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Generalized α-V-univex functions for multiobjective variational control problems*†

by

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Abstract: The purpose of this paper is to introduce a new class of α-V-univex/ generalized α-V-univex functions for a class of multiobjective variational control problems. Moreover, sufficient optimality conditions and Mond-Weir type duality results, associated with the multiobjective variational control problem, are established under aforesaid assumptions.

Keywords: control problem; α-V-univexity; efficiency; sufficiency; duality

1. Introduction

Multiobjective optimization deals with solving problems having several conflicting objectives simultaneously, while the control problem consists in transferring the state variable from an initial state to a final state so as to optimize a given functional, subject to constraints on the control and state variables.

Thus, multiobjective control problem is a wide field of research having extensive applications in real world situations, ranging from engineering to economics, and many more. For example, multiobjective control problems are used in flight control design, in the control of space structures, in industrial process control and other diverse fields.

Convexity plays a significant role in optimization as it gives global validity to propositions otherwise only locally true. However, in the real world of mathematical and economic models, convexity appears to be a restrictive condition. As a result, nonconvex optimization problems have been studied by various authors. Naniewicz and Puchala (2012) studied a nonconvex optimization problem

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in the case when the functional to be minimized has integrand expressed as a minimum of two quadratic functions by constructing an appropriate minimizing sequence. Tabor and Tabor (2012) showed that the Takagi class can serve as an important source of examples and counterexamples for paraconvex and semi-convex functions. By similar motivation, several researchers have generalized the concept of convexity.

Firstly, Hanson (1981) introduced the concept of invexity to extend the validity of the sufficiency of the Kuhn-Tucker conditions. Mond and Smart (1988) obtained duality results for a control problem using invexity and showed that for invex functions, the necessary conditions are also sufficient. Bhatia and Kumar (1995) introduced multiobjective control problems and proved duality results under generalized ρ-invexity assumptions. Nahak and Nanda (1997, 2007) further extended this concept by proving duality results for multiobjective variational control problems, under the assumption of (F, ρ)-convexity and V-invexity, respectively.

Gulati, Husain and Ahmed (2005) derived optimality conditions and duality results for multiobjective control problems involving generalized convexity. Also, Ahmad and Gulati (2005) proved results for mixed type dual for multiobjective variational problems under generalized (F, ρ)-convexity. Later, Ahmad and Sharma (2010) extended the notion of generalized $(F, \alpha, \rho, \theta)$-V-convex functions to variational control problems. Recently, Kailey and Gupta (2013) further extended the concept of generalized (F, α, ρ, d)-convexity and proved duality results for a class of symmetric non-differentiable multiobjective fractional variational problems.

The concept of univex functions as a generalization of invex functions was introduced by Bector, Suneja and Gupta (1992). Later on, many authors (Arana-Jiménez, Ruiz-Garzón, Rufián-Lizana and Osuna-Gómez, 2012; Chen, 2002; de Oliveira, Silva and Rojas-Medar, 2009; Khazafi and Rueda, 2009; Khazafi, Rueda and Enflo, 2010; Zhian and Qingkai, 2001) extended the concept of generalized convexity. Being inspired by Bector, Suneja and Gupta (1992), Noor (2004), Nahak and Nanda (1997, 2007), and Preda, Stancu-Minasian, Beldiman and Stancu (2009), we introduce the concept of α-V-univex functions for a multiobjective variational control problem and obtain sufficient optimality conditions and duality results.

The rest of the paper is organized as follows: In Section 2, we introduce the definitions of α-V-univex and generalized α-V-univex functions, and recall a set of necessary optimality conditions. In Section 3, we prove the sufficient optimality conditions, and in Section 4, we present the Mond-Weir type multiobjective variational control dual problem and derive weak and strong duality results. Finally, we conclude our paper in Section 5.

2. Notations and preliminaries

Let R^n denote the n-dimensional Euclidean space. Let $y, z \in R^n$, we denote: $y \leqq z \Leftrightarrow y_i \leqq z_i$, $i = 1, 2, \ldots, n$; $y \leq z \Leftrightarrow y \leqq z$ and $y \neq z$; $y < z \Leftrightarrow y_i < z_i$, $i =$

$1, 2, \ldots, n$.

Let $I = [a, b]$ be a real interval. Let $f_i : I \times R^n \times R^n \times R^m \times R^m \to R$, $i \in P = \{1, 2, \ldots, p\}$, $g_j : I \times R^n \times R^n \times R^m \times R^m \to R$, $j \in M = \{1, 2, \ldots, m\}$ and $h_k : I \times R^n \times R^n \times R^m \times R^m \to R$, $k \in N = \{1, 2, \ldots, n\}$ be continuously differentiable functions. Consider the function $f(t, x(t), \dot{x}(t), u(t), \dot{u}(t))$, where t is the independent variable, $x : I \to R^n$ is the state variable and $u : I \to R^m$ is the control variable. $u(t)$ is related to $x(t)$ via the state equation $h(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) = 0$, where the dot denotes the derivative with respect to t. f_{ix}, $f_{i\dot{x}}$, f_{iu} and $f_{i\dot{u}}$ denote the partial derivatives of f_i with respect to $x, \dot{x}, u$ and $\dot{u}$, respectively. For instance,

$$f_{ix} = \left(\frac{\partial f_i}{\partial x_1}, \frac{\partial f_i}{\partial x_2}, \ldots, \frac{\partial f_i}{\partial x_n} \right), \; f_{i\dot{x}} = \left(\frac{\partial f_i}{\partial \dot{x}_1}, \frac{\partial f_i}{\partial \dot{x}_2}, \ldots, \frac{\partial f_i}{\partial \dot{x}_n} \right).$$

Similarly, g_{jx}, $g_{j\dot{x}}$, g_{ju}, $g_{j\dot{u}}$ and h_{kx}, $h_{k\dot{x}}$, h_{ku}, $h_{k\dot{u}}$ can be defined. For notational convenience, we use $x, \dot{x}, u, \dot{u}$ in place of $x(t), \dot{x}(t), u(t), \dot{u}(t)$, respectively. Let the differentiation operator D be given by

$$z = Dx \Leftrightarrow x(t) = \gamma + \int_a^t z(s)ds,$$

where γ is a given boundary value. Therefore, $D = d/dt$ except at discontinuities. Let X denote the space of all piecewise smooth functions $x : I \mapsto R^n$ with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$ and Y denote the space of all piecewise smooth functions $u : I \mapsto R^m$ with norm $\|u\|_\infty$.

