σ_1=2.0,$\sigma_2 = 1.5$	assumption
$\begin{cases}\mu_1 \sim N(\eta_1, \sigma_{\eta_1}) \\ \mu_2 \sim N(\eta_2, \sigma_{\eta_2})\end{cases}$	The drift parameters	$\begin{cases}\mu_1 \sim N(10.15，0.2) \\ \mu_2 \sim N(15.3, 0.15)\end{cases}$	Ref. [14]
l_1, l_2	The soft threshold of degradation process	l_1=4100,l_2 = 4500	Refs. [13,30]
λ	The shock rate	3×10^{-3}	Refs. [13,30]
k	The number of the shock	3	assumption
P	The probability of fatal shock occurring	1/3	Ref. [31]
$W_j \sim N(\mu_W, \sigma_W)$	The shock loads	$\mu_W = 72.6 \sigma_W = 6.3$	Ref. [32]
c	the coefficient of the degradation increment	0.75	assumption
D_0	The initial value of the hard failure threshold	92	Refs. [13,30]

Similar with the analysis process of numerical case, based on the parameters of Table 5 and the reliability calculation process of formula (16), the reliability results under two degradation processes are obtained, which is shown in Fig.7.

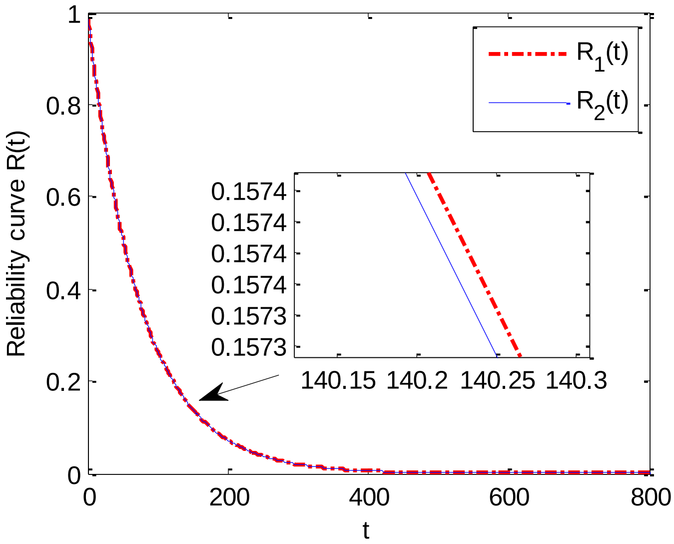

Fig.7. The different reliability curve based on the two degradation processes.

In Fig.7, the *X*-axis is the operation time of the MEMS system, the *Y*-axis is the reliability curve with the time. The red line and blue line are representing the reliability based on the first and second degradation process, respectively. The decline rate of the reliability curve is particularly fast when the time t=180. The reliability results of MEMS corresponding to the two degradation processes are extremely similar, indicating that the operational reliability results of the two resonators in MEMS are almost the same without considering correlation.

Then, to analyze the correlation of two degradation process, the reliability with the multiple degradation process and random shock with dynamic failure threshold is calculated. Based on the formulas (16) and (22), the optimal Copula function in reliability modeling is obtained by the log-likelihood results and AIC, BIC of the two reliability R_1, R_2. The joint distribution of the reliability under different degradation processes are calculated. The calculation results of different Copula function are shown in Table 6.

Table 6. The copula function fitting results on the MEMS two-components.

Copula function	**LL[a]**	**AIC[b]**	**BIC[c]**	**parameter**	**Rank**
Norm	17501	35005.5	35010.2	1	5
Clyton	-3577	-7152	-7147.4	164.9625	4
Frank	-2317	-4632	-4627.95	31.8054	2
Gumbel	**-6533**	**-13065**	**-13060.8**	**70.8149**	**1**
T	-3956	-7909	-7900.2	[0.9,2.1]	3

In Table 6, it is obvious that the AIC and BIC of the Gumbel function is the smallest compared with the other Copula functions, AIC=-13065, BIC=-13060.8. The norm function is worst in the Copula function fitting process. Hence, the Gumbel function is chosen to calculate the joint distribution of two degradation processes in the MEMS case.

