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Exact and approximate controllability conditions for the micro-swimmers deflection governed by electric field on a plane: The Green's function approach

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We study the exact and approximate controllabilities of the Langevin equation describing the Brownian motion of particles with a white noise. The Langevin equation is shown to describe also the bacterial run-and-tumble motion. Applying the Green's function approach to the Green's function representation of the Langevin equation, we obtain necessary and sufficient conditions for exact controllability in the form of a finite-dimensional problem of moments. For the approximate controllability, we obtain only sufficient conditions. The sets of resolving controls are characterized in both cases. The theoretical derivations are supported by a numerical analysis.

Key words: run, tumble, micro-swimmers, Green's function of nonlinear equation, discontinuous control.

1. Introduction

The necessity of controlling micro-swimmers– artificial microscopic particles which exhibit a self-propelling feature in a fluid – is related with their wide applications ranging from targeted drug delivery to design of efficient microsensors and micro-actuators [1, 2]. In the mentioned applications, the controlled motion of the micro-swimmers is investigated: given the initial state (position and velocity) of the particles, the problem is to control the external influence or taxis (e.g., chemical, thermal, electromagnetic [3]) to transmit the particle to a desired state [4]. The state of the micro-swimmers in time is fully described by the particles speed, v, and deflection, x, (change of the direction), which are stochastic quantities:

$$\boldsymbol{v}(t) = v(t)\boldsymbol{e}(t),$$

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$$\boldsymbol{e}(t) = x(t)\boldsymbol{e}_0 + \sqrt{1 - x^2(t)}\,\boldsymbol{n}_0, \quad x = \cos\varphi.$$

Here $\boldsymbol{e}$ is the orientation vector, $\boldsymbol{e}_0 = \boldsymbol{e}t(0)$; $\boldsymbol{n}_0 = \boldsymbol{n}(0)$, $\boldsymbol{n}$ is the normal vector; φ is the rotation angle of the particle (see Figure 1).

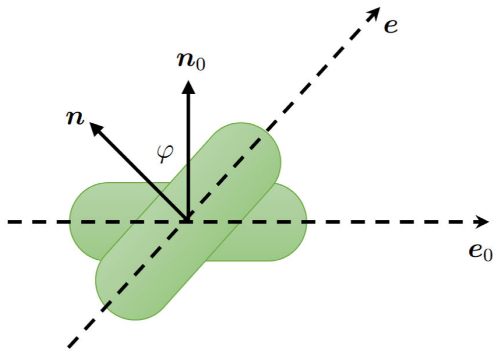

Figure 1: Schematic representation of particle orientation change

Nevertheless, a single stochastic model is recovered from experimental data in [5, 6]: the speed is uniquely defined by the deflection. Therefore, the biggest challenge is in controlling the *direction* of the particles. The mathematical model consists of a Langevin equation with a hyperbolic potential and a white noise with zero mean:

$$\frac{dx}{dt} = -\frac{dU}{dx} + \eta_x(t), \qquad t > 0. \tag{1}$$

Here U is a phenomenological potential given by

$$U(x) = U_0 - \rho\left[x - \frac{\gamma}{\delta}\cosh(\delta x)\right],$$

$U_0 = const$ and the parameters ρ, γ and δ are constrained by

$$\rho^2\delta\sqrt{1+\gamma^2} = c_0\,, \tag{2}$$

where c_0 is a given constant. For specific values of γ and δ, the constraint (2) defines the values of ρ corresponding to either run or tumble modes of the particles motion [6].

A feedback control problem in terms of the micro-swimmers deflection has been recently studied in [4]. Control is carried out by means of a time-harmonic AC electric field applied to the particles. At this, the field intensity and the frequency are the control parameters. The particles state obeys a coupled system of Langevin equations with a white noise with zero mean. The key point of the method consists in instantaneous switches between active Brownian motion and rotations by changing the electric field frequency. The deflection of the particles is measured at each instant, and depending on the accepted value, the frequency is set to a corresponding value. An optimal feedback strategy is developed for the particular form of the electric field considered.