In this paper, we consider the following multiobjective variational control problem:

(**CP**) Minimize $\displaystyle\int_a^b f(t, x, \dot{x}, u, \dot{u})dt = (\int_a^b f_1(t, x, \dot{x}, u, \dot{u})dt,$
$\displaystyle\int_a^b f_2(t, x, \dot{x}, u, \dot{u})dt, \ldots, \int_a^b f_p(t, x, \dot{x}, u, \dot{u})dt)$

subject to

$$x(a) = \gamma, \; x(b) = \delta,$$
$$g(t, x, \dot{x}, u, \dot{u}) \leqq 0, \; t \in I, \quad (1)$$
$$h(t, x, \dot{x}, u, \dot{u}) = 0, \; t \in I. \quad (2)$$

We denote the set of all feasible solutions to (CP) by X°, i.e.,

$$X^\circ = \{(x, u) \in (X, Y) : x(a) = \gamma, x(b) = \delta, g(t, x, \dot{x}, u, \dot{u}) \leqq 0, h(t, x, \dot{x}, u, \dot{u}) = 0\}.$$

Definition 1 *A point $(\bar{x}, \bar{u}) \in X^\circ$ is said to be an efficient solution of (CP), if there exists no other point $(x, u) \in X^\circ$ such that*

$$\int_a^b f(t, x, \dot{x}, u, \dot{u})dt \leq \int_a^b f(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})dt.$$

Definition 2 *A feasible point* $(\bar{x},\bar{u}) \in X^{\circ}$ *is said to be a weakly efficient solution of (CP), if there exists no other point* $(x,u) \in X^{\circ}$ *such that*

$$\int_a^b f(t,x,\dot{x},u,\dot{u})dt < \int_a^b f(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})dt.$$

Let $b_{\circ}(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \in R_{+}$, $\alpha(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \in R^p_{+}\backslash\{0\}$, $\eta(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \in R^n$, $\xi(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \in R^m$ and $\phi_{\circ}: R \to R$. For notational convenience, we use $b_{\circ}$ for $b_{\circ}(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$,
α_i for $\alpha_i(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$, η for $\eta(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$, and ξ for $\xi(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$. Let $\psi: I \times X \times X \times Y \times Y \mapsto R^p$ be a vector functional.

Now, we introduce the concept of α-V-univexity as follows.

Definition 3 *A vector functional* $\int_a^b \psi(t,x,\dot{x},u,\dot{u})dt$ *is said to be* α*-V-univex at* $(\bar{x},\bar{u})$ *with respect to the functions* $b_{\circ},\phi_{\circ},\alpha,\eta$ *and* ξ*, if for all* $(x,u) \in (X,Y)$ *and* $i \in P$*,*

$$b_{\circ}\int_a^b \phi_{\circ}[\psi_i(t,x,\dot{x},u,\dot{u}) - \psi_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]dt$$
$$\geqq \int_a^b \alpha_i[(\psi_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\psi_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\eta$$
$$+ (\psi_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\psi_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\xi]dt.$$

Remark 1 *If we take* $b_{\circ} = 1$, $\phi_{\circ}(a) = a$ *and* $D\psi_{i\dot{u}} = 0$*, then* α*-V-univex function reduces to* V*-invex function given by Nahak and Nanda (2007).*

Now, we present the following example which is α-V-univex but not V-invex.

Example 1 *Let* $I = [0,1]$ *and* $X = Y = C([0,1],R_{+})$*. We define the function* $\psi: I \times X \times X \times Y \times Y \mapsto R^2$ *as*

$$\psi(t,x,\dot{x},u,\dot{u}) = (-4(x^2(t)+x(t)+u(t)), -x^2(t)-2x(t)-3u(t)).$$

Further, let $\phi_{\circ}: R \mapsto R$ *be given as* $\phi_{\circ}(a) = -2a$*. Define*

$$\alpha_1 = \frac{\bar{x}(t)\bar{u}(t)+1}{2},\ \alpha_2 = \frac{\bar{x}(t)\bar{u}(t)+3}{2},$$
$$\eta = \frac{x^2(t)+3u(t)}{4},\ \xi = \frac{2x^2(t)+3x(t)+\bar{x}(t)\bar{u}(t)}{4},$$

and take $b_{\circ} = 2$, $\bar{x}(t) = t$ *and* $\bar{u}(t) = t^2$*. Then,* $\int_0^1 \psi(t,x,\dot{x},u,\dot{u})dt$ *is* α*-V-univex at* $(\bar{x},\bar{u}) = (0,0)$ *but not* V*-invex for the same functions* α*,* η *and* ξ *as can be seen below.*

Explanation: Firstly, we have to show that

$$b_\circ \int_0^1 \phi_\circ[\psi_1(t,x,\dot{x},u,\dot{u}) - \psi_1(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]dt$$

$$\geqq \int_0^1 \alpha_1[(\psi_{1x}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\psi_{1\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\eta$$

$$+ (\psi_{1u}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\psi_{1\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\xi]dt, \quad \text{at } (\bar{x},\bar{u}) = (0,0).$$

