Technical Sciences 16(3), 2013, 179–192

EXAMPLE OF ROBUST FREE ADJUSTMENT OF HORIZONTAL NETWORK COVERING DETECTION OF OUTLYING POINTS

Marek Hubert Zienkiewicz, Tomasz Bałuta

Institute of Geodesy
University of Warmia and Mazury in Olsztyn

Received 27 November 2012; accepted 31 July 2013; avilable on line 2 August 2013

Key words: free adjustment, robust estimation.

Abstract

This paper presents the method of detecting outlying reference points by applying robust free adjustment. The article contains the theoretical basis of the robust free adjustment. Theoretical considerations are supplemented by a numerical example showing the possible practical applications. In this paper is also included example, which presents detecting of reference points contaminated by gross error, in the case of existence gross errors in observation sets.

Introduction

Theory of the robust adjustment of observation set is one of the most rapidly developing estimation methods. Development of methods of the geodetic networks adjustment concern, especially the robust adjustment for gross errors. General idea of the robust estimation is identify and reduce influence of outliers in the adjustment solution. Reduce the influence of the outliers or their complete elimination can be achieved by damping weights of observations (e.g. HUBER 1981, YANG 1994). It is a very useful property, especially in the analysis of deformation. In the measurements of displacement and deformations, gross errors, may occur not only in the observation but also, in the coordinates of reference points. This may be related to the situation

Corespondence: Marek Zienkiewicz, Instytut Geodezji, Uniwersytet Warmińsko-Mazurski, ul. Oczapowskiego 1, 10-719 Olsztyn, phone: +48 89 523 60 25, e-mail: marek.zienkiewicz@uwm.edu.pl

when the reference point is displaced. It is obvious that performing the correct interpretation of the results is dependent of the reference points. There are known a lot of robust methods concerned a control of reference mark stability (e.g. DUCHNOWSKI 2010). In practice, there might be also other situations, that are related with the change of the reference points coordinates. It happens when the coordinates of the point are contaminated by gross errors due to incorrectly inputted data. It is known that if we take that point as a fixed point during adjustment, then the results of the adjustment will be incorrect. In such cases detection of the contaminated point of the network can be obtain by the robust free adjustment. The identification of the contaminated points and reduction of their influence on the final result of the adjustment was proposed in (WIŚNIEWSKI 2002). Making the geodetic network "free" and applying the robust free adjustment, large increments to the outlying coordinate can be expected. Reduce influence of the contaminated coordinates on the final adjustment results can be obtain by modifying elements of the coordinates weight matrix. Simultaneously are designated the values of outlying coordinates. The principles of the robust free adjustment are related to the classical method of the robust M – estimation (WIŚNIEWSKI 2005). In practice, there might be a situation where besides gross errors in the coordinates, a gross errors in the observations may also occur. In such case it is recommend to apply the hybrid M – estimation (CZAPLEWSKI 2004). In this method occurs both, damping weights of the coordinates weight matrix and damping weights of the observations weight matrix.

Theoretical foundation of free adjustment of the geodetic network

The classical estimation methods assume, that the coefficients matrix **A** is a column, full rank matrix (WIŚNIEWSKI 2005). Then the matrix $\mathbf{A}^T\mathbf{PA}$ (where **P** is weight matrix of observations) is a non-singular matrix, and its inverse is following $(\mathbf{A}^T\mathbf{PA})^{-1}$. In case of the free geodetic network, rank of the matrix $\underset{n,m}{\mathbf{A}}$ (where n is the number of observations and m is the number of the parameters) is less then quantity of its columns and difference between both values is equal to the defect (further: d) of matrices, i.e. $d = m - \text{ranki}(\mathbf{A})$. In case of geodetic networks defect of matrices **A**, can be identified with the freedom degrees relative to the adopted coordinate system (external degrees of freedom). A properly chosen network cannot have internal degrees of freedom (e.g. similarity). Thus, angular – linear network (2D network) has three degrees of freedom (further: SW) – displacement due to X, Y axes and rotation. Assume that at the network shown in Figure 1 the points 1^o, 2^o, 3^o have the approximate coordinates, and the points $\hat{1}$, $\hat{2}$, $\hat{3}$, have estimated coordinates. Due to the fact, that network is free ($SW = 3$), all of the adjustment points may be displaced relative to the approximate points. The main aim of the free adjustment is optimal fitting adjustment network to approximate network.

