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Robust $\mathcal{L}_2$ consensus of high-order swarm systems with time-varying delays[†]

by

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Abstract: Consensus problems for high-order continuous-time swarm systems in directed networks with time delays, uncertainties and external disturbances are investigated. Firstly, the state space of a swarm system is decomposed into a consensus subspace (CS) and a complement consensus space (CCS). A necessary and sufficient condition for the system with time delays and uncertainties to achieve consensus is presented based on the state projection on CCS, and an explicit expression of the consensus function is shown on the basis of the state projection on CS. Then, a sufficient condition for the system to achieve consensus with a desired $\mathcal{L}_2$ performance is given. Finally, numerical simulations are shown to demonstrate theoretical results.

Keywords: consensus, swarm system, uncertainty, time delay, disturbance rejection

1. Introduction

Recently, research on swarm systems has received significant attention due to numerous potential applications in different fields such as formation control, flocking, attitude alignment of clusters of satellites, and congestion control of distributed sensor networks (Xiao et al., 2009; Olfati–Saber, 2006; Lawton and Beard, 2002; Yu et al., 2009; Ren, 2010), etc.

For swarm systems to accomplish complicated tasks, a group of agents may need to interact with each other and asymptotically achieve an agreement over some variables of interest. This problem is usually called a consensus problem. Vicsek et al. (1995) proposed a simple but interesting discrete-time model, and

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by numerical simulations, it was shown that the system can achieve consensus on heading via a local updated rule. A theoretical explanation of the consensus behavior of the Vicsek's model was presented in Jadbabaie et al. (2003). Consensus problems for first-order continuous-time swarm systems were studied in Olfati-Saber and Murray (2004). Ren (2004) relaxed the conditions for consensus in Jadbabaie et al. (2003) and Olfati-Saber and Murray (2004), and pointed out that the communication topology having a spanning tree is critical for a swarm system to achieve consensus. In recent years, the study of consensus problems has developed fast, and many research topics were addressed. For example, consensus over random networks was discussed in Porfiri et al. (2008), formation controllability of swarm systems based on consensus techniques was addressed in Cai and Zhong (2010), consensus problems for swarm systems with time delays and/or uncertainties were dealt with in Sun et al. (2008), Xi et al. (2013), Lin et al. (2008) in terms of linear matrix inequalities (LMIs), consensus problems for second–order swarm systems were studied in Zhu et al. (2009) and Zhu (2011), and high–order consensus problems were considered in Ren et al. (2007), Wang et al. (2008), Xiao and Wang (2007), Cai and Zhong (2011), Xi et al. (2011, 2010), Liu and Jia (2009). It is worth mentioning that consensus-type techniques have been successfully used in flocking (Olfati-Saber, 2006) and formation control (Xiao et al., 2009; Ren, 2010; cai and Zhong, 2010), only to name just a few.

It is well-known that time delays, uncertainties and external disturbances may degrade the performance of control systems. In a swarm system, information delays and external disturbances appear naturally in the process of information transmission among agents. Uncertainties originate from variations of the strength of communication, which mean that the topology of a swarm system may be time-varying. Consensus problems for first-order swarm systems with time delays and/or uncertainties were studied in Sun et al. (2018), Xi et al. (2013)Lin et al. (2008) based on LMIs techniques, while many swarm systems in the real world are of high order. Swarm systems with constant time delays were dealt with in Zhu et al. (2009) and Zhu (2011), where it was assumed that the dynamics of each agent is described by a second-order integrator. A high-order swarm system with the dynamics of each agent described by a special controllability canonical form was studied in Ren et al. (2007). Wang et al. (2008) considered consensus problems for high-order swarm systems with less structural limitations, where it was assumed that communication topologies are undirected. A general high-order swarm system was studied in Xiao and Wang (2007), and a necessary and sufficient condition for consensus was given under the assumption that the consensus function, which is the agreement state of each agent, is time-invariant. We considered consensus problems of high-order swarm systems with time-varying consensus functions in Cai and Zhong (2011), Xi et al. (2011, 2010). Liu and Jia (2009) studied high-order swarm systems with external disturbances based on H_∞ theory and LMIs techniques.