L.H.S.:

$$b_\circ \int_0^1 \phi_\circ[\psi_1(t,x,\dot{x},u,\dot{u}) - \psi_1(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]dt$$

$$= 2\int_0^1 \phi_\circ[-4(x^2(t)+x(t)+u(t)) + 4(\bar{x}^2(t)+\bar{x}(t)+\bar{u}(t))]dt$$

$$= 2\int_0^1 (-2)[-4(x^2(t)+x(t)+u(t)) + 4(\bar{x}^2(t)+\bar{x}(t)+\bar{u}(t))]dt$$

$$= 16\int_0^1 (x^2(t)+x(t)+u(t))dt.$$

R.H.S.:

$$\int_0^1 \alpha_1[(\psi_{1x}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\psi_{1\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\eta$$

$$+(\psi_{1u}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\psi_{1\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\xi]dt$$

$$= \int_0^1 \frac{\bar{x}(t)\bar{u}(t)+1}{2}[-4(2\bar{x}(t)+1)\frac{x^2(t)+3u(t)}{4} - 4\frac{2x^2(t)+3x(t)+\bar{x}(t)\bar{u}(t)}{4}]dt$$

$$= -\int_0^1 \frac{\bar{x}(t)\bar{u}(t)+1}{2}[(2\bar{x}(t)+1)(x^2(t)+3u(t)) + 2x^2(t)+3x(t)+\bar{x}(t)\bar{u}(t)]dt$$

$$= -\frac{3}{2}\int_0^1 (x^2(t)+x(t)+u(t))dt.$$

Therefore, it follows that

$$b_\circ \int_0^1 \phi_\circ[\psi_1(t,x,\dot{x},u,\dot{u}) - \psi_1(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]dt$$

$$\geqq \int_0^1 \alpha_1[(\psi_{1x}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\psi_{1\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\eta$$

$$+ (\psi_{1u}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\psi_{1\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\xi]dt, \quad \text{at } (\bar{x},\bar{u}) = (0,0).$$

Similarly, it can be shown that

$$b_\circ \int_0^1 \phi_\circ[\psi_2(t,x,\dot{x},u,\dot{u}) - \psi_2(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]dt$$

$$\geqq \int_0^1 \alpha_2[(\psi_{2x}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\psi_{2\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\eta$$
$$+ (\psi_{2u}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\psi_{2\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\xi]dt, \quad \text{at } (\bar{x},\bar{u}) = (0,0).$$

Therefore, $\int_0^1 \psi(t,x,\dot{x},u,\dot{u})dt$ is α-V-univex at $(\bar{x},\bar{u}) = (0,0)$. Again,

$$\int_0^1 [\psi_1(t,x,\dot{x},u,\dot{u}) - \psi_1(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]dt$$
$$= \int_0^1 [-4(x^2(t) + x(t) + u(t)) + 4(\bar{x}^2(t) + \bar{x}(t) + \bar{u}(t))]dt$$
$$= -4\int_0^1 (x^2(t) + x(t) + u(t))dt.$$

Also,

$$\int_0^1 \alpha_1[(\psi_{1x}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\psi_{1\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\eta + \psi_{1u}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\xi]dt$$
$$= \int_0^1 \frac{\bar{x}(t)\bar{u}(t)+1}{2}[-4(2\bar{x}(t)+1)\frac{x^2(t)+3u(t)}{4} - 4\frac{2x^2(t)+3x(t)+\bar{x}(t)\bar{u}(t)}{4}]dt$$
$$= -\int_0^1 \frac{\bar{x}(t)\bar{u}(t)+1}{2}[(2\bar{x}(t)+1)(x^2(t)+3u(t)) + 2x^2(t) + 3x(t) + \bar{x}(t)\bar{u}(t)]dt$$
$$= -\frac{3}{2}\int_0^1 (x^2(t) + x(t) + u(t))dt.$$

Hence, it follows that

$$\int_0^1 [\psi_1(t,x,\dot{x},u,\dot{u}) - \psi_1(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]dt$$
$$\ngeqq \int_0^1 \alpha_1[(\psi_{1x}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\psi_{1\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\eta + \psi_{1u}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\xi]dt,$$

which shows that $\int_0^1 \psi(t,x,\dot{x},u,\dot{u})dt$ is not V-invex at $(\bar{x},\bar{u}) = (0,0)$.

DEFINITION 4 *A vector functional $\int_a^b \psi(t,x,\dot{x},u,\dot{u})dt$ is said to be (strictly) α-V-pseudounivex at $(\bar{x},\bar{u})$ with respect to the functions $b_\circ, \phi_\circ, \alpha, \eta$ and ξ, if for all $(x,u) \in (X,Y)$ and $i \in P$,*

$$\int_a^b [(\sum_{i\in P}\psi_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\sum_{i\in P}\psi_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\eta$$
$$+ (\sum_{i\in P}\psi_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\sum_{i\in P}\psi_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\xi]dt \geqq 0$$
$$\Rightarrow b_\circ \int_a^b \phi_\circ[\sum_{i\in P}\alpha_i\psi_i(t,x,\dot{x},u,\dot{u}) - \sum_{i\in P}\alpha_i\psi_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]dt (>) \geqq 0.$$

DEFINITION 5 *A vector functional $\int_a^b \psi(t,x,\dot{x},u,\dot{u})dt$ is said to be (strictly) α-V-quasiunivex at $(\bar{x},\bar{u})$ with respect to the functions $b_\circ, \phi_\circ, \alpha, \eta$ and ξ, if for all $(x,u) \in (X,Y)$ and $i \in P$,*

$$b_\circ \int_a^b \phi_\circ[\sum_{i\in P} \alpha_i \psi_i(t,x,\dot{x},u,\dot{u}) - \sum_{i\in P} \alpha_i \psi_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]dt \leqq 0$$

$$\Rightarrow \int_a^b [(\sum_{i\in P} \psi_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\sum_{i\in P} \psi_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\eta$$

$$+ (\sum_{i\in P} \psi_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\sum_{i\in P} \psi_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\xi]dt(<) \leqq 0.$$