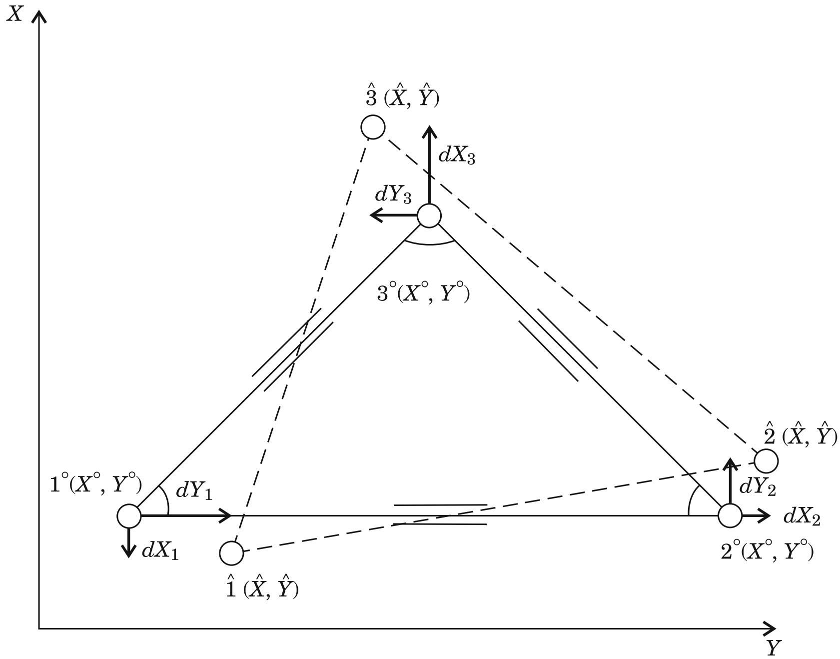

Fig. 1. Geometric interpretation of the free adjustment

Under consideration in the free adjustment of geodetic networks, where defect is different from zero, should be additional criterion for increments dX_i, dY_i for all points in the network. Provided that criterion has a form $\Phi_X(d\mathbf{X}) = d\mathbf{X}^T\mathbf{P_X}d\mathbf{X} = \min.$,solution of the free adjustment problem can be represented by the following equations (WIŚNIEWSKI 2005).

$$\begin{cases} \mathbf{V} = \mathbf{A}d\mathbf{X} + \mathbf{L} \\ \Phi(d\mathbf{X}) = \mathbf{V}^T\mathbf{PV} = \min. \\ \Phi_X(d\mathbf{X}) = d\mathbf{X}^T\mathbf{P_X}d\mathbf{X} = \min. \end{cases} \tag{1}$$

where $\mathbf{P_X}$ is matrix of parameters weights $\mathbf{X}$, and thus the weights matrix of demanding increments $d\mathbf{X}$. Solving this problem it is necessary to take into account only the aim function $\Phi(d\mathbf{X}) = \mathbf{V}^T\mathbf{PV}$ which gives normal equations

$\mathbf{A}^T\mathbf{PA}d\mathbf{X} + \mathbf{A}^T\mathbf{PL} = \mathbf{0}$ with a singular matrix of coefficients $\mathbf{A}^T\mathbf{PA}$. Having matrix $\mathbf{A}$ and vector $d\mathbf{X}$ in the following forms:

$$\underset{n,m}{\mathbf{A}} = \left[\underset{n,r}{\mathbf{A}_r}\ \underset{n,d}{\mathbf{A}_d}\right],\ d\mathbf{X} = \left[\underset{1,r}{d\mathbf{X}_r^T}\ \underset{1,d}{d\mathbf{X}_d^T}\right],$$

where $r = \text{rank}(\underset{\text{mxm}}{\mathbf{A}^T\mathbf{PA}})$, normal equations can be written in following way:

$$\mathbf{A}^T\mathbf{PA}d\mathbf{X} + \mathbf{A}^T\mathbf{PL} = \mathbf{0} \Leftrightarrow \tag{2}$$

$$\begin{bmatrix}\mathbf{A}_r^T\\ \mathbf{A}_d^T\end{bmatrix}\mathbf{P}[\mathbf{A}_r\ \mathbf{A}_d]\begin{bmatrix}d\mathbf{X}_r\\ d\mathbf{X}_d\end{bmatrix} + \begin{bmatrix}\mathbf{A}_r^T\\ \mathbf{A}_d^T\end{bmatrix}\mathbf{PL} = 0 \Leftrightarrow \tag{3}$$

$$\begin{bmatrix}\mathbf{A}_r^T\ \mathbf{PA}_r & \mathbf{A}_r^T\ \mathbf{PA}_d\\ \mathbf{A}_d^T\ \mathbf{PA}_r & \mathbf{A}_d^T\ \mathbf{PA}_d\end{bmatrix}\begin{bmatrix}d\mathbf{X}_r\\ d\mathbf{X}_d\end{bmatrix} + \begin{bmatrix}\mathbf{A}_r^T\mathbf{PL}\\ \mathbf{A}_d^T\mathbf{PL}\end{bmatrix} = 0 \tag{4}$$

In this way we get a system consisting of two matrix equations. They contain r and d of equations respectively. Due to the fact that the rank $(\underset{\text{mxm}}{\mathbf{A}^T\mathbf{PA}}) = r$, so this matrix is singular. So if the vector of increments $d\mathbf{X} = \begin{bmatrix}d\mathbf{X}_r\\ d\mathbf{X}_d\end{bmatrix}$ has solution of the first matrix system, then we can certainly say that it has also solution of the second matrix system. Therefore, a system of two matrix equations may be replaced by the first of them (SZUBRYCHT, WIŚNIEWSKI 2004, WOLF 1972).