For a given swarm system, two important consensus problems should be considered: (i) What are the conditions for consensus? (ii) How to determine the consensus function which may be time-varying? To the best of our knowledge, for a general high-order swarm system with time-varying delays, uncertainties and external disturbances, there is no general method provided in the literature to deal with the above two consensus problems.

In the current paper, consensus problems for high-order swarm systems with time delays, uncertainties and external disturbances are dealt with. These systems consist of N agents of order d. Two subspaces, the consensus subspace (CS) and the complement consensus subspace (CCS), are introduced, the direct sum of which is the $N \times d$-dimensional complex Euclidean space $\mathbb{C}^{Nd}$. First, a swarm system with time delays and uncertainties is considered. The state of the system is projected onto CS and CCS, and two subsystems associated with the state projection on CS and CCS, respectively, can be obtained by a linear transformation. It is shown that the asymptotic stability of the subsystem associated with the state projection on CCS is a necessary and sufficient condition for the system to achieve consensus, and the subsystem associated with the state projection on CS determines the consensus function. Furthermore, the structures of the consensus function are investigated according to different impacts of time delays and uncertainties. Based on the aforementioned necessary and sufficient condition, a sufficient condition in terms of LMIs is presented for the system to achieve consensus with a desired $\mathcal{L}_2$ performance.

Compared with the existing studies on consensus problems of high-order swarm systems, the current paper has the following three novel features. Firstly, in the current paper, the dynamics of each agent in a swarm system is a general high-order linear model, and the communication topology is an arbitrary directed graph. Moreover, the consensus function can be time-varying. In Zhu et al. (2009, 2011), Ren et al. (2007), Wang et al. (2008), some limitations on either the dynamics of each agent or the structure of communication topology are imposed. In Xiao and Wang (2007), swarm systems with fewer limitations were considered, but the respective method cannot be used to deal with swarm systems with time-varying consensus functions. Secondly, the current paper presents an explicit expression of the consensus function. Determining the consensus function is one of basic problems for high-order swarm systems, however, to the best of our knowledge, there was not a general method to determine consensus functions in the literature. Thirdly, in the current paper, the influence of time delays, uncertainties and external disturbances is dealt with. In our previous works (Cai and Zhong, 2011; Xi et al., 2011, 2010), these factors were not considered. Zhu et al. (2009) studied second-order swarm systems with constant delays by frequency domain methods, but their methods are no longer valid when time delays are time-varying. Liu and Jia (2009) dealt with high-order swarm systems with $\mathcal{L}_2$ external disturbances, while they assumed that the consensus function is the average of states of all agents.

This paper is organized as follows. In Section 2, some basic definitions and results in graph theory are presented, two subspaces are introduced, and the problem description is given. In Section 3, the main results about consensus problems are presented. Numerical simulations are shown in Section 4. Finally, concluding remarks are stated in Section 5.

In the current paper, for simplicity of notation, 0 is applied to denote zero matrices of any size with zero vectors and zero number as special cases and also to denote subspaces consisting of zero matrices. In symmetric block matrices, an asterisk ($*$) is used to represent a term which is induced by symmetry.

2. Preliminaries and problem description

In this section, first some basic concepts and results in graph theory are briefly summarized which are related to our later analysis. Then as the foundation of our method, two subspaces of $\mathbb{C}^{Nd}$ are introduced and their properties are analyzed. Finally, the problem description is presented.