DEFINITION 6 *A vector functional $\int_a^b \psi(t,x,\dot{x},u,\dot{u})dt$ is said to be prestrictly α-V-quasiunivex at $(\bar{x},\bar{u})$ with respect to the functions $b_\circ, \phi_\circ, \alpha, \eta$ and ξ, if for all $(x,u) \in (X,Y)$ and $i \in P$,*

$$b_\circ \int_a^b \phi_\circ[\sum_{i\in P} \alpha_i \psi_i(t,x,\dot{x},u,\dot{u}) - \sum_{i\in P} \alpha_i \psi_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]dt < 0$$

$$\Rightarrow \int_a^b [(\sum_{i\in P} \psi_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\sum_{i\in P} \psi_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\eta$$

$$+ (\sum_{i\in P} \psi_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\sum_{i\in P} \psi_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\xi]dt \leqq 0.$$

REMARK 2 *In the proofs of theorems, sometimes it may be more convenient to use certain alternative, but equivalent, forms of the above definitions. For example: a vector functional $\int_a^b \psi(t,x,\dot{x},u,\dot{u})dt$ is said to be α-V-pseudounivex with respect to the functions $b_\circ, \phi_\circ, \alpha, \eta$ and ξ, if for all $(x,u) \in (X,Y)$ and $i \in P$,*

$$b_\circ \int_a^b \phi_\circ[\sum_{i\in P} \alpha_i \psi_i(t,x,\dot{x},u,\dot{u}) - \sum_{i\in P} \alpha_i \psi_i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]dt < 0$$

$$\Rightarrow \int_a^b [(\sum_{i\in P} \psi_{ix}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\sum_{i\in P} \psi_{i\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\eta$$

$$+ (\sum_{i\in P} \psi_{iu}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - D\sum_{i\in P} \psi_{i\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\xi]dt < 0.$$

LEMMA 1 *(Kuhn-Tucker type necessary conditions) (Ahmad and Sharma, 2010) Let $(\bar{x},\bar{u})$ solve the following single objective problem:*

$$\textit{Minimize} \int_a^b \phi(t, x, \dot{x}, u, \dot{u})dt$$

subject to

$$x(a) = \gamma, \ x(b) = \delta,$$
$$g(t, x, \dot{x}, u, \dot{u}) \leqq 0, \ t \in I,$$
$$h(t, x, \dot{x}, u, \dot{u}) = 0, \ t \in I.$$

If the Frëchet derivative $[D - H_x(\bar{x}, \bar{u})]$ is surjective and the optimal solution $(\bar{x}, \bar{u})$ is normal, then there exist piecewise smooth functions $\bar{\mu} : I \mapsto R^m$ and $\bar{\nu} : I \mapsto R^n$ satisfying the following conditions:

$$\phi_x(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j(t) g_{jx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k(t) h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$
$$= D[\phi_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j(t) g_{j\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k(t) h_{k\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})],$$
$$\phi_u(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j(t) g_{ju}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k(t) h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$
$$= D[\phi_{\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j(t) g_{j\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k(t) h_{k\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})],$$
$$\sum_{j\in M} \bar{\mu}_j(t) g_j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) = 0,$$
$$\bar{\mu}(t) \geqq 0, \quad \forall t \in I.$$

LEMMA 2 *(Chankong and Haimes, 1983) $(\bar{x}, \bar{u})$ is an efficient solution for (CP) if and only if $(\bar{x}, \bar{u})$ solves*

$$\textbf{(CP)}_s \ \textit{Minimize} \int_a^b f_k(t, x, \dot{x}, u, \dot{u})dt$$

subject to

$$x(a) = \gamma, \ x(b) = \delta,$$
$$\int_a^b f_i(t, x, \dot{x}, u, \dot{u})dt \leq \int_a^b f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})dt, \quad \forall i \in P, \quad i \neq k,$$
$$g(t, x, \dot{x}, u, \dot{u}) \leq 0, t \in I,$$
$$h(t, x, \dot{x}, u, \dot{u}) = 0, t \in I.$$

3. Sufficient optimality conditions

In the sequel of the paper, $\int_a^b \bar{\lambda} f(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})dt$ denotes the vector $(\int_a^b \bar{\lambda}_1 f_1(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})dt, \int_a^b \bar{\lambda}_2 f_2(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})dt, \ldots, \int_a^b \bar{\lambda}_p f_p(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})dt)$. Similarly, $\int_a^b \bar{\mu}(t) g(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})dt$ and $\int_a^b \bar{\nu}(t) h(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})dt$ can be defined.

THEOREM 1 *(Sufficiency) Let* $(\bar{x}, \bar{u})$ *be a feasible solution to (CP). Suppose that there exist scalars* $\bar{\lambda}_i \geqq 0,\ \sum_{i\in P} \bar{\lambda}_i = 1,\ \bar{\mu}_j(t) \geqq 0,\ j \in M$, *such that for all* $t \in I$,

$$\begin{aligned}&\sum_{i\in P} \bar{\lambda}_i f_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j(t) g_{jx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k(t) h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\\ &= D[\sum_{i\in P} \bar{\lambda}_i f_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) +\\ &\qquad \sum_{j\in M} \bar{\mu}_j(t) g_{j\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k(t) h_{k\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})],\end{aligned} \tag{3}$$

$$\begin{aligned}&\sum_{i\in P} \bar{\lambda}_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j(t) g_{ju}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k(t) h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\\ &= D[\sum_{i\in P} \bar{\lambda}_i f_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j\in M} \bar{\mu}_j(t) g_{j\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\\ &\qquad + \sum_{k\in N} \bar{\nu}_k(t) h_{k\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})],\end{aligned} \tag{4}$$

$$\int_a^b \sum_{j\in M} \bar{\mu}_j g_j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) dt = 0, \tag{5}$$

$$\int_a^b \sum_{k\in N} \bar{\nu}_k h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) dt = 0. \tag{6}$$