$$\mathbf{A}^T\mathbf{PA}d\mathbf{X} + \mathbf{A}^T\mathbf{PL} = \mathbf{0} \Leftrightarrow$$

$$\mathbf{A}_r^T\mathbf{PA}_r d\mathbf{X}_r + \mathbf{A}_r^T\mathbf{PA}_d d\mathbf{X}_d + \mathbf{A}_r^T\mathbf{PL} = \mathbf{0} \tag{5}$$

with nonsingular matrix of coefficients $\mathbf{A}_r^T\mathbf{PA}_r$. This is consistent with the conception of minimum norm g – inverse (RAO 1973). The first system of normal equations consist of r independent equations with $m = r + d$ unknowns. Because of the datum defect exist, so this is a system of equation with more quantity of unknowns than equations. The quantity of equation in the second system of equations is equal to the defect of network. It means that equations forming second system of equations are independent from the equations of first system (SZUBRYCHT, WIŚNIEWSKI 2004). Therefore, the system (4) can be written in the form (5). Derivation matrix

$$\mathbf{B} = [\mathbf{A}_r^T\mathbf{PA}_r\quad \mathbf{A}_r^T\mathbf{PA}_d] \tag{6}$$

equations (5) can also be written as

$$\mathbf{B}d\mathbf{X} + \mathbf{A}_r^T\mathbf{P}\mathbf{L} = \mathbf{0} \tag{7}$$

After partial (only with function $\Phi(d\mathbf{X}) = \mathbf{V}^T\mathbf{P}\mathbf{V} = \min.$) solving the task (1), the final optimization form is obtained

$$\begin{cases} \mathbf{B}d\mathbf{X} + \mathbf{A}_r^T\mathbf{P}\mathbf{L} = \mathbf{0} \\ \Phi_X(d\mathbf{X}) = d\mathbf{X}^T\mathbf{P}_\mathbf{X}d\mathbf{X} = \min. \end{cases} \tag{8}$$

This is conditional equation and using Lagrange's functions can be replaced by formula

$$\begin{cases} \mathbf{B}d\mathbf{X} + \mathbf{A}_r^T\mathbf{P}\mathbf{L} = \mathbf{0} \\ \Phi_{XL}(d\mathbf{X}) = \Phi_X(d\mathbf{X}) - 2\mathbf{K}^T(\mathbf{B}d\mathbf{X} + \mathbf{A}_r^T\mathbf{P}\mathbf{L}) = \\ = d\mathbf{X}^T\mathbf{P}_\mathrm{X}d\mathbf{X} - 2\mathbf{K}^T(\mathbf{B}d\mathbf{X} + \mathbf{A}_r^T\mathbf{P}\mathbf{L}) = \min. \end{cases} \tag{9}$$

where **K** is correlative vector (Lagrange's multipliers). The solution (9) is a vector (WIŚNIEWSKI 2005):

$$d\hat{\mathbf{X}} = -\mathbf{P}_\mathrm{X}^{-1}\mathbf{B}^T(\mathbf{B}\mathbf{P}_\mathrm{X}^{-1}\mathbf{B}^T)^{-1}\mathbf{A}_r^T\mathbf{P}\mathbf{L} \tag{10}$$

Accuracy analysis of the free adjustment is similar to the classic adjustment. During calculation estimator of the variance coefficient, it should be taken into consideration that the number of redundant observation is increased by defect network. This estimator has a form:

$$m_0^2 = \frac{\mathbf{V}^T\mathbf{P}\mathbf{V}}{n - m + d} \tag{11}$$

Calculation mean errors of the coordinates were conducted in a similar to the classical method, however the covariance matrix of adjustment estimated parameters, has in a free adjustment following form (SZUBRYCHT, WIŚNIEWSKI 2004, WIŚNIEWSKI 2005).

$$\mathbf{C}_{\hat{\mathbf{X}}} = m_0^2\mathbf{P}_\mathbf{X}^{-1}\mathbf{B}^T(\mathbf{B}\mathbf{P}_\mathbf{X}^{-1}\mathbf{B}^T)^{-1}\mathbf{A}_r^T\mathbf{P}\mathbf{A}_r(\mathbf{B}\mathbf{P}_\mathbf{X}^{-1}\mathbf{B}^T)^{-1}\mathbf{B}\mathbf{P}_\mathbf{X}^{-1} \tag{12}$$

In the robust free adjustment, particular importance has covariance matrix of the increments assuming erroneous of the approximate coordinates vector $\mathbf{C}_{\hat{\mathbf{X}}(zb)}$. Taking into account that $d\hat{\mathbf{X}}$ and $\mathbf{X}^0$ are two independent random variables in the vector $\hat{\mathbf{X}} = \mathbf{X}^0 + d\hat{\mathbf{X}}$, we can derive the formula of covariance matrix $\mathbf{C}_{\hat{\mathbf{X}}(zb)}$ for $m_0 = 1$ (WIŚNIEWSKI 2005):

$$\mathbf{C}_{\hat{\mathbf{X}}(zb)(\boldsymbol{m}_0=1)} = \mathbf{C}_{\mathbf{X}^o}\mathbf{B}^T(\mathbf{B}\mathbf{P}_{\mathbf{X}}^{-1}\mathbf{B}^T)^{-1}\mathbf{B}\mathbf{P}_{\bar{\mathbf{X}}}^{-1} + \mathbf{C}_{\hat{\mathbf{X}}} \tag{13}$$

where $\mathbf{C}_{\mathbf{X}^o} = m_0^2\mathbf{P}_{\bar{\mathbf{X}}}^{-1}$, is the covariance matrix of the vector of the approximate coordinates $\mathbf{X}^0$. The diagonal values of the matrix $\mathbf{C}_{\hat{\mathbf{X}}(zb)(m_0=1)}$, are mean squared errors of the adjusted estimates.