2.1. Basic concepts and results in graph theory

A directed graph G consists of a node set $\mathcal{V}(G) = \{v_1, v_2, \cdots, v_N\}$, an edge set $\mathcal{E}(G) \subseteq \{(v_i, v_j) : v_i, v_j \in \mathcal{V}(G)\}$ and a weighted adjacency matrix $\tilde{W} = [w_{ij}] \in \mathbb{R}^{N\times N}$ with $w_{ij} \geqslant 0$. If (v_i, v_j) is an edge of G, v_i and v_j are defined as the *parent* and *child nodes* respectively. If $w_{ji} > 0$, then $(v_i, v_j) \in \mathcal{E}(G)$. Moreover, it is assumed that $w_{ii} = 0$ for all $i \in \{1, 2, \cdots, N\}$. The set of *neighbors* of v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V}(G) : (v_j, v_i) \in \mathcal{E}(G)\}$. The *in-degree* of v_i is defined as $\deg_{in}(v_i) = \sum_{j\in\mathcal{N}_i} w_{ij}$. Let $\tilde{D}$ be the *degree matrix* of G, which is defined as a diagonal matrix with the in-degree of each node along its diagonal. The *Laplacian matrix* of G is defined as $L = \tilde{D} - \tilde{W}$. A directed graph having a spanning tree means that there exists at least one node having a directed path to all the other nodes. More details on graph theory can be found in Godsil and Royal (2001) and Merris (1998). The following lemmas show some basic properties of the Laplacian matrix L.

LEMMA 1 *Ren (2004), Godsil and Royal (2001) Let L be the Laplacian matrix of a communication graph G and $\mathbf{1} = [1, 1, \cdots, 1]^T \in \mathbb{R}^N$, then*

(i) L at least has a zero eigenvalue, and $\mathbf{1}$ is the associated eigenvector, that is, $L\mathbf{1} = 0$;

(ii) If G has a spanning tree, then 0 is a simple eigenvalue of L, and all the other $N-1$ eigenvalues have positive real-parts.

LEMMA 2 *Xi et al. (2010) If G does not have a spanning tree, then L at least has two zero eigenvalues with the geometric multiplicity being not less than 2.*

2.2. State space decomposition

Let $U = \left[\mathbf{1}, \bar{U}\right] \in \mathbb{C}^{N\times N}$ be nonsingular, $c_j \in \mathbb{R}^d$ $(j = 1, 2, \cdots, d)$ be linearly independent, I_d be a $d \times d$ identity matrix and $e_i \in \mathbb{R}^N$ $(i = 1, 2, \cdots, N)$ with a 1 as its ith component and 0 elsewhere. The following two subspaces of $\mathbb{C}^{Nd}$ are introduced.

Definition 1 *Let* $p_j = (U \otimes I_d)(e_1 \otimes c_j) = \mathbf{1} \otimes c_j$ $(j = 1, 2, \cdots, d)$ *and* $p_j = (U \otimes I_d)(e_i \otimes c_k)$ $(j = (i-1)d + k; i = 2, \cdots, N; k = 1, 2, \cdots, d)$. *A consensus subspace (CS) is defined as the subspace* $\mathbb{C}(U)$ *spanned by* $p_1, p_2, \cdots, p_d$ *and a complement consensus subspace (CCS) as the subspace* $\bar{\mathbb{C}}(U)$ *spanned by* $p_{d+1}, p_{d+2}, \cdots, p_{Nd}$.

Since p_j $(j = 1, 2, \cdots, Nd)$ are linearly independent, the following lemma can be easily obtained.

Lemma 3 $\mathbb{C}(U) \oplus \bar{\mathbb{C}}(U) = \mathbb{C}^{Nd}$.

2.3. Problem description

Consider a swarm system with N agents which interact with each other via local information exchanges. A directed graph G can be used to describe the communication topology of the swarm system. For $i, j \in \{1, 2, \cdots, N\}$, the node v_i in G represents agent i, the edge $(v_i, v_j) \in \mathcal{E}(G)$ corresponds to the information channel from agent i to agent j, and w_{ji} denotes the transmitting strength of the channel (v_i, v_j).