Based on the Gumbel function and the formula (22), the reliability of the MEMS with a competitive failure process and dynamic failure threshold in the lifecycle is calculated. To demonstrate the advantage of the related multiple degradation process of the reliability model, the reliability model with an independent failure process is adopted for experiment comparison, which is shown in Fig.8.

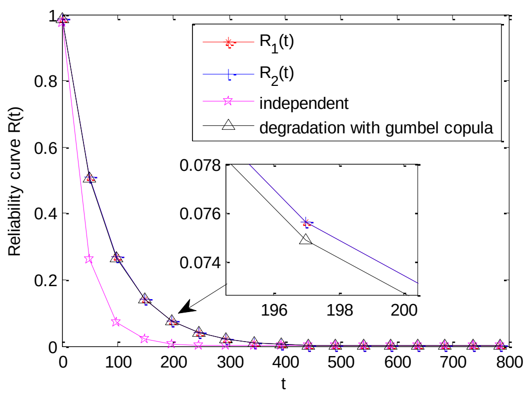

Fig. 8. The reliability comparison of MEMS system.

Fig.8 depict the comparison results of the reliability under the different situation. The black line is the reliability under the multiple degradation process with the Copula -'Gumbel' function. The magenta line is the reliability that consider the independent failure process. Clearly, the reliability with the independent process underestimates the real state of the MEMS. Furthermore, based on the dependent degradation processes, the results are similar to the reliability $R_1(t)$ and $R_2(t)$. Since the reliability is obtained by the correlation calculation of $R_1(t)$ and $R_2(t)$, the corresponding reliability trend of the two degradation processes is learned. Specifically, the reliability result of the model proposed in this paper is slightly lower than the reliability corresponding to the two degradation processes ($R_1(t)$ and $R_2(t)$), which is also proved that the results are consistent with the real state.

Since the dynamic failure threshold is proposed in the process of reliability model construction, to analyze the sensitivity of the parameters in the reliability results, the different threshold $D(t)$ and shock rate λ are set in the reliability calculation process. Fig. 9 and Fig. 10 represent the impact of random shock rate λ and the hard failure threshold $D(t)$ on the reliability.

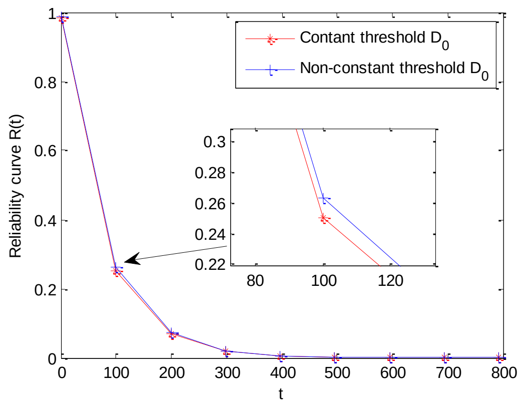

Fig. 9. The reliability under different D(t).

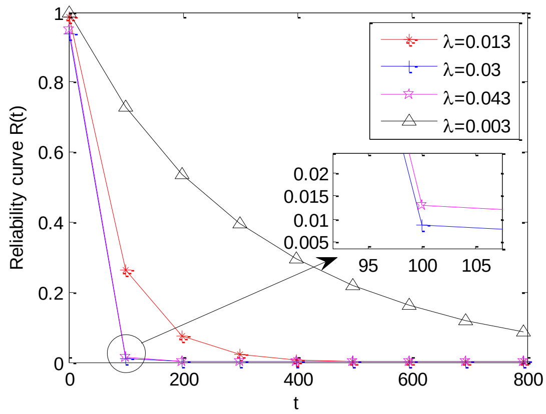

Fig. 10. The reliability under different λ.