Robust free adjustment

The ideas of robust free adjustment are location and reduction the influence of outlying increments to geodetic coordinate of the points. The location of such coordinates in case of an gross error is not difficult. In free geodetic network occurs significantly increase value of coordinates increments burdened by gross error in relation to other increments. Reducing outliers increments in similar to the robust M – estimations (KAMIŃSKI, WIŚNIEWSKI 1992a, 1992b), is conducted by using damping of weight matrix, in our case weight matrix of coordinates $\mathbf{P}_{\mathbf{X}}$ (WIŚNIEWSKI 2005). For this purpose, we can applied one of the damping functions, such as Huber, Hampel or Danish function (KAMIŃSKI, WIŚNIEWSKI 1992a, 1992b, WIŚNIEWSKI 2005). In this study Danish damping function was used to increments $d\mathbf{X}$ and $d\mathbf{Y}$ as result we have following form (KAMIŃSKI, WIŚNIEWSKI 1992a, 1992b, KRARUP, KUBIK 1983):

$$t_X(d\overline{X}) = \begin{cases} 1 & \text{for } d\overline{X} \in \Delta_X = \langle -k_X; k_X \rangle \\ \exp\{-l_X(|d\overline{X}| - k_X)^{g_X}\} & \text{for } |d\overline{X}| > k_X \end{cases} \tag{14}$$

$$t_Y(d\overline{Y}) = \begin{cases} 1 & \text{for } d\overline{Y} \in \Delta_Y = \langle -k_Y; k_Y \rangle \\ \exp\{-l_Y(|d\overline{Y}| - k_X)^{g_Y}\} & \text{for } |d\overline{Y}| > k_Y \end{cases} \tag{15}$$

where:
$d\overline{X}$ – standardized increment for coordinate X,
$d\overline{Y}$ – standardized increment for coordinate Y,
$\Delta_X = \langle -k_X; k_X \rangle$ – acceptable range for standardized increments $d\overline{X}$,
$\Delta_Y = \langle -k_Y; k_Y \rangle$ – acceptable range for standardized increments $d\overline{Y}$,
l_X, l_Y, g_X, g_Y – control parameters.

Free robust adjustment solution is iterative. The first point of this process is to calculate the free increments $d\hat{\mathbf{X}}$ according to equation (10), and their covariance matrix in the form (13). On this basis standardized increments are calculated:

$$d\bar{\hat{X}}_j = \frac{d\hat{\mathbf{X}}_j}{m_{\hat{\mathbf{X}}_j}}, \quad d\bar{\hat{Y}}_j = \frac{d\hat{\mathbf{Y}}_j}{m_{\hat{\mathbf{Y}}_j}} \tag{16}$$

where $m_{\hat{\mathbf{X}}_j}$, $m_{\hat{\mathbf{Y}}_j}$ are mean errors of the estimators $d\hat{\mathbf{X}}_j$, $d\hat{\mathbf{Y}}_j$ (square root of suitable elements of the matrix $\hat{\mathbf{C}}_{\hat{\mathbf{X}}(zb)(\boldsymbol{m}_0=1)}$). Basis on value (16) and dumping function (14) and (15), equivalent weight matrix of increments vectors $d\hat{\mathbf{X}}$ is calculated. Denoting this matrix by $\hat{\mathbf{P}}_{\mathbf{X}}$, we can write:

$$\hat{\mathbf{P}}_{\mathbf{X}} = \mathbf{P}_{\mathbf{X}}\mathbf{T}(d\bar{\hat{\mathbf{X}}}) \tag{17}$$

where:

$$\mathbf{T}(d\bar{\hat{X}}) = \text{Diag}\,\{t_X(d\bar{\hat{X}}_1),\, t_Y(d\bar{\hat{Y}}_1),\ldots,\, t_X(d\bar{\hat{X}}_{n_p}),\, t_Y d\bar{\hat{X}}_{n_p})\} \tag{18}$$

is diagonal damping matrix. If all standardized increments $d\hat{\mathbf{X}}_j$, $d\hat{\mathbf{Y}}_j$, $j = 1,\ldots,n_p$, are in their acceptable ranges (they are random), then the damping matrix $\mathbf{T}(d\hat{\mathbf{X}})$ is the unit matrix. Afterwards we get $\hat{\mathbf{P}}_{\mathbf{X}} = \mathbf{P}_{\mathbf{X}}$ which ends the adjustment process. Otherwise, the next step is the iteration, in which vector of free increments is recalculated