Nevertheless, the controllability property of the micro-swimmers deflection for general taxis and arbitrary terminal states remains open. Inspired by this challenge, in this paper we formulate a two-point and a multi-point control problems for the Langevin equation with respect to the particle deflection with a known initial value. Considering a general control field, exact and approximate controllability conditions are derived using the Green's function-based explicit solution reported earlier in [5]. Note that controllability of standard Brownian motion has been considered in [7].

2. Control of the Langevin equation

The main aim of the present paper is the establishment of exact and approximate controllability for (1). To this end, let us assume that the control field, u, enters into (1) linearly, i.e.,

$$\frac{dx}{dt} = -\frac{dU}{dx} + \eta_x(t) + u(t). \tag{3}$$

Let the particle deflection at the initial instant $t = 0$ is given:

$$x(0,u) = x_0\,. \tag{4}$$

Here, u plays the role of the time-dependent intensity of an external electric field applied to the micro-swimmers to change their deflection. At this, we formally include the control function into the argument of the deflection to illustrate its implicit dependence on u. This dependence will be made explicit below.

Two basic control problems can be considered for (3), (4): two- and multi-point control problems.

2.1. Two-point control problem

The two-point problem consists in characterization of all the admissible controls $u \in \mathscr{U}$ providing the desired terminal deflection

$$x(T,u) = x_T \tag{5}$$

within a given time T for a given x_T.

2.2. Multi-point control problem

The multi-point control problem requires to find all the admissible controls $u \in \mathscr{U}$ providing the system of intermediate constraints

$$x(t_k, u) = x_{t_k}, \qquad k = 1, \ldots, K, \tag{6}$$

at the given instants t_k restricted by

$$0 \leqslant t_1 < t_2 \cdots < t_K \leqslant T,$$

for any given x_{t_k}. Obviously, (6) contains the initial and terminal values above if $t_1 = 0$ and $t_K = T$. Apparently, the two-point control problem is a particular case of the general multi-point control problem when $K = 2$, $t_1 = 0$, and $t_K = T$.

2.3. The set of admissible controls

In applied problems, the admissible controls need to be bounded. Moreover, they need to stop operating at the terminal instant $t = T$. Besides, switches at required instants must be allowed. Therefore, we constrain the consideration by the admissible controls

$$\mathscr{U} = \left\{u \in L^2[0,T],\ |u| \leqslant \epsilon,\ \mathrm{supp}(u) \subseteq [0,T]\right\}, \tag{7}$$

where ϵ is a given positive constant and $\mathrm{supp}(u) = \overline{\{t \in \mathbb{R}^+,\ u(t) \neq 0\}}$ denotes the support of u. As soon as we need to use impulsive controls, the space of distributions must be considered as control function space.

2.4. The set of resolving controls: exact and approximate controllability and lack of controllability

We would say that (3), (4) is exactly controllable at $t = T$ if there exists a $u \in \mathscr{U}$ such that (6) holds exactly for given K, t_k, x_{t_k}. The set of admissible controls providing (6), i.e.,

$$\mathscr{U}_{res}^{ex} = \{u \in \mathscr{U},\ \ (6)\},$$

is called the set of exactly resolving controls. Thus, the system is exactly controllable if and only if $\mathscr{U}_{res}^{ex} \neq \varnothing$.

Similarly, if for given K, t_k, x_{t_k}, (6) is satisfied with a required precision, (3), (4) is referred to as approximately controllable (with a required precision) at $t = T$. Thus, the set of approximately resolving controls is defined as follows:

$$\mathscr{U}_{res}^{ap} = \{u \in \mathscr{U},\ \ |x(t_k,u) - x_{t_k}| \leqslant \epsilon,\ k = 1,2,\ldots,K\}.$$

The system is approximately controllable if and only if $\mathscr{U}_{res}^{ap} \neq \varnothing$.

Apparently, exactly controllable systems are approximately controllable with arbitrarily small precision, whereas, in general, approximate controllability does not imply exact controllability.

Finally, note that if $\mathscr{U}_{res}^{ex} = \varnothing$ ($\mathscr{U}_{res}^{ap} = \varnothing$), the system lacks to be exactly (resp. approximately) controllable.

