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SIMPLE EIGENVECTORS OF UNBOUNDED OPERATORS OF THE TYPE "NORMAL PLUS COMPACT"

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Abstract. The paper deals with operators of the form $A = S + B$, where B is a compact operator in a Hilbert space H and S is an unbounded normal one in H, having a compact resolvent. We consider approximations of the eigenvectors of A, corresponding to simple eigenvalues by the eigenvectors of the operators $A_n = S + B_n$ $(n = 1, 2, \ldots)$, where B_n is an n-dimensional operator. In addition, we obtain the error estimate of the approximation.

Keywords: Hilbert space, linear operators, eigenvectors, approximation, integro-differential operators, Schatten-von Neumann operators.

Mathematics Subject Classification: 47A75, 47B10, 65F15.

1. INTRODUCTION AND NOTATIONS

Let H be a separable Hilbert space with a scalar product $(\cdot, \cdot)$, the norm $\|\cdot\| = \sqrt{(\cdot, \cdot)}$ and the unit operator I. Let S be a normal operator in H, having a compact resolvent, and B be a compact operator in H. Besides, we do not assume that B is normal. Our main object is the operator

$$A = S + B. \tag{1.1}$$

Numerous integro-differential operators can be represented in the form (1.1) (cf. [1, 3, 4]). This paper deals with approximations of the eigenfunctions of the operators defined as in (1.1).

The literature devoted to approximations of the eigenvectors of various concrete operators is rather rich. In particular, in the paper, [12] approximations of Schrödinger eigenfunctions are explored by canonical perturbation theory. In [5] the author investigates eigenvectors of Toeplitz matrices under higher order three term recurrence and circulant perturbations. The paper [9] deals with approximations of eigenfunctions of the periodic Schrödinger operators. The paper [16] introduces an algorithm to numerically approximate eigenfunctions of Sturm-Liouville problems corresponding to eigenvalues in a given region. In the papers [2,13–15], the authors investigate stability and approximation properties of the eigenfunctions of Neumann and Dirichlet Laplacians. In particular, the lowest nonzero eigenvalue and corresponding eigenfunction is studied. Certainly, we could not survey the whole subject here and refer the reader to the above listed publications and references given therein.

To the best of our knowledge, the approximations of the eigenfunctions of the operators of the form (1.1) were not investigated in the available literature.

We introduce the notation. For a linear unbounded operator A in H, $Dom(A)$ is the domain, A^* is the adjoint of A; $\sigma(A)$ denotes the spectrum of A and A^{-1} is the inverse to A, $R_\lambda(A) = (A - I\lambda)^{-1}$ $(\lambda \notin \sigma(A))$ is the resolvent; $\lambda_k(A)$ are the eigenvalues of A taken with their multiplicities; $\rho(A, \lambda) = \inf_{s\in\sigma(A)} |\lambda - s|$ - the distance between $\lambda \in \mathbb{C}$ and $\sigma(A)$. If A is bounded, then $\|A\|$ means its operator norm.

We will say that an eigenvalue of a linear operator is *simple*, if its algebraic multiplicity is equal to one. By $e(\lambda(A))$ we denote the *normalized eigenvector* corresponding to an eigenvalue $\lambda(A)$.

For an integer $p \geq 1$, SN_p is the Schatten-von Neumann ideal of compact operators K in H with the finite norm $N_p(K) = [\text{Trace}\,(KK^*)^{p/2}]^{1/p}$.

2. PRELIMINARIES

Let T_1 and T_2 be two linear operators in H with $Dom(T_2) = Dom(T_1)$ and $q := \|T_1 - T_2\| < \infty$. Assume that

$$\|R_\lambda(T_1)\| \leq \phi(1/\rho(T_1, \lambda)) \text{ for all regular } \lambda \text{ of } T_1, \tag{2.1}$$

where $\phi(x)$ is a monotonically increasing non-negative continuous function of a non-negative variable x, such that $\phi(0) = 0$ and $\phi(\infty) = \infty$. Put

$$\Omega(c, r) := \{z \in \mathbb{C} : |z - c| \leq r\} \text{ and } \partial\Omega(c, r) := \{z \in \mathbb{C} : |z - c| = d\} \ (c \in \mathbb{C}, r > 0).$$

Under condition (2.1), let T_1 have an eigenvalue $\lambda(T_1)$ and

$$d := \frac{1}{2}\text{distance}\{\lambda(T_1), \sigma(T_1)\backslash\lambda(T_1)\} > 0. \tag{2.2}$$