$$d\hat{\mathbf{X}} = -\hat{\mathbf{P}}_{\mathbf{X}}^{-1}\mathbf{B}^T(\mathbf{B}\hat{\mathbf{P}}_{\mathbf{X}}^{-1}\mathbf{B}^T)^{-1}\mathbf{A}_r^T\mathbf{PL} \tag{19}$$

with covariance matrix

$$\mathbf{C}_{\hat{\mathbf{X}}(zb)(m_0=1)} = \hat{\mathbf{C}}_{\mathbf{X}^0}\mathbf{B}^T(\mathbf{B}\hat{\mathbf{P}}_{\mathbf{X}}^{-1}\mathbf{B}^T)^{-1}\mathbf{B}\hat{\mathbf{P}}_{\mathbf{X}}^{-1} + \hat{\mathbf{C}}_{\hat{\mathbf{X}}} \tag{20}$$

Iterative process is continued so long until all the standardized increments will have values in their limits, respectively Δ_X and Δ_Y.

Characteristics of the geodetic test network

The first step was designed and staked out regular geodetic network, consisting five points on the area Kortowo II. The network have a shape of a square of side 200 [m] (Fig. 2). Network measurement was carried out in a single measurement session, obtained the observations presented in Figure 2.

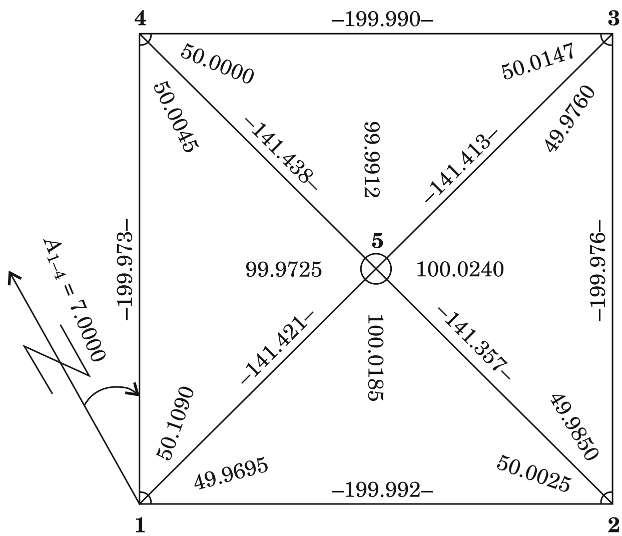

Fig. 2. The measurement results of the test network

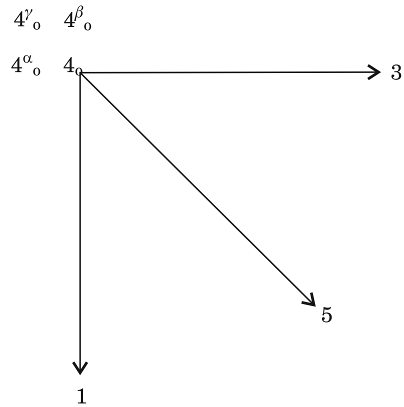

Fig. 3. The variants of the point no. 4 position

Let assume that the coordinates of the fourth point were contaminated by gross error. Then assumed that point as a fixed point during adjustment will have a negative influence on the obtained results of the adjustment. For the research purposes three variants of the position of the 4 (denoted by 4^{α}, 4^{β} and 4^{γ}) were designed, which simulate gross error in the coordinates (Fig. 3.). Values of the coordinates of point no. 4 are presented in Table 1. The aim of the research is to examine detecting of contaminated coordinates in the network by apply the robust free adjustment. Additionally the damping effect of weights matrix of coordinates on the final results of the network adjustment will be examined. Displacement ΔX and ΔY will be treated as gross errors of the coordinates.

Table 1

The variants of the approximate coordinates of the point no. 4

Δ	Variant		
	4^{α}	4^{β}	4^{γ}
ΔX[m]	$\Delta X^{\alpha} = 0.00$	$\Delta X^{\beta} = 0.30$	$\Delta X^{\gamma} = 0.31$
ΔY[m]	$\Delta Y^{\alpha} = 0.30$	$\Delta Y^{\beta} = 0.00$	$\Delta Y^{\gamma} = 0.24$

Based on data from measurements approximate coordinates of pointsin the local coordinate system were calculated. To calculate the approximate coordinates, let assume following coordinates of point no. 1 $X = 1000$ [m], $Y = 1000$ [m] and azimuth $A_{1\text{-}4} = 7.0000^{[g]}$. Obtained, approximate coordinates are shown in Table 2.

Table 2

Summary of approximate coordinates

	No. point	1	2	3	4	4^{α}	4^{β}	4^{γ}	5
Computed approximate coordinates	X[m]	1,000.00	978.09	1,176.83	1,198.76	1,198.80	1,199.05	1,199.09	1,088.39
	Y[m]	1,000.00	1,198.79	1,220.74	1,021.94	1,021.65	1,021.98	1,021.71	1,110.40

Results of robust free adjustment

The results of robust free adjustment are the estimates of increments to the approximate coordinates of all points. The test network was adjusted freely for all of the four variants of the approximate coordinates. The network was adjusted for following means errors $m_k = 35^{cc}$, $m_d = 0.025$ [m]. The results of such adjustment are summarized in Table 3.