3. Controllability of the Langevin equation

In this section we study the cases of occurrence of the controllability property. We consider merely the case of the multi-point control, since, as it is shown above, the two-point control problem is its particular case.

3.1. Exact controllability

First, we begin with the analysis of exact controllability. Then, the following theorem holds true.

Theorem 1 *For the exact controllability of (3), (4), i.e., for the exact satisfaction of (6) at a given T, it is necessary and sufficient for u to satisfy the following system of equality type constraints*[1]*:*

$$\int_0^{t_k} G(t_k,\tau)u(\tau)\,\mathrm{d}\tau = M_k, \tag{8}$$

for all $k = 1,\ldots,K$.

Proof. In terms of the general controllability theory [8], the multi-point control problem is equivalent to the following system of constraints:

$$\mathscr{R}_k(u) = |x(t_k,u) - x_k| = 0, \quad k = 1,\ldots,K.$$

Thus, if for at least one admissible control $u \in \mathscr{U}$, the following equality holds:

$$\mathscr{R}_k(u) = 0, \tag{9}$$

for all $k = 1,\ldots,K$, then the Langevin equation is exactly controllable.

In order to reduce constraints on the control function providing (9) exactly, we involve the explicit form of the solution of (3), (4) obtained in [5]:

$$x(t,u) = x_s + (x_0 - x_s)G(t,0) + \int_0^t G(t,\tau)\,[\eta_x(\tau) + u(\tau)]\,\mathrm{d}\tau, \tag{10}$$

where x_s is the steady-state solution, and G is the Green's function.

[1]Notations are explained in the proof

Let (9) (or, equivalently, (6)) holds. Then, evaluating (10) for $t = t_k$ and substituted $x(t_k, u)$ by x_k, we straightforwardly obtain (8) with

$$M_k = x_k - x_s - (x_0 - x_s) G(t_k, 0) - \int_0^{t_k} G(t_k, \tau) \eta_x(\tau) \mathrm{d}\tau.$$

Now let us prove the sufficiency. Let (8) be satisfied. Then, we have

$$x_s + (x_0 - x_s)\, G(t_k, 0) + \int_0^{t_k} G(t_k, \tau)\, [\eta_x(\tau) + u(\tau)]\, \mathrm{d}\tau - x_k = 0.$$

This expression is nothing else but

$$x(t, u)\Big|_{t=t_k} - x_k = 0, \quad k = 1, \ldots, K,$$

which implies (9).

Thus, the problem of the exact controllability is reduced to characterization of the set of exactly resolving controls

$$\mathscr{U}_{res}^{ex} = \{u \in \mathscr{U}, \ \ (8)\}.$$

3.1.1. The Green's function

The Green's function of (3), (4) is rigorously determined in [5]. It turns out that it has different forms for the run and tumble modes. More specifically, in the run mode

$$x_s = 1, \quad G(t, \tau) = \exp\left[-\frac{|t - \tau|}{\chi}\right],$$

where

$$\chi = \frac{1}{\rho \delta \sqrt{1 + \gamma^2}}$$

is the characteristic time.

On the other hand, in the tumble mode

$$x_s = \frac{1}{\delta} \operatorname{arcsinh} \frac{1}{\gamma}, \quad G(t, \tau) = \frac{\ln\left(1 - \beta \exp\left[-M(t, \tau)\right]\right)}{\ln\left(1 - \beta \exp\left[-m(t, \tau)\right]\right)},$$

where

$$\beta = -\frac{4}{\gamma^2} \frac{\gamma \exp[-\delta] + 1 - \sqrt{1 + \gamma^2}}{\gamma \exp[-\delta] + 1 + \sqrt{1 + \gamma^2}},$$

$$M(t, \tau) = \frac{1}{\chi} \max\{t, \tau\}, \qquad m(t, \tau) = \frac{1}{\chi} \min\{t, \tau\}.$$

See [5] for details of the derivation.

3.1.2. Heuristic characterization of $\mathscr{U}_{res}^{ex}$

There exist several approaches towards the solution of (8). For example, it can be treated as a system of finite-dimensional problems of moments. The existence of the explicit L^2-optimal solution of (8) is among its advantages. The method of heuristic determination of control also can be applied efficiently (see [8,9] for details and for a proof of equivalency of these two methods).