Suppose that

$$q\phi(1/d) < 1. \tag{2.3}$$

Since $R_\lambda(T_1) - R_\lambda(T_2) = R_\lambda(T_1)(T_2 - T_1)R_\lambda(T_2)$, from (2.1) and (2.3) it follows that

$$\|R_\lambda(T_2)\| \leq \frac{\|R_\lambda(T_1)\|}{1 - q\phi(1/d)} \leq \frac{\phi(1/d)}{1 - q\phi(1/d)} < \infty \quad (\lambda \in \partial\Omega(\lambda(T_1), d)). \tag{2.4}$$

Put

$$P(T_1) = -\frac{1}{2\pi i} \int\limits_{\partial\Omega(\lambda(T_1),d)} R_\lambda(T_1)d\lambda \quad \text{and} \quad P(T_2) = -\frac{1}{2\pi i} \int\limits_{\partial\Omega(\lambda(T_1),d)} R_\lambda(T_2)d\lambda,$$

that is, $P(T_1)$ and $P(T_2)$ are the Riesz projections onto the eigenspaces of T_1 and T_2, respectively, corresponding to the points of the spectra, which belong to $\Omega(\lambda(T_1), d)$.

Lemma 2.1. *Let T_1 satisfy condition* (2.1)*, with an eigenvalue $\lambda(T_1)$ of the algebraic multiplicity ν and the condition*

$$q\phi(1/d)[1 + \phi(1/d)d] < 1 \tag{2.5}$$

holds, where d is defined by (2.2)*. Then $\dim P(T_1)H = \dim P(T_2)H = \nu$ and*

$$\|P(T_1) - P(T_2)\| \le \delta, \quad \textit{where} \quad \delta := \frac{qd\phi^2(1/d)}{1 - q\phi(1/d)} < 1. \tag{2.6}$$

Proof. From (2.5) we have $q\phi(1/d) < 1$; furthermore, $q\phi^2(1/d)d < 1 - q\phi(1/d)$, this implies $\delta < 1$. From (2.4) with $\partial\Omega = \partial\Omega(\lambda(T_1), d)$ it follows that

$$\begin{aligned}\|P(T_1) - P(T_2)\| &\le \frac{1}{2\pi} \int\limits_{\partial\Omega} \|R_\lambda(T_1) - R_\lambda(T_2)\||d\lambda| \\ &\le \frac{1}{2\pi} \int\limits_{\partial\Omega} \|R_\lambda(T_2)\|q\phi(1/d)|d\lambda| \le \frac{q\phi^2(1/d)d}{1 - q\phi(1/d)} = \delta.\end{aligned}$$

Now due to (2.6) and the well-known result [10, p. 156, Problem III.2.1] we have $\dim P(T_1)H = \dim P(T_2)H = \nu$, as claimed. □

Lemma 2.2. *Suppose T_1 has a simple eigenvalue $\lambda(T_1)$, and conditions* (2.1) *and* (2.5) *hold. Then T_2 has in $\Omega(\lambda(T_1), d)$ a simple eigenvalue, say $\lambda(T_2)$. Moreover,*

$$\|e(\lambda(T_2)) - e(\lambda(T_1))\| \le \frac{2\delta}{1 - \delta}.$$

Proof. For simplicity put $e = e(\lambda(T_1))$. Due to the previous lemma T_2 has in $\Omega(\lambda(T_1), d)$ a simple eigenvalue and $\|P(T_1) - P(T_2)\| \le \delta < 1$. Consequently, $P(T_2)e \ne 0$, since $P(T_1)e = e$. Thanks to the relation $T_2P(T_2)e = \lambda(T_2)P(T_2)e$, $P(T_2)e$ is an eigenvector of T_2. Put $\eta = \|P(T_2)e\|$. Then $e(\lambda(T_2)) = \frac{1}{\eta}P(T_2)e$ is a normalized eigenvector of T_2. For simplicity put $e(\lambda(T_2)) = f$. So

$$e - f = P(T_1)e - \frac{1}{\eta}P(T_2)e = e - \frac{1}{\eta}e + \frac{1}{\eta}(P(T_1) - P(T_2))e.$$

But

$$\eta \ge \|P(T_1)e\| - \|(P(T_1) - P(T_2))e\| \ge 1 - \delta.$$

Hence $\frac{1}{\eta} \leq (1-\delta)^{-1}$ and

$$\|e-f\| \leq \Big(\frac{1}{\eta}-1\Big)\|e\| + \frac{1}{\eta}\|P(T_1)-P(A_2)\|\|e\|$$
$$\leq (1-\delta)^{-1} - 1 + (1-\delta)^{-1}\delta = 2\delta(1-\delta)^{-1}, \qquad \square$$

as claimed.