These results show a clear difference between the received estimates of parameters. In case of the classical adjustments, establishing the 4^{α} 4^{β} or 4^{γ} as a fixed point, cause that the results of the adjusted coordinates are even more distorted. In free robust adjustment Danish damping function was used. In the acceptable ranges $\Delta_X = < -k_X; k_X >$ and $\Delta_Y = < -k_Y; k_Y >$ for standardized increments $d\overline{X}$ and $d\overline{Y}$, was taken $k_X = k_Y = 1$. Then, with the probability equal 0.68 can be said that standardized increments which do not belong to the acceptable ranges are called outliers increments. In process of robust free

Table 3

Results of the classical free adjustment

No. point	Variant			
	I	II	III	IV
$\hat{X}_1$ [m]	1,000.00	999.97	1,000.10	1,000.07
$\hat{Y}_1$ [m]	1,000.00	999.98	999.98	999.95
$\hat{X}_2$ [m]	978.07	978.10	978.09	978.13
$\hat{Y}_2$ [m]	1,198.79	1,198.77	1,198.76	1,198.74
$\hat{X}_3$ [m]	1,176.85	1,176.90	1,176.86	1,176.91
$\hat{Y}_3$ [m]	1,220.72	1,220.63	1,220.76	1,220.68
$\hat{X}_4$ [m]	1,198.77	1,198.74	1,198.86	1,198.84
$\hat{Y}_4$ [m]	1,021.95	1,021.86	1,022.00	1,021.91
$\hat{X}_5$ [m]	1,088.39	1,088.40	1,088.45	1,088.46
$\hat{Y}_5$ [m]	1,110.40	1,110.35	1,110.41	1,110.36

adjustment, let assume the following parameters $g_X = g_Y = 2$ and $l_X = l_Y = 0.6$. Such a strict qualification has the relation to main aim, namely the effective identification of point displacement. Robust free adjustment is iterative. In the first step we assumed that the weight matrix of increment vector $d\mathbf{X}$ as the unit matrix. Therefore starting step is classical free adjustment is with optimization criterion $d\mathbf{X}^T\mathbf{P_X}d\mathbf{X} = d\mathbf{X}^Td\mathbf{X} = \min$. Then $d\mathbf{X}^{(0)} = d\hat{\mathbf{X}}$, and next $d\overline{X}^{(0)} = d\hat{\overline{X}}$. In each next step, the j-th iteration matrix of weights $\mathbf{P_X}$ is replaced by the equivalent matrix $\hat{\mathbf{P}}_{\mathbf{X}}^{(j)} = \mathbf{P_X}^{(j-1)}\mathbf{T}(d\overline{X}^{(j-1)})$, where $\mathbf{T}(d\overline{X}^{(j-1)})$ is diagonal

Table 4

Results of the robust free adjustment

Specification	Variant					
	II		III		IV	
	$\hat{\mathbf{X}}$	$d\hat{\mathbf{X}}$	$\hat{\mathbf{X}}$	$d\hat{\mathbf{X}}$	$\hat{\mathbf{X}}$	$d\hat{\mathbf{X}}$
$\hat{X}_1$ [m]	1,000.01	0.01	1,000.04	0.04	1,000.01	0.01
$\hat{Y}_1$ [m]	1,000.00	0.00	1,000.00	0.00	1,000.00	0.00
$\hat{X}_2$ [m]	978.07	-0.02	978.07	-0.02	978.08	-0.01
$\hat{Y}_2$ [m]	1,198.78	-0.01	1,198.78	-0.01	1,198.78	-0.01
$\hat{X}_3$ [m]	1,176.86	0.03	1,176.85	0.02	1,176.86	0.03
$\hat{Y}_3$ [m]	1,220.71	-0.03	1,220.74	0.00	1,220.71	-0.03
$\hat{X}_4$ [m]	1,198.77	**-0.03** ΔX = 0.00	1,198.80	**-0.25** ΔX= -0.30	1,198.78	**-0.31** ΔX = -0.31
$\hat{Y}_4$ [m]	1,021.94	**0.29** ΔY = 0.30	1,021.98	**0.00** ΔY = 0.00	1,021.94	**0.23** ΔY = 0.24
$\hat{X}_5$ [m]	1,088.40	0.01	1,088.41	0.02	1,088.40	0.01
$\hat{Y}_5$ [m]	1,110.40	0.00	1,110.41	0.01	1,110.39	-0.01

damping matrix. After ten iterative steps, the following results of robust free adjustment were obtained.

The results of the computations clearly show which coordinates of the points are contaminated by gross terror. Of course, the method is the most effective for single outlying point in the network. With a greater number of contaminated points, their detection would be difficult or even impossible. The coordinates obtained in robust free adjustment were compared with the coordinates obtained in the classical free adjustment (variant I in Tab. 3.). The results of this comparison including the differences between the respective coordinates are summarized in Table 5.