In Fig.9, the blue line represents the reliability results under the dynamic failure threshold $D(t)$. It is obvious that the reliability $R(t)$ of the constant failure threshold is lower than that of the dynamic failure process. Since the resistance ability of the MEMS system will decrease when the random shock act on the product, and the reliability will decrease further. So, this is line in with the actual operating situation.

Similarly, the reliability will increase when the random shock rate is reduced in Fig.10. When the random shock rate λ=0.043, the shock impact on the MEMS system is significant. So its reliability results rapidly decrease in the early stage during MEMS operation due to the high impact rate, leading to the MEMS system directly fail. But, when λ= 0.003, the reliability trend of MEMS during its lifecycle is relatively mild. Hence, it is necessary to pay attention to the impact of the external environment on the system.

In this paper, the reliability model that considers the multiple degradation process and random shock with a dynamic failure threshold is constructed. The reliability results can serve as the analytical basis for the MEMS's intelligent decision-making.

6. Conclusions

In this paper, we propose a reliability model for a multi-degradation processes system operating in a complex environment. The model is based on multiple wiener degradation-shock competing failure process and dynamic failure threshold. The Wiener process with the random effects is employed to capture the heterogeneity of the degradation process between subsystems. The hard failure process consists of an extreme shock model and a dynamic hard failure threshold. To enhance the accuracy of the reliability model, the copula function is adopted to obtain the joint distribution of reliability under multi-degradation processes. The reliability is then calculated by considering multiple dependent degradation processes and random shock processes. In the experiment results-both numerical and in MEMS application- it is observed that relying on independent degradation processes hampers result accuracy. Instead, using the wiener process with random effects and dynamic hard failure threshold better reflects the real operation state. Additionally, it is important to consider the impact of the external environment on the system, such as the shock rate (λ) and different hard failure thresholds (D(t)), when modeling system reliability. Hence, our proposed method not only enhances the credibility of reliability results but also provides support for further system maintenance strategies.

Acknowledgments

This work was supported by the National Science Foundation of China (No. U2034209, No. 62120106011), and the Doctoral Dissertation Innovation Fund of Xi'an University of Technology(252072218).

Appendix A

$$R_i^2(t)=\sum_{k=1}^{\infty}p\left(O_i(t)<l_i,\ \bigcap_{j=1}^{N_2(t)}W_j<D(t)|N_2(t)=k\right)\times p(N_2(t)=k)\quad=\sum_{k=1}^{\infty}p\left(X_i(t)+S_i(t)<l_i,\ \bigcap_{j=1}^{N_2(t)}W_j<D(t)|N_2(t)=k\right)\times p(N_2(t)=k)$$

$$=\sum_{k=1}^{\infty}\int_0^{l_i}f_{X_i}(x)dx\int_0^{l_i-x_i}f_{S_i}(s)ds\times\Phi\left(\frac{D_0-\mu_W-c\mu_{I_i}}{\sqrt{\sigma_W^2+c\sigma_{I_i}^2}}\right)^k\times\frac{e^{-\lambda qt}\cdot(\lambda qt)^k}{k!}\times p(N_2(t)=k)$$

$$=\sum_{k=1}^{\infty}\int_0^{l_i}[F_{S_i}(l_i-x)-F_{S_i}(0)]\cdot f_{X_i}(x)dx\times\Phi\left(\frac{D_0-\mu_W-c\mu_{I_i}}{\sqrt{\sigma_W^2+c\sigma_{I_i}^2}}\right)^k\times\frac{e^{-\lambda qt}\cdot(\lambda qt)^k}{k!}$$