The result is similar to the latter lemma in the case of bounded operators and is proved in [6] (see also [7, Lemma 4.3.2]).

3. THE MAIN RESULTS

Let $\{e_k\}_{k=1}^{\infty}$ be the normalized eigenvectors of a normal operator S having a compact resolvent, and let a compact operator B be represented in the basis $\{e_k\}_{k=1}^{\infty}$ by a matrix $(b_{jk})_{j,k=1}^{\infty}$. Then operator A defined by (1.1) is represented by the matrix (a_{jk}) with $a_{jj} = \lambda_j(S) + b_{jj}$ and $a_{jk} = b_{jk}$ $(j \neq k)$.

For an integer $n < \infty$, put $\hat{b}_{jk}^{(n)} = b_{jk}$ if $1 \leq j,k \leq n$ and $\hat{b}_{jk}^{(n)} = 0$ otherwise. Denote by B_n the operator represented in the basis $\{e_k\}_{k=1}^{\infty}$ by matrix $(b_{jk}^{(n)})_{j,k=1}^{\infty}$. So B_n has a range no more than n. We will approximate the spectrum of A by the spectrum of the operators $A_n = S + B_n$ $(n = 1,2,\ldots)$. So $A_n = S_n \oplus C_n$, where

$$C_n = (b_{jk})_{j,k=1}^{n} + diag(\lambda_k(S))_{k=1}^{n} \quad \text{and} \quad S_n = diag(\lambda_k(S))_{k=n+1}^{\infty}.$$

Consequently, C_n has in the basis $\{e_k\}_{k=1}^{n}$ the entries $c_{jj} = \lambda_j(S) + b_{jj}$ and $c_{jk} = b_{jk}$ $(j \neq k; 1 \leq j,k \leq n)$.

Note that the resolvent

$$R_\lambda(A) = (S - \tau + B - (\lambda-\tau)I)^{-1} = \big(I + (B-(\lambda-\tau)I)(S-\tau I)^{-1})(S-\tau I)\big)^{-1}$$
$$= R_\tau(S)\,(I + (B-(\lambda-\tau)I)R_\tau(S)))^{-1} \quad (\tau \notin \sigma(S))$$

is compact for any regular λ of A, and therefore, the spectrum of A is discrete. Since B is compact, we have

$$q_n := \|A_n - A\| = \|B_n - B\| \to 0 \quad \text{as} \quad n \to \infty.$$

Introduce the quantity

$$g(C_n) = \Big[N_2^2(C_n) - \sum_{k=1}^{n} |\lambda_k(C_n)|^2\Big]^{1/2}.$$

The following relations are checked in [7, Section 2.1].

$$g^2(C_n) \leq N_2^2(C_n) - |\mathrm{Trace}\, C_n^2| \quad \text{and} \quad g^2(C_n) \leq 2N_2^2(C_{\sqrt{-1},n}),$$

where $C_{\sqrt{-1},n} = (C_n - C_n^*)/2i$. If C_n is a normal matrix: $C_n C_n^* = C_n^* C_n$, then $g(C_n) = 0$. Assume that

$$A_n \text{ have a simple eigenvalue } \lambda_0(A_n) \tag{3.1}$$

and put

$$d_{0,n} := \frac{1}{2}\text{distance}\{\lambda_0(A_n), \sigma(A_n)\backslash\lambda_0(A_n)\},$$

and

$$\Phi_n(C_n, x) := \sum_{j=0}^{n-1} \frac{g^j(C_n)}{x^{j+1}\sqrt{j!}} \quad (x > 0).$$

Theorem 3.1. *Let condition* (3.1) *hold and*

$$q_n \Phi_n(C_n, d_{0n})\,[1 + \Phi_n(C_n, d_{0n}) d_{0n}] < 1.$$

Then A has in $\Omega(\lambda_0(A_n), d_{0n})$ a unique simple eigenvalue, denoted by $\lambda_0(A)$. Besides,

$$\|e(\lambda_0(A)) - e(\lambda_0(A_n))\| \le \frac{2\hat{\delta}_n}{1 - \hat{\delta}_n}, \ \textit{where } \hat{\delta}_n := \frac{q_n d_{0n} \Phi_n^2(C_n, d_{0n})}{1 - q_n \Phi_n(C_n, d_{0n})}. \tag{3.2}$$