Table 5

Summary of differences adjustment of network coordinates

Specification	$\hat{X}_1$ [m]	$\hat{Y}_1$ [m]	$\hat{X}_2$ [m]	$\hat{Y}_2$ [m]	$\hat{X}_3$ [m]	$\hat{Y}_3$ [m]	$\hat{X}_4$ [m]	$\hat{Y}_4$ [m]	$\hat{X}_5$ [m]	$\hat{Y}_5$ [m]
$\hat{\mathbf{X}}_{II} - \hat{\mathbf{X}}_{I}$	0.01	0.00	0.00	-0.01	0.01	-0.01	0.00	-0.01	0.01	0.00
$\hat{\mathbf{X}}_{III} - \hat{\mathbf{X}}_{I}$	0.04	0.00	0.00	-0.01	0.00	0.02	0.03	0.03	0.02	0.01
$\hat{\mathbf{X}}_{IV} - \hat{\mathbf{X}}_{I}$	0.01	0.00	0.01	-0.01	0.01	-0.01	0.01	-0.01	0.01	-0.01

The largest differences occur between the results of the classical free adjustment and the results of the robust free adjustment in III variant (for point 4 with approximate coordinates 4^{β}). Let consider a situation when in network, besides outlying points exist also observations contaminated by gross error. For the purpose of numerical example, an observation d1-4 was contaminated by gross error 0.400 m ($d_{1\text{-}4}$ = 199.573 m). The adjustment is conducted according to principles of hybrid M – estimation (CZAPLEWSKI 2004). This method is characterized by the simultaneous damping weights of observation weight matrix and damping weights of coordinates weight matrix. Therefore, besides the standardized increments the standardized corrections is determined.

$$\bar{\hat{\mathbf{V}}}_i = \frac{\mathbf{V}_i}{m_{\hat{\mathbf{V}}_i}}, \qquad i = 1,2,..., 20 \tag{21}$$

To determine these standardize corrections, it is necessary to specify the form of the covariance matrix of the corrections vector $\hat{\mathbf{C}}_{\mathbf{V}}$. The values on the diagonal on the covariance matrix are the squares of the mean errors of corrections. The covariance matrix of the corrections vector derivation is presented in the paper (CZAPLEWSKI 2004). Below the final form of covariance matrix of the corrections vector is presented:

$$\hat{\mathbf{C}}_{\mathbf{V}} = m_0^2 \mathbf{M}_+ (\mathbf{A}\mathbf{P}_{\mathbf{X}}^{-1}\mathbf{A}^T + \mathbf{P}^{-1})\mathbf{M}_+^T \tag{22}$$

where

$$\mathbf{M}_+ = \mathbf{I} - \mathbf{A}\mathbf{P}_{\mathbf{X}}^{-1}\mathbf{B}^T(\mathbf{B}\mathbf{P}_{\mathbf{X}}^{-1}\mathbf{B}^T)^{-1}\mathbf{A}^T\mathbf{P} \tag{23}$$

To identify and reduce the influence of observation contaminated by gross errors, the same value of parameters g, l and k as in the identification process of contaminated coordinates of the points is used. The adjustment procedure is analogous like in the previous example. In this case two equivalent weights matrices: $\hat{\mathbf{P}}_{\mathbf{X}}$ and $\hat{\mathbf{P}}_{\mathbf{X}}^{(j)} = \hat{\mathbf{P}}_{\mathbf{X}}^{(j-1)}\mathbf{T}(\mathbf{L}^{(j-1)})$, where $\mathbf{T}(\mathbf{L}^{(j-1)})$ is diagonal damping matrix are determined. After ten iterative steps, obtained the following results of geodetic network adjustment by applying hybrid M – estimation.

Table 6

The results of robust free adjustment (hybrid M – estimation)

Specification	Variant					
	II		III		IV	
	$\hat{\mathbf{X}}$	$d\hat{\mathbf{X}}$	$\hat{\mathbf{X}}$	$d\hat{\mathbf{X}}$	$\hat{\mathbf{X}}$	$d\hat{\mathbf{X}}$
$\hat{X}_1$ [m]	1,000.01	0.01	1,000.03	0.03	1,000.00	-0.00
$\hat{Y}_1$ [m]	1,000.00	0.00	1,000.00	0.00	1,000.00	0.00
$\hat{X}_2$ [m]	978.08	-0.01	978.07	-0.02	978.07	-0.02
$\hat{Y}_2$ [m]	1,198.78	-0.01	1,198.78	-0.01	1,198.78	-0.01
$\hat{X}_3$ [m]	1,176.86	0.03	1,176.85	0.02	1,176.85	0.02
$\hat{Y}_3$ [m]	1,220.72	-0.02	1,220.75	0.01	1,220.72	-0.02
$\hat{X}_4$ [m]	1,198.77	**-0.03**	1,198.79	**-0.26**	1,198.77	**-0.32**
$\hat{Y}_4$ [m]	1,021.94	**0.29**	1,021.97	**-0.01**	1,021.94	**0.23**
$\hat{X}_5$ [m]	1,088.39	0.00	1,088.40	0.01	1,088.39	-0.00
$\hat{Y}_5$ [m]	1,110.39	-0.01	1,110.41	0.01	1,110.40	-0.00

The results of the computations clearly show, that the coordinates were contaminated by gross error. Similar to the free robust adjustment, with an increasing number of observation contaminated by gross errors, it is more difficult to identified the contaminated points. It should be noted, that quantities of outliers are not the only problem with regard to the identification of outliers points. Unfortunately, this method is most effective in case of large errors in approximate coordinates.