Further, assume that

(i) $\int_a^b \bar{\lambda} f(t, \cdot, \cdot, \cdot, \cdot) dt$ *is* $\hat{\alpha}$-*V-pseudounivex at* $(\bar{x}, \bar{u})$ *with respect to* $b_\circ$, $\phi_\circ$, $\hat{\alpha}$, η *and* ξ;

(ii) $\int_a^b \bar{\mu}(t) g(t, \cdot, \cdot, \cdot, \cdot) dt$ *is* $\tilde{\alpha}$-*V-quasiunivex at* $(\bar{x}, \bar{u})$ *with respect to* b_1, ϕ_1, $\tilde{\alpha}$, η *and* ξ;

(iii) $\int_a^b \bar{\nu}(t) h(t, \cdot, \cdot, \cdot, \cdot) dt$ *is* α^*-*V-quasiunivex at* $(\bar{x}, \bar{u})$ *with respect to* b_2, ϕ_2, α^*, η *and* ξ;

(iv) $\phi_2(0) = 0$ *and for any scalar function* $p(t)$,
$\int_a^b p(t) dt < 0 \Rightarrow \int_a^b \phi_\circ(p(t)) dt < 0$,
$\int_a^b \phi_1(p(t)) dt > 0 \Rightarrow \int_a^b p(t) dt > 0$;

(v) $b_\circ > 0$, $b_1 > 0$.

Then $(\bar{x}, \bar{u})$ *is a weakly efficient solution of (CP).*

Proof. Suppose, contrary to the result, that $(\bar{x}, \bar{u}) \in X^\circ$ is not a weakly efficient solution to (CP). Then there exists $(x, u) \in X^\circ$ such that

$$\int_a^b f(t, x, \dot{x}, u, \dot{u}) dt < \int_a^b f(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) dt,$$

which by $\bar{\lambda}_i \geqq 0$, $\sum_{i\in P} \bar{\lambda}_i = 1$, $\hat{\alpha}_i > 0$, $i \in P$, gives

$$\int_a^b \sum_{i\in P} \hat{\alpha}_i \bar{\lambda}_i f_i(t, x, \dot{x}, u, \dot{u})dt < \int_a^b \sum_{i\in P} \hat{\alpha}_i \bar{\lambda}_i f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})dt. \tag{7}$$

From the assumptions (iv), (v) and inequality (7), it follows that

$$b_\circ \int_a^b \phi_\circ(\sum_{i\in P} \hat{\alpha}_i \bar{\lambda}_i f_i(t, x, \dot{x}, u, \dot{u}) - \sum_{i\in P} \hat{\alpha}_i \bar{\lambda}_i f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}))dt < 0.$$

Therefore, by hypothesis (i) and the above inequality, we get

$$\int_a^b [(\sum_{i\in P} \bar{\lambda}_i f_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - D \sum_{i\in P} \bar{\lambda}_i f_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}))\eta$$
$$+ (\sum_{i\in P} \bar{\lambda}_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - D \sum_{i\in P} \bar{\lambda}_i f_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}))\xi]dt < 0. \tag{8}$$

Now, from the feasibility of (x, u) to (CP), hypothesis (6), and using $\bar{\nu} \neq 0$, we have

$$\int_a^b \sum_{k\in N} \bar{\nu}_k(t) h_k(t, x, \dot{x}, u, \dot{u})dt = \int_a^b \sum_{k\in N} \bar{\nu}_k(t) h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})dt. \tag{9}$$

Again, by hypothesis (iii) and (9), we have

$$\int_a^b [(\sum_{k\in N} \bar{\nu}_k(t) h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - D \sum_{k\in N} \bar{\nu}_k(t) h_{k\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}))\eta$$
$$+ (\sum_{k\in N} \bar{\nu}_k(t) h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - D \sum_{k\in N} \bar{\nu}_k(t) h_{k\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}))\xi]dt \leqq 0. \tag{10}$$

On adding inequalities (8) and (10), we obtain

$$\int_a^b [(\sum_{i\in P} \bar{\lambda}_i f_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k(t) h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - D(\sum_{i\in P} \bar{\lambda}_i f_{i\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$
$$+ \sum_{k\in N} \bar{\nu}_k(t) h_{k\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})))\eta + (\sum_{i\in P} \bar{\lambda}_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k(t) h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})$$
$$- D(\sum_{i\in P} \bar{\lambda}_i f_{i\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k\in N} \bar{\nu}_k(t) h_{k\dot{u}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})))\xi]dt < 0.$$

The above inequality together with relations (3) and (4), yields

$$\int_a^b [(\sum_{j\in M} \bar{\mu}_j(t) g_{jx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - D \sum_{j\in M} \bar{\mu}_j(t) g_{j\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}))\eta$$

$$+(\sum_{j\in M}\bar{\mu}_j(t)g_{ju}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-D\sum_{j\in M}\bar{\mu}_j(t)g_{j\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))\xi]dt>0,$$

which along with the hypothesis (ii), gives

$$b_1\int_a^b\phi_1(\sum_{j\in M}\tilde{\alpha_j}\bar{\mu}_j(t)g_j(t,x,\dot{x},u,\dot{u})-\sum_{j\in M}\tilde{\alpha_j}\bar{\mu}_j(t)g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))dt>0. \quad (11)$$

Hence, it follows from inequality (11), assumptions (iv), (v) and $\tilde{\alpha_j}>0$, $j\in M$, that

$$\int_a^b(\sum_{j\in M}\bar{\mu}_j(t)g_j(t,x,\dot{x},u,\dot{u})-\sum_{j\in M}\bar{\mu}_j(t)g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))dt>0. \quad (12)$$

On the other hand, from the feasibility of (x,u) to (CP) and (5), we have

$$\int_a^b\sum_{j\in M}\bar{\mu}_j(t)g_j(t,x,\dot{x},u,\dot{u})dt\leqq\int_a^b\sum_{j\in M}\bar{\mu}_j(t)g_j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})dt,$$

which contradicts (12). Hence $(\bar{x},\bar{u})$ is a weakly efficient solution to (CP). This completes the proof. □

The proofs of the following two theorems are similar to that of Theorem 1 and hence are being omitted.