Following to [9], let us represent the solution of (8) in the following form:

$$u(t) = \begin{cases} \sum_{m=1}^{M} u_m \sin(\omega_m t + \sigma_m), & t \in [0,T], \\ 0, & \text{else}, \end{cases}$$

where M, u_m, ω_m and σ_m are free parameters chosen to satisfy (8) exactly. Apparently, in this case $u \in \mathscr{U}$ as soon as $|u_m| \leqslant \epsilon$.

Note for comparison that [4] uses a single mode ($M = 1$) of the presented solution with $\sigma_m = 0$. However, obviously, the case of multi-point control can not be studied by only one mode.

In many applications including control of heating process, switching regimes are considered [8]:

$$u(t) = \sum_{m=1}^{M} u_m \theta(t - \mu_m), \quad t \in [0,T]. \tag{11}$$

Above θ is the Heaviside function, and the free parameters are M, u_m and the switching instants μ_m. In order to prevent overlapping of the modes, the constraint

$$0 \leqslant \mu_1 < \mu_2 < \cdots < \mu_M \leqslant T \tag{12}$$

is added to the system of constraints reduced from (8) for M, u_m, and μ_m. Apparently, in this case as well, $u \in \mathscr{U}$ as soon as $|u_m| \leqslant \epsilon$.

Depending on the intermediate constraints at $t = t_k$, it might be the case that $\mathscr{U}_{res}^{ex} = \varnothing$. This can be overcome by extending $\mathscr{U}$ to contain impulsive regimes as follows [8]:

$$u(t) = \sum_{m=1}^{M} u_m \delta(t - \mu_m), \tag{13}$$

where δ is the Dirac function. Evidently, by virtue of (12), $\operatorname{supp}(u) = [0,T]$ and in the definition of $\mathscr{U}$ (7), L^2 must be substituted by the space of distributions.

In the above cases, the solvability strongly depends on the relation between K and M.

It is noteworthy that with the above heuristic solutions, a cost functional of the form $\kappa[u]$ can be extremized with respect to the free parameters contained in each form of u.

3.2. Approximate controllability

In practice, the achieving of the approximate controllability of a dynamic system is more probable compared with the exact controllability in the sense that the set of resolving controls providing the approximate controllability is much wider than that providing the exact controllability. In our case, due to the presence of a noise, which results a slight distortion of the predicted deflection [4] making the exact achievement of a required position unrealistic, it is more curious and sensible to consider the *approximate* controllability of the particle. The approximate controllability of the Langevin equation is equivalent to the system of inequality type constraints

$$\mathscr{R}_k(u) \leqslant \epsilon_k \ \text{ on } \ \mathscr{U}, \tag{14}$$

for desired accuracies ϵ_k and all k. In this case, the set of approximately resolving controls reads as follows:

$$\mathscr{U}_{res}^{ap} = \{u \in \mathscr{U},\ (14)\}.$$

Knowing the exact solution (10), by virtue of the triangle inequality we straightforwardly derive that system

$$\left|\int_0^{t_k} G(t_k,\tau)u(\tau)\,d\tau\right| \leqslant \epsilon_k - |M_k|, \quad k = 1,\ldots,K, \tag{15}$$

is sufficient for the approximate controllability of the Langevin equation with multi-point constraints. Note that since M_k depends on the system parameters, the right hand side of the last inequality, in principle, can be made non-negative, i.e., $\epsilon_k - |M_k| \geqslant 0$, $k = 1,\ldots,K$. Then,

$$\widetilde{\mathscr{U}_{res}^{ap}} = \{u \in \mathscr{U},\ (15)\} \subseteq \mathscr{U}_{res}^{ap}.$$