If, in addition,

$$A_{\sqrt{-1}} = (A - A^*)/2i \in SN_2 \tag{3.3}$$

and with the notation

$$\hat{\Phi}(A, x) := \sum_{j=0}^{\infty} \frac{(\sqrt{2} N_2(A_{\sqrt{-1}}))^j}{x^{j+1}\sqrt{j!}} \quad (x > 0),$$

the inequality

$$q_n \hat{\Phi}(A, d_{0n})\,[1 + \hat{\Phi}(A, d_{0n}) d_{0n}] < 1 \tag{3.4}$$

is fulfilled, then $\hat{\delta}_n \to 0$.

This theorem is proved in the next section.

Now assume that a condition more general than (3.3) hold:

$$A - A^* \in SN_{2p} \quad (p = 1, 2, \ldots). \tag{3.5}$$

Under this condition we establish a result, which in the case (3.3) is less sharp than Theorem 3.1. To this end put

$$\beta_p := \begin{cases} 2(1 + ctg\,(\frac{\pi}{4p})\,) & \text{if } p = 2^{m-1},\ m = 1, 2, \ldots, \\ 2(1 + \frac{2p}{exp(2/3)ln2}) & \text{otherwise} \end{cases}$$

and

$$\hat{\psi}_p(A, x) = \sum_{m=0}^{p-1}\sum_{k=0}^{\infty} \frac{N_{2p}^{kp+m}(\beta_p A_{\sqrt{-1}})}{x^{kp+m+1}\sqrt{k!}} \quad (x > 0).$$

In addition, for $n = jp$ $(j = 1, 2, \ldots)$ denote

$$\psi_{p,n}(C_n, x) = \sum_{m=0}^{p-1} \sum_{k=0}^{j} \frac{N_{2p}^{kp+m}(\beta_p C_{\sqrt{-1},n})}{x^{kp+m+1}\sqrt{k!}}.$$

Theorem 3.2. *Under conditions* (3.1) *and* (3.5) *with* $n = jp$ $(j = 1, 2, \ldots)$, *let*

$$q_n \psi_{p,n}(C_n, d_{0n})[1 + \psi_{p,n}(C_n, d_{0n}) d_{0n}] < 1.$$

Then A *has in* $\Omega(\lambda_0(A_n), d_{0n})$ *a simple eigenvalue, denoted by* $\lambda_0(A)$. *Moreover,*

$$\|e(\lambda_0(A)) - e(\lambda_0(A_n))\| \leq \frac{2\hat{\Delta}_{p,n}}{1 - \hat{\Delta}_{p,n}}, \ \text{where } \hat{\Delta}_{p,n} := \frac{q_n d_{0n} \psi_{p,n}(C_n, d_{0n})}{1 - q_n \psi_{p,n}(C_n, d_{0n})}. \tag{3.9}$$

If, in addition,

$$q_n \hat{\psi}_p(A, d_{0n}) \, [1 + \hat{\psi}_p(A, d_{0n}) d_{0n}] < 1, \tag{3.6}$$

then $\hat{\Delta}_{p,n} \to 0$.

This theorem is also proved in the next section.

4. PROOFS OF THEOREMS 3.1 AND 3.2

Put $Q_n = \sum_{k=1}^n (\cdot, e_k) e_k$. Then $C_n = Q_n A Q_n$ and $S_n = (I - Q_n)S = S(I - Q_n)$. Clearly, $S_n C_n = C_n S_n = 0$ and

$$\sigma(A_n) = \sigma(C_n) \cup \{\lambda_k(S)\}_{k=n+1}^{\infty}. \tag{4.1}$$

Thus

$$\|R_\lambda(A_n)\| = \max\{\|Q_n R_\lambda(C_n)\|, \|(I - Q_n) R_\lambda(S_n)\|\}. \tag{4.2}$$

Assume that

$$\|R_\lambda(C_n)\| \leq \sum_{k=0}^{n-1} \frac{c_k}{\rho^{k+1}(C_n, \lambda)} = p_n(1/\rho(C_n, \lambda)) \quad (\lambda \notin \sigma(C_n)), \tag{4.3}$$

where $c_k = const \geq 0$, $c_0 = 1$, and

$$p_n(x) = \sum_{k=0}^{n-1} c_k x^{k+1}.$$

Since S_n is normal, (4.1) and (4.2) imply the inequality

$$\|R_\lambda(A_n)\| \leq \max\{p_n(1/\rho(C_n, \lambda)), 1/\rho(S_n, \lambda)\}.$$

But due to (4.1) $\rho(C_n, \lambda) \geq \rho(A_n, \lambda)$ and $\rho(S_n, \lambda) \geq \rho(A_n, \lambda)$. In addition, $p_n(x) \geq x$ for $x \geq 0$. Thus

$$\|R_\lambda(A_n)\| \leq p_n(1/\rho(A_n, \lambda)).$$

Now Lemma 2.2 implies the following result.