Theorem 2 *Let $(\bar{x},\bar{u})$ be a feasible solution of (CP). Suppose that there exist scalars $\bar{\lambda}_i\geqq 0$, $\sum_{i\in P}\bar{\lambda}_i=1$, $\bar{\mu}_j(t)\geqq 0$, $j\in M$ satisfying the conditions (3) to (6).*
Further, assume that

- (i) *$\int_a^b\bar{\lambda}f(t,\cdot,\cdot,\cdot,\cdot)dt$ is prestrictly $\hat{\alpha}$-V-quasiunivex at $(\bar{x},\bar{u})$ with respect to $b_\circ$, $\phi_\circ$, $\hat{\alpha}$, η and ξ;*
- (ii) *$\int_a^b\bar{\mu}(t)g(t,\cdot,\cdot,\cdot,\cdot)dt$ is $\tilde{\alpha}$-V-quasiunivex at $(\bar{x},\bar{u})$ with respect to b_1, ϕ_1, $\tilde{\alpha}$, η and ξ;*
- (iii) *$\int_a^b\bar{\nu}(t)h(t,\cdot,\cdot,\cdot,\cdot)dt$ is strictly α^*-V-quasiunivex at $(\bar{x},\bar{u})$ with respect to b_2, ϕ_2, α^*, η and ξ;*
- (iv) *$\phi_2(0)=0$ and for any scalar function $p(t)$,*
  $\int_a^b p(t)dt<0\Rightarrow\int_a^b\phi_\circ(p(t))dt<0$,
  $\int_a^b\phi_1(p(t))dt>0\Rightarrow\int_a^b p(t)dt>0$;
- (v) $b_\circ>0$, $b_1>0$.

Then $(\bar{x},\bar{u})$ is a weakly efficient solution of (CP).

THEOREM 3 *Let $(\bar{x}, \bar{u})$ be a feasible solution of (CP). Suppose that there exist scalars $\bar{\lambda}_i > 0, \sum_{i \in P} \bar{\lambda}_i = 1, \bar{\mu}_j(t) \geqq 0, j \in M$ satisfying the conditions (3) to (6). Further, assume that*

(i) $\int_a^b \bar{\lambda} f(t,\cdot,\cdot,\cdot,\cdot)dt$ *is $\hat{\alpha}$-V-pseudounivex at $(\bar{x}, \bar{u})$ with respect to $b_\circ$, $\phi_\circ$, $\hat{\alpha}$, η and ξ;*

(ii) $\int_a^b \bar{\mu}(t) g(t,\cdot,\cdot,\cdot,\cdot)dt$ *is $\tilde{\alpha}$-V-quasiunivex at $(\bar{x}, \bar{u})$ with respect to b_1, ϕ_1, $\tilde{\alpha}$, η and ξ;*

(iii) $\int_a^b \bar{\nu}(t) h(t,\cdot,\cdot,\cdot,\cdot)dt$ *is α^*-V-quasiunivex at $(\bar{x}, \bar{u})$ with respect to b_2, ϕ_2, α^*, η and ξ;*

(iv) *$\phi_2(0) = 0$ and for any scalar function $p(t)$,*
$\int_a^b p(t)dt < 0 \Rightarrow \int_a^b \phi_\circ(p(t))dt < 0,$
$\int_a^b \phi_1(p(t))dt > 0 \Rightarrow \int_a^b p(t)dt > 0;$

(v) $b_\circ > 0$, $b_1 > 0$.

Then $(\bar{x}, \bar{u})$ is an efficient solution of (CP).

4. Duality

In this section, we present the following Mond-Weir type dual program (Ahmad and Sharma, 2010) for (CP) and prove some duality results.

(MD) Maximize $\int_a^b f(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ)dt$

subject to

$$x^\circ(a) = \gamma, \; x^\circ(b) = \delta,$$

$$\sum_{i \in P} \lambda_i f_{ix^\circ}(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) + \sum_{j \in M} \mu_j(t) g_{jx^\circ}(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) + \sum_{k \in N} \nu_k(t) h_{kx^\circ}(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) = D[\sum_{i \in P} \lambda_i f_{i\dot{x}^\circ}(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) + \sum_{j \in M} \mu_j(t) g_{j\dot{x}^\circ}(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) + \sum_{k \in N} \nu_k(t) h_{k\dot{x}^\circ}(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ)], t \in I, \tag{13}$$

$$\sum_{i \in P} \lambda_i f_{iu^\circ}(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) + \sum_{j \in M} \mu_j(t) g_{ju^\circ}(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) + \sum_{k \in N} \nu_k(t) h_{ku^\circ}(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) = D[\sum_{i \in P} \lambda_i f_{i\dot{u}^\circ}(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) + \sum_{j \in M} \mu_j(t) g_{j\dot{u}^\circ}(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) + \sum_{k \in N} \nu_k(t) h_{k\dot{u}^\circ}(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ)], t \in I, \tag{14}$$

$$\int_a^b \sum_{j\in M} \mu_j(t) g_j(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) dt \geqq 0, \tag{15}$$

$$\int_a^b \sum_{k\in N} \nu_k(t) h_k(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) dt = 0, \tag{16}$$

$$\lambda_i \geqq 0, \ \sum_{i\in P} \lambda_i = 1, \ \mu_j(t) \geqq 0, \, j \in M, \, t \in I.$$