$$=\sum_{k=1}^{\infty}\int_0^{l_i}\left[F_{S_i}(l_i-x)f_{X_i}(x)dx-F_{S_i}(0)\int_0^{l_i}f_{X_i}(x)dx\right]\times\Phi\left(\frac{D_0-\mu_W-c\mu_{I_i}}{\sqrt{\sigma_W^2+c\sigma_{I_i}^2}}\right)^k\times\frac{e^{-\lambda qt}\cdot(\lambda qt)^k}{k!}$$

$$=\sum_{k=1}^{\infty}\int_0^{l_i}\left[\Phi\left(\frac{l_i-x-ka_i\mu_W}{\sqrt{k}a_i\sigma_W}\right)f_{X_i}(x)dx-\Phi\left(\frac{-ka_i\mu_W}{\sqrt{k}a_i\sigma_W}\right)\int_0^{l_i}f_{X_i}(x)dx\right]\times\Phi\left(\frac{D_0-\mu_W-c\mu_{I_i}}{\sqrt{\sigma_W^2+c\sigma_{I_i}^2}}\right)^k\times\frac{e^{-\lambda qt}\cdot(\lambda qt)^k}{k!}$$

Where, $f_{X_i}(x)$ is the probability density function when $X_i(t)=x$,

$$f_{X_i}(x)=\sqrt{\frac{x^2}{\sqrt{2\pi\left(\sigma^2+\sigma_\eta^2t\right)t^3}}}exp\left[-\frac{(x-\eta t)^2}{2(\sigma^2t+\sigma_\eta^2t^2)}\right]$$

Then

$$\int_0^{l_i}f_{X_i}(x)dx=1-\left(\phi\left(\frac{\eta_it-l_i}{\sqrt{\sigma_{\eta_i}^2t^2+\sigma_i^2t}}\right)+exp\left(\frac{2\eta l_i}{\sigma_{\eta_i}^2}+\frac{2\sigma_i^2l_i}{\sigma_i^2}\right)\cdot\phi\left(\frac{2\sigma_{\eta_i}^2tl_i+\sigma_i^2(\eta_it+l_i)}{\sigma_i^2\sqrt{\sigma_{\eta_i}^2t^2+\sigma_i^2t}}\right)\right)$$

When$A=\int_0^{l_i}\left[\Phi\left(\frac{l_i-x-ka_i\mu_W}{\sqrt{k}a_i\sigma_W}\right)f_{X_i}(x)dx-\Phi\left(\frac{-ka_i\mu_W}{\sqrt{k}a_i\sigma_W}\right)\int_0^{l_i}f_{X_i}(x)dx\right]\times\Phi\left(\frac{D_0-\mu_W-c\mu_{I_i}}{\sqrt{\sigma_W^2+c\sigma_{I_i}^2}}\right)^k\times\frac{e^{-\lambda qt}\cdot(\lambda qt)^k}{k!}$

Hence,

$$A=\int_0^{l_i}\left[\phi\left(\frac{l_i-x-ka_i\mu_w}{\sqrt{k}a_i\sigma_w}\right)\int_0^{l_i}f_{X_i}(x)dx\right]-\phi\left(\frac{-ka_i\mu_w}{\sqrt{k}a_i\sigma_w}\right)\int_0^{l_i}f_{X_i}(x)dx\times\phi\left(\frac{D_0-\mu_w-c\mu_{I_i}}{\sqrt{\sigma_w^2+c\sigma_{I_i}^2}}\right)^k\times\frac{\exp(-\lambda qt)\cdot(\lambda qt)^k}{k!}$$