Lemma 4.1. *Let the conditions* (3.1), (4.3) *and*

$$q_n p_n(1/d_{0n})\,[1 + p_n^2(1/d_{0n})d_{0n}] < 1$$

hold. Then A has in $\Omega(\lambda_0(A_n), d_{0n})$ a unique simple eigenvalue $\lambda_0(A)$ and

$$\|e(\lambda_0(A)) - e(\lambda_0(A_n))\| \le \frac{2\delta_n}{1-\delta_n}, \;\; \textit{where}\; \delta_n := \frac{q_n d_{0n} p_n^2(1/d_{0n})}{1 - q_n p_n(1/d_{0n})}.$$

Note that according to (4.1) either $\lambda_0(A_n) \in \sigma(S_n)$ or $\lambda_0(A_n) \in \sigma(C_n)$.

Proof of Theorem 3.1. Thanks to Corollary 2.1.2 of [7] we have

$$\|R_\lambda(C_n)\| \le \sum_{k=0}^{n-1} \frac{g^k(C_n)}{\sqrt{k!}\rho^{k+1}(C_n,\lambda)} \text{ for any regular point } \lambda \text{ of } C_n.$$

Hence, inequality (3.2) is due to the previous lemma.

Furthermore, as it was mentioned, $g(C_{\sqrt{-1},n}) \le \sqrt{2}N_2(C_{\sqrt{-1},n})$. In addition, $N_2(C_{\sqrt{-1},n}) \le N_2(A_{nI}) \le N_2(A_{\sqrt{-1}})$ and $\Phi_n(C_n,x) \le \hat{\Phi}(A,x)$ $(x > 0)$. Now letting, $n \to \infty$ we obtain that $\hat{\delta}_n \to 0$, provided conditions (3.4) and (3.3) hold. This proves the theorem. □

To prove Theorem 3.2 we need the following result.

Lemma 4.2. *Let T be a linear operator acting in a Euclidean space $\mathbb{C}^n$ with $n = jp$ and integers $p \ge 1, j \ge 1$. Then*

$$\|R_\lambda(T)\| \le \sum_{m=0}^{p-1}\sum_{k=0}^{j} \frac{N_{2p}^{kp+m}(\beta_p T_{\sqrt{-1}})}{\rho^{pk+m+1}(T,\lambda)\sqrt{k!}} \quad (\lambda \notin \sigma(T)),$$

where $T_{\sqrt{-1}} = (T - T^*)/2i$.

Proof. Due to the algebraic Schur theorem (cf. [11]) $T = D + V$ $(\sigma(T) = \sigma(D))$, where D is a normal matrix and V is a nilpotent matrix. Besides, D and V have the same invariant subspaces, and V is called the nilpotent part of T. Thanks to [7, Lemma 6.8.3],

$$\|R_\lambda(T)\| \le \sum_{m=0}^{p-1}\sum_{k=0}^{j} \frac{N_{2p}^{kp+m}(V)}{\rho^{pk+m+1}(T,\lambda)\sqrt{k!}} \quad (\lambda \notin \sigma(T)),$$

where V is the nilpotent part of $\sqrt{-1}$. Making use of Lemma 7.9.2 from [7], we get the inequality $N_{2p}(V) \le \beta_p N_{2p}(T_{\sqrt{-1}})$ for appropriately chosen β_p. This proves the lemma. □

Proof of Theorem 3.2. The previous lemma and Lemma 4.1 imply inequality (3.2).

Furthermore, take into account that $N_{2p}(C_{\sqrt{-1},n}) \le N_{2p}(A_{nI}) \le N_{2p}(A_{\sqrt{-1}})$ and $\psi_{p,n}(C_n,x) \le \hat{\psi}_p(A,x)$ $(x > 0)$. Now letting, $n \to \infty$ we obtain that $\hat{\Delta}_{p,n} \to 0$, provided conditions (3.5) and (3.6) hold. This proves the theorem. □

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