Furthermore, since for any $u \in \mathscr{U}$, the following inequality holds:

$$\left|\int_0^{t_k} G(t_k,\tau)u(\tau)\,d\tau\right| \leqslant \epsilon \cdot \int_0^{t_k} |G(t_k,\tau)|\,d\tau$$

and, on the other hand,

$$|M_k| \leqslant \int_0^{t_k} |G(t_k,\tau)| \cdot |\eta_x(\tau)|\,d\tau + |x_s + (x_0 - x_s)G(t_k,0) - x_k| := N_k,$$

then the inequality

$$\epsilon \leqslant \max_{k=1,\dots,K} \frac{N_k}{G_k}, \tag{16}$$

where

$$G_k = \int_0^{t_k} |G(t_k,\tau)|\,\mathrm{d}\,\tau,$$

defines the set

$$\overline{\mathscr{U}}^{ap}_{res} = \{u \in \mathscr{U},\ (16)\} \subseteq \mathscr{U}^{ap}_{res}.$$

In other words, (16) is sufficient for approximate controllability.

It is noteworthy that, if for chosen values of the system parameters, (16) does not hold, then (15) needs to be verified. However, (15) is also a sufficient condition meaning that, in general, it does not define the set $\mathscr{U}^{ap}_{res}$ entirely.

4. Numerical analysis

In this section we carry out a numerical analysis of controllability for a single particle to verify the controllability conditions derived above. Following to [6], we consider the values $\delta_t = 9.062$, $\gamma_t = 6.63 \cdot 10^{-3}$, $\rho_t = 1$ for the tumble mode and $\delta_r = 4.71 \cdot 10^{-2}$, $\gamma_r = 4.98$, $\rho_r = 6.21$ for the run mode. For the sake of simplicity, we assume that $\eta_x \equiv 0$.

Consider the following three-point control problem: find an admissible control $u \in \mathscr{U}$ such that the following conditions hold:

$$x(0.5,u) = 0.5,\ \ x(2,u) = 0,\ \ T = 2,$$

provided that

$$x(0,u) = 0.$$

Then, (8) takes the following form:

$$\int_0^{0.5} G(0.5,\tau)u(\tau)\mathrm{d}\,\tau = 0.5 + x_s\left[G(0.5,0) - 1\right], \tag{17}$$

$$\int_0^{2} G(2,\tau)u(\tau)\mathrm{d}\,\tau = x_s\left[G(2,0) - 1\right]. \tag{18}$$

Choosing the control function in the following form (cf. (11)):

$$u(t) = u_1\theta(t-\mu_1) + u_2\theta(t-\mu_2),\ \ t \in [0,2],\ \ 0 < \mu_1 < \mu_2 < 2,$$

we ensure $u \in L^2[0,2]$ and $\mathrm{supp}(u) = [0,2]$. Here u_1, u_2, μ_1 and μ_2 are free parameters determined to satisfy (17), (18).

Assume that the particle is in a tumble mode. Substituting u into (17), (18), we obtain $u_1 = 1$, $\mu_1 = 0.55$, and $u_2 = -3.1$, $\mu_2 = 1.45$. It is evident from Figure 2 that the trajectory of the particle is continuous. However, since the applied electric field has discontinuities, the trajectory is not differentiable at isolated points (the velocity changes instantaneously).

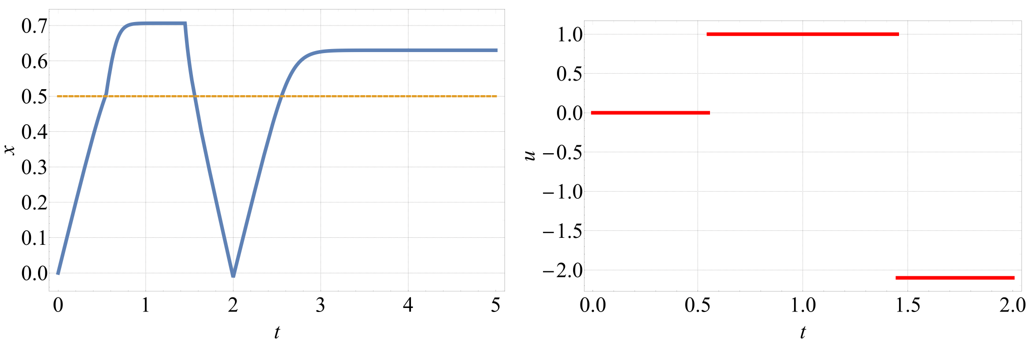

Figure 2: Plot of the particle trajectory (left) and the piecewise constant control (right)