$$=\int_0^{l_i}\left[\phi\left(\frac{l_i-x-ka_i\mu_w}{\sqrt{k}a_i\sigma_w}\right)\int_0^{l_i}f_{X_i}(x)dx\right]-\phi\left(\frac{-ka_i\mu_w}{\sqrt{k}a_i\sigma_w}\right)\left[1-\left(\phi\left(\frac{\eta_it-l_i}{\sqrt{\sigma_{\eta_i}^2t^2+\sigma_i^2t}}\right)+exp\left(\frac{2\eta l_i}{\sigma_{\eta_i}^2}+\frac{2\sigma_i^2l_i}{\sigma_i^2}\right)\cdot\phi\left(\frac{2\sigma_{\eta_i}^2tl_i+\sigma_i^2(\eta_it+l_i)}{\sigma_i^2\sqrt{\sigma_{\eta_i}^2t^2+\sigma_i^2t}}\right)\right)\right]\times\phi\left(\frac{D_0-\mu_w-c\mu_{I_i}}{\sqrt{\sigma_w^2+c\sigma_{I_i}^2}}\right)^k\times$$

$$\frac{exp(-\lambda qt)\cdot(\lambda qt)^k}{k!}$$

$$=\frac{1}{\sqrt{\left(2\pi\left(\sigma_i^2+\sigma_{\eta_i}^2t\right)t^3\right)^{\frac{1}{2}}}}\cdot\int_0^{l_i}\phi\left(\frac{l_i-x-ka_i\mu_w}{\sqrt{k}a_i\sigma_w}\right)\cdot x\cdot exp\left[-\frac{(x-)^2}{2\left(\sigma_{\eta_i}^2t^2+\sigma_i^2t\right)}\right]dx-\phi\left(\frac{-ka_i\mu_w}{\sqrt{k}a_i\sigma_w}\right)\left[1-\left(\phi\left(\frac{\eta_it-l_i}{\sqrt{\sigma_{\eta_i}^2t^2+\sigma_i^2t}}\right)+exp\left(\frac{2\eta l_i}{\sigma_{\eta_i}^2}+\frac{2\sigma_i^2l_i}{\sigma_i^2}\right)\cdot\right.\right.$$

$$\phi\left(\frac{2\sigma_{\eta_i}^2 tl_i+\sigma_i^2(\eta_i t+l_i)}{\sigma_i^2\sqrt{\sigma_{\eta_i}^2 t^2+\sigma_i^2 t}}\right)\Bigg)\Bigg]\times\phi\left(\frac{D_0-\mu_w-c\mu_{l_i}}{\sqrt{\sigma_w^2+c\sigma_{l_i}^2}}\right)^k\times\frac{exp\,(-\lambda qt)\cdot(\lambda qt)^k}{k!}$$

Furthermore, $R_i^2(t)$ is

$$R_i^2(t)=\sum_{k=1}^{\infty}\frac{1}{\sqrt{\left(2\pi\left(\sigma_i^2+\sigma_{\eta_i}^2 t\right)t^3\right)^{\frac{1}{2}}}}\cdot\int_0^{l_i}\phi\left(\frac{l_i-x-ka_i\mu_w}{\sqrt{k}a_i\sigma_w}\right)\cdot x\cdot exp\left[-\frac{\left(x-\eta_i t\right)^2}{2\left(\sigma_{\eta_i}^2 t^2+\sigma_i^2 t\right)}\right]dx$$

$$-\phi\left(\frac{-ka_i\mu_w}{\sqrt{k}a_i\sigma_w}\right)\left[1-\left(\phi\left(\frac{\eta_i t-l_i}{\sqrt{\sigma_{\eta_i}^2 t^2+\sigma_i^2 t}}\right)+exp\left(\frac{2\eta l_i}{\sigma_{\eta_i}^2}+\frac{2\sigma_i^2 l_i}{\sigma_i^2}\right)\cdot\phi\left(\frac{2\sigma_{\eta_i}^2 tl_i+\sigma_i^2(\eta_i t+l_i)}{\sigma_i^2\sqrt{\sigma_{\eta_i}^2 t^2+\sigma_i^2 t}}\right)\right)\right]\times\phi\left(\frac{D_0-\mu_w-c\mu_{l_i}}{\sqrt{\sigma_w^2+c\sigma_{l_i}^2}}\right)^k\times\frac{\exp\,(-\lambda qt)\cdot(\lambda qt)^k}{k!}$$

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