Theorem 4 *(Weak duality) Let (x, u) and $(x^\circ, u^\circ, \lambda, \mu(t), \nu(t))$ be the feasible solutions to (CP) and (MD), respectively. If*

(*i*) $\int_a^b \lambda f(t, \cdot, \cdot, \cdot, \cdot) dt$ *is $\hat{\alpha}$-V-pseudounivex at (x°, u°) with respect to $b_\circ$, $\phi_\circ$, $\hat{\alpha}$, η and ξ;*

(*ii*) $\int_a^b \mu(t) g(t, \cdot, \cdot, \cdot, \cdot) dt$ *is $\tilde{\alpha}$-V-quasiunivex at (x°, u°) with respect to b_1, ϕ_1, $\tilde{\alpha}$, η and ξ;*

(*iii*) $\int_a^b \nu(t) h(t, \cdot, \cdot, \cdot, \cdot) dt$ *is α^*-V-quasiunivex at (x°, u°) with respect to b_2, ϕ_2, α^*, η and ξ;*

(*iv*) $\phi_2(0) = 0$ *and for any scalar function $p(t)$,*
$\int_a^b \phi_\circ(p(t)) dt = \int_a^b p(t) dt$,
$\int_a^b \phi_1(p(t)) dt = \int_a^b p(t) dt$;

(*v*) $b_\circ > 0$, $b_1 > 0$.

Then

$$\int_a^b f(t, x, \dot{x}, u, \dot{u}) dt \not< \int_a^b f(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) dt.$$

Proof. We proceed by contradiction. Suppose that

$$\int_a^b f(t, x, \dot{x}, u, \dot{u}) dt < \int_a^b f(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) dt,$$

which, by $\lambda_i \geqq 0$, $\sum_{i\in P} \lambda_i = 1$, $\hat{\alpha}_i > 0$, $i \in P$, implies

$$\int_a^b \sum_{i\in P} \hat{\alpha}_i \lambda_i f_i(t, x, \dot{x}, u, \dot{u}) dt < \int_a^b \sum_{i\in P} \hat{\alpha}_i \lambda_i f_i(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) dt.$$

From the assumptions (*iv*), (*v*) and the above inequality, it follows that

$$b_\circ \int_a^b \phi_\circ\Big(\sum_{i\in P} \hat{\alpha}_i \lambda_i f_i(t, x, \dot{x}, u, \dot{u}) - \sum_{i\in P} \hat{\alpha}_i \lambda_i f_i(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ)\Big) dt < 0. \tag{17}$$

Therefore, by hypothesis (*i*) and inequality (17), we get

$$\int_a^b \Big[\Big(\sum_{i\in P} \lambda_i f_{ix^\circ}(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ) - D \sum_{i\in P} \lambda_i f_{i\dot{x}^\circ}(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ)\Big)\eta$$

$$+(\sum_{i\in P}\lambda_i f_{iu^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ) - D\sum_{i\in P}\lambda_i f_{i\dot{u}^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ))\xi]dt < 0. \quad (18)$$

Now, from the feasibility of (x,u) and $(x^\circ,u^\circ,\lambda,\mu(t),\nu(t))$ to (CP) and (MD), respectively, and $\nu \neq 0$, we have

$$\int_a^b \sum_{k\in N}\nu_k(t)h_k(t,x,\dot{x},u,\dot{u})dt = \int_a^b \sum_{k\in N}\nu_k(t)h_k(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ)dt. \quad (19)$$

Again, by hypothesis (iii), (iv) and (19), we have

$$\int_a^b [(\sum_{k\in N}\nu_k(t)h_{kx^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ) - D\sum_{k\in N}\nu_k(t)h_{k\dot{x}^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ))\eta$$
$$+(\sum_{k\in N}\nu_k(t)h_{ku^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ) - D\sum_{k\in N}\nu_k(t)h_{k\dot{u}^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ))\xi]dt \leqq 0. \quad (20)$$

On adding inequalities (18) and (20), we obtain

$$\int_a^b [(\sum_{i\in P}\lambda_i f_{ix^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ) + \sum_{k\in N}\nu_k(t)h_{kx^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ)$$
$$-D(\sum_{i\in P}\lambda_i f_{i\dot{x}^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ) + \sum_{k\in N}\nu_k(t)h_{k\dot{x}^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ)))\eta$$
$$+(\sum_{i\in P}\lambda_i f_{iu^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ) + \sum_{k\in N}\nu_k(t)h_{ku^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ)$$
$$-D(\sum_{i\in P}\lambda_i f_{i\dot{u}^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ) + \sum_{k\in N}\nu_k(t)h_{k\dot{u}^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ)))\xi]dt < 0.$$

The above inequality, together with relations (13) and (14), yields

$$\int_a^b [(\sum_{j\in M}\mu_j(t)g_{jx^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ) - D\sum_{j\in M}\mu_j(t)g_{j\dot{x}^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ))\eta$$
$$+(\sum_{j\in M}\mu_j(t)g_{ju^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ) - D\sum_{j\in M}\mu_j(t)g_{j\dot{u}^\circ}(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ))\xi]dt > 0,$$

which, along with the hypothesis (ii), gives

$$b_1\int_a^b \phi_1(\sum_{j\in M}\tilde{\alpha_j}\mu_j(t)g_j(t,x,\dot{x},u,\dot{u}) - \sum_{j\in M}\tilde{\alpha_j}\mu_j(t)g_j(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ))dt > 0. \quad (21)$$

Hence, it follows from inequality (21), assumptions (iv), (v) and $\tilde{\alpha_j} > 0$, $j \in M$, that

$$\int_a^b (\sum_{j\in M}\mu_j(t)g_j(t,x,\dot{x},u,\dot{u}) - \sum_{j\in M}\mu_j(t)g_j(t,x^\circ,\dot{x}^\circ,u^\circ,\dot{u}^\circ))dt > 0. \quad (22)$$