Table 7

Results of the robust free adjustment

Specification	Variant									
	V		VI		VII		VIII		IX	
	$\hat{\mathbf{X}}$	$d\hat{\mathbf{X}}$	$\hat{\mathbf{X}}$	$d\hat{\mathbf{X}}$	$\hat{\mathbf{X}}$	$d\hat{\mathbf{X}}$	$\hat{\mathbf{X}}$	$d\hat{\mathbf{X}}$	$\hat{\mathbf{X}}$	$d\hat{\mathbf{X}}$
$\hat{X}_1$ [m]	1,000.00	0.00	1,000.01	0.01	1,000.01	0.01	1,000.01	0.01	1,000.02	0.02
$\hat{Y}_1$ [m]	1,000.00	0.00	1,000.00	0.00	1,000.00	0.00	1,000.00	0.00	1,000.00	0.00
$\hat{X}_2$ [m]	978.07	-0.02	978.07	-0.02	978.07	-0.02	978.07	-0.02	978.07	-0.02
$\hat{Y}_2$ [m]	1,198.79	0.00	1,198.79	-0.00	1,198.79	0.00	1,198.79	0.00	1,198.79	0.00
$\hat{X}_3$ [m]	1,176.85	0.02	1,176.85	0.02	1,176.85	0.02	1,176.85	0.02	1,176.85	0.02
$\hat{Y}_3$ [m]	1,220.72	-0.02	1,220.72	-0.02	1,220.72	-0.02	1,220.72	-0.02	1,220.73	-0.01
$\hat{X}_4$ [m]	1,198.77	**0.00**	1,198.78	**-0.00**	1,198.78	**-0.01**	1,198.78	**-0.02**	1,198.78	**-0.03**
$\hat{Y}_4$ [m]	1,021.96	**0.01**	1,021.96	**0.01**	1,021.96	**0.01**	1,021.96	**0.01**	1,021.96	**0.01**
$\hat{X}_5$ [m]	1,088.39	0.00	1,088.39	0.00	1,088.40	0.01	1,088.40	0.01	1,088.40	0.01
$\hat{Y}_5$ [m]	1,110.41	0.01	1,110.41	0.01	1,110.41	0.01	1,110.41	0.01	1,110.41	0.01

For small changes in the approximate coordinates, the coefficients matrix **A** is changing very slightly and has little effect on the final value of the vector $d\hat{\mathbf{X}}$. Table 7. presents the results of the robust free adjustment obtained for following variants of approximate coordinates (variant V – X_4^0 = 119,877 and Y_4^0 = 1,021.95, variant VI – X_4^0 = 119,878 and Y_4^0 = 1,021.95, variant VII – X_4^0 = 119,879 and Y_4^0 = 1,021.95, variant VIII – X_4^0 = 119,880 and Y_4^0 = 1,021.95, variant IX – X_4^0 = 119,881 and Y_4^0 = 1,021.95).

Conclusions

Developing the estimation methods has a significant influence on geodesy and related sciences. Implementation new computational algorithms, engineers gain new opportunities for solution geodetic observation. In recent years the development concern methods of robust estimation for gross errors. In this paper we have shown that the practical importance has also robust free adjustment. Classical free adjustment is most commonly used to adjustment free realization network (or other special-purpose networks). Such adjustment allow for optimum fitting adjustment of network to approximate network (network with known approximate coordinates). It is also possible objective analysis of the accuracy, independent of the variant elimination degrees of freedom network. Robust free adjustment remains mentioned above features of classical free adjustment, however allows identifying of points whose coordinates are burdened with gross errors. The reason of such errors may be different, for example, resulting from unrecognized displacements points of network. In our test network, values of movements were known. This enabled the evaluation of effectiveness of robust free adjustment. The paper shows very interesting properties of the hybrid M – estimation which concerned damping weight of observation weight matrix and damping weight of coordinates weight matrix. In each of analyzed variant, a single outlying point has been detected. Although the experiment was carried out successfully this method has some limitations of the approach with regard to practical applications. These limitations concern among others the values of errors of the approximate coordinates that need to be in the range of linearity for the coefficients of matrices **A**. Additionally, it should be noted that method presented in this paper is most effective in a situation where only one point is outlying. In the case of existing more than one outlying point in the network, the detection problems of these points may appear. Robust free adjustment can be further developed and detailed comparative analysis. This may particular apply selections of other damping function. Detailed research of this method should concern the reliability against a larger number observation and outlying points.

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