Consider now the following four-point control problem:

$$x(0,u) = 0, \;\; x(1,u) = 2, \;\; x(2,u) = 4, \;\; x(4,u) = 2, \;\; T = 4.$$

The system of constraints in this case is as follows:

$$\int_0^1 G(1,\tau)u(\tau)\mathrm{d}\tau = 2 + x_s\left[G(1,0) - 1\right], \tag{19}$$

$$\int_0^2 G(2,\tau)u(\tau)\mathrm{d}\tau = 4 + x_s\left[G(2,0) - 1\right], \tag{20}$$

$$\int_0^4 G(4,\tau)u(\tau)\mathrm{d}\tau = 2 + x_s\left[G(4,0) - 1\right]. \tag{21}$$

Now we employ the impulsive regime (13) with $M = 3$. Let us consider the trajectory of a single particle in the run mode. Then, the resolving control will be:

$$u(t) = -1.5\delta(t-1) + \delta(t-2) - 2.55\delta(t-4).$$

It is evident from Figure 3 that, indeed, the particle passes through the required points at the given time instants.

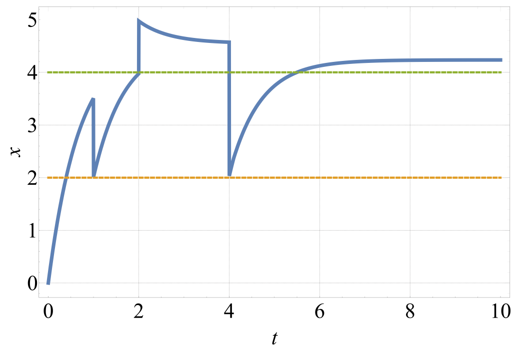

Figure 3: Plot of the particle trajectory controlled by impulsive regime (13)

Note that in this case $u \notin L^2[0,4]$. Nevertheless, the integrals in (19)–(21) exist due to the translational property of the Dirac delta:

$$\int_0^{t_k} G(\cdot,\tau)\delta(\tau-\mu_k)\,\mathrm{d}\tau = G(\cdot,\mu_k).$$

In the same way, using heuristic solutions of (8), we can study the controllability of particles in a more general statement. In particular, using the advantages of the impulsive regime (13), we can achieve a periodicity of particle motion. Indeed, applying (13), we see from Figure 4 that the control regime

$$u(t) = -0.63\sum_{m=1}^{M}\delta(t-m), \quad m \leqslant [T],$$

where $[T]$ is the integer part of T, ensure a periodic motion of the particle.

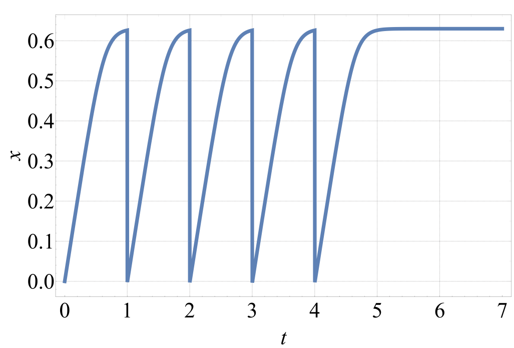

Figure 4: Periodic particle trajectory achieved with the aim of (13)

Note that for $t > T$, the trajectory becomes constant, i.e., the particle moves uniformly. This is a consequence of the fact that the external influence (electric field) vanishes when $t > T$.

5. Conclusions

We derive necessary and sufficient conditions for exact controllability and sufficient conditions for approximate controllability of the Langevin equation describing the Brownian motion of micro-swimmers subjected to an external electric field. We employ the Green's function representation of the Langevin equation solution in order to derive controllability conditions according to the recently developed Green's function approach. In the case of exact controllability, we characterize the set of resolving controls through a finite-dimensional problem of moments. Further, parametric families of controls resolving the problem of moments are derived. Numerical analysis supports our theoretical derivations.

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