On the other hand, from the feasibility of (x, u) and $(x^\circ, u^\circ, \lambda, \mu(t), \nu(t))$ to (CP) and (MD), respectively, we have

$$\int_a^b \sum_{j\in M} \mu_j(t) g_j(t, x, \dot{x}, u, \dot{u})dt \leqq \int_a^b \sum_{j\in M} \mu_j(t) g_j(t, x^\circ, \dot{x}^\circ, u^\circ, \dot{u}^\circ)dt,$$

which contradicts (22). This completes the proof. □

Theorem 5 *(Strong duality) Let (x°, u°) be an efficient solution to (CP) at which a constraint qualification is satisfied. Then there exist piecewise smooth $\lambda \in R^p$, $\mu : I \mapsto R^m$ and $\nu : I \mapsto R^n$ such that $(x^\circ, u^\circ, \lambda, \mu(t), \nu(t))$ is feasible for (MD). Furthermore, if weak duality (Theorem 4) holds between (CP) and (MD), then $(x^\circ, u^\circ, \lambda, \mu(t), \nu(t))$ is an efficient solution of the problem (MD).*

Proof. Since (x°, u°) is an efficient solution for (CP), then from Lemma 2, (x°, u°) solves $(\mathrm{CP})_s$. As (x°, u°) satisfies the constraint qualification for $(\mathrm{CP})_s$, it follows from Lemma 1 that there exist piecewise smooth $\bar{\lambda} \in R^{p-1}, \bar{\mu} : I \mapsto R^m$ and $\bar{\nu} : I \mapsto R^n$ such that for all t ∈ I,

$$\begin{aligned}
&f_{kx}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) + \sum_{\substack{i=1\\ i\neq k}}^{p} \bar{\lambda}_i f_{ix}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) \\
&\qquad + \sum_{j\in M} \bar{\mu}_j g_{jx}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) + \sum_{k\in N} \bar{\nu}_k h_{kx}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) \\
&= D\Big[f_{k\dot{x}}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) + \sum_{\substack{i=1\\ i\neq k}}^{p} \bar{\lambda}_i f_{i\dot{x}}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) \\
&\qquad + \sum_{j\in M} \bar{\mu}_j g_{j\dot{x}}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) + \sum_{k\in N} \bar{\nu}_k h_{k\dot{x}}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ})\Big], \\
&f_{ku}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) + \sum_{\substack{i=1\\ i\neq k}}^{p} \bar{\lambda}_i f_{iu}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) \\
&\qquad + \sum_{j\in M} \bar{\mu}_j g_{ju}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) + \sum_{k\in N} \bar{\nu}_k h_{ku}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) \\
&= D\Big[f_{k\dot{u}}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) + \sum_{\substack{i=1\\ i\neq k}}^{p} \bar{\lambda}_i f_{i\dot{u}}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) \\
&\qquad + \sum_{j\in M} \bar{\mu}_j g_{j\dot{u}}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) + \sum_{k\in N} \bar{\nu}_k h_{k\dot{u}}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ})\Big], \\
&\int_a^b \sum_{j\in M} \bar{\mu}_j g_j(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ})dt = 0, \\
&\bar{\mu} \geq 0.
\end{aligned}$$

Let $\frac{1}{\alpha} = 1 + \sum_{\substack{i=1 \\ i \neq k}}^{p} \bar{\lambda}_i$. Then, we get

$$\sum_{i \in P} \lambda_i f_{ix}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) + \sum_{j \in M} \mu_j g_{jx}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ})$$
$$+ \sum_{k \in N} \nu_k h_{kx}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) = D\Big[\sum_{i \in P} \lambda_i f_{i\dot{x}}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ})$$
$$+ \sum_{j \in M} \mu_j g_{j\dot{x}}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) + \sum_{k \in N} \nu_k h_{k\dot{x}}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ})\Big],$$
$$\sum_{i \in P} \lambda_i f_{iu}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) + \sum_{j \in M} \mu_j g_{ju}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ})$$
$$+ \sum_{k \in N} \nu_k h_{ku}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) = D\Big[\sum_{i \in P} \lambda_i f_{i\dot{u}}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ})$$
$$+ \sum_{j \in M} \mu_j g_{j\dot{u}}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) + \sum_{k \in N} \nu_k h_{k\dot{u}}(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ})\Big],$$
$$\int_a^b \sum_{j \in M} \mu_j g_j(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) dt = 0,$$
$$\mu \geq 0,$$

where $\lambda_k = \alpha > 0, \lambda_i = \alpha \sum_{i \in P} \bar{\lambda}_i, i \neq k, \sum_{j \in M} \mu_j = \alpha \sum_{j \in M} \bar{\mu}_j, \sum_{k \in N} \nu_k = \alpha \sum_{k \in N} \bar{\nu}_k$. Also we have $\int_a^b \sum_{k \in N} \nu_k h_k(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) dt = 0$. Therefore $(x^\circ, u^\circ, \lambda, \mu(t), \nu(t))$ is a feasible solution for (MD).

Moreover, if we assume that $(x^\circ, u^\circ, \lambda, \mu(t), \nu(t))$ is not an efficient solution to (MD), then there exists a feasible solution $(x, u, \lambda^\circ, \mu^\circ, \nu^\circ)$ to (MD) such that

$$\int_a^b f(t, x, \dot{x}, u, \dot{u}) dt \geq \int_a^b f(t, x^\circ, \dot{x^\circ}, u^\circ, \dot{u^\circ}) dt,$$

which contradicts the weak duality Theorem 4. Hence, $(x^\circ, u^\circ, \lambda, \mu(t), \nu(t))$ is an efficient solution to (MD). □

5. Conclusion

In this paper, we have considered a multiobjective variational control problem and its Mond-Weir type dual problem. Using the concept of efficiency, weak and strong duality theorems have been proved under the assumptions of generalized α-V-univexity. There is a rich scope to extend these notions to the class of non-differentiable multiobjective variational problems. This will orient the future research of the authors.

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