Assume that all the agents share a common state space $\mathbb{R}^d$, and let $x_i(t) \in \mathbb{R}^d$ denote the state of agent i $(i \in \{1, 2, \cdots, N\})$ which needs to be coordinated, then the dynamics of agent i can be described by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + B_\varpi \varpi_i(t), \tag{1}$$

where $A \in \mathbb{R}^{d \times d}$, $B \in \mathbb{R}^{d \times m_1}$, $B_\varpi \in \mathbb{R}^{d \times m_2}$, $u_i(t)$ is the consensus protocol, and $\varpi_i(t) \in \mathcal{L}_{2e}$ (Vidjasagar, 1993) is the external disturbance.

Definition 2 *For a swarm system with N agents, the system is said to achieve consensus if for any given bounded initial condition, there exists a vector-valued function* $c(t) \in \mathbb{R}^d$ *dependent of the initial condition such that* $\lim_{t\to\infty}(x_i(t) - c(t)) = 0$ $(i = 1, 2, \cdots, N)$, *where* $c(t)$ *is called a consensus function.*

In that follows, consider a consensus protocol of the form:

$$u_i(t) = K \sum_{v_j \in \mathcal{N}_i} (w_{ij} + \Delta w_{ij}(t))(x_j(t - \tau_{ij}(t)) - x_i(t - \tau_{ij}(t))), \tag{2}$$

where $i \in \{1, 2, \cdots, N\}$, $K \in \mathbb{R}^{m_1 \times d}$, $\tau_{ij}(t)$ is a time-varying delay from agent j to agent i, and $\Delta w_{ij}(t)$ is the time-varying uncertainty of the transmitting strength w_{ij} of (v_j, v_i). Suppose that there exist r different time delays in G. Let $\tau_k(t) \in \{\tau_{ij}(t) : i, j \in \{1, 2, \cdots, N\}\}$ $(k = 1, 2, \cdots, r)$. It is assumed that the time-varying delays satisfy:

(A1): $0 \leqslant \tau_k(t) \leqslant \bar{\tau}_k < \infty$, $|\dot{\tau}_k(t)| \leqslant d_k < 1$ for $t \geqslant 0$, where $\bar{\tau}_k$ and d_k $(k = 1, 2, \cdots, r)$ are known positive constants.

The uncertainty $\Delta w_{ij}(t)$ satisfies the following assumption

(A2): $|\Delta w_{ij}(t)| = \begin{cases} \leqslant a_{ij} & i \neq j \text{ and } w_{ij} \neq 0, \\ 0 & \text{otherwise}, \end{cases}$

where a_{ij} is a known positive constant for $i, j \in \{1, 2, \cdots, N\}$ and $\Delta w_{ij}(t)$ $(i, j \in$

$\{1, 2, \cdot, N\}$) is a piecewise continuous function of time t.
From the definition of the Laplacian matrix, one sees that the uncertainty matrix ΔL of L satisfies $\Delta L\mathbf{1} = 0$.

Now define matrices $L_k = [l_{kij}] \in \mathbb{R}^{N\times N}$ as follows:

$$l_{kij} = \begin{cases} -w_{ij}, & j \neq i, \tau_{ij}(\cdot) = \tau_k(\cdot), \\ 0, & j \neq i, \tau_{ij}(\cdot) \neq \tau_k(\cdot), \\ -\sum_{m=1,m\neq i}^{N} l_{kim}, & j = i, \end{cases}$$

where $k = 1, 2, \cdots, r$. One sees that $L = \sum_{k=1}^{r} L_k$, $L_k\mathbf{1} = 0$, and $\Delta L_k\mathbf{1} = 0$ $(k = 1, 2, \cdots, r)$.
Let $x(t) = \left[x_1^T(t), x_2^T(t), \cdots, x_N^T(t)\right]^T$ and $\varpi(t) = \left[\varpi_1^T(t), \varpi_2^T(t), \cdots, \varpi_N^T(t)\right]^T$. Under the above consensus protocol, the state of a swarm system with N agents evolves according to the following system

$$\begin{cases} \dot{x}(t)=(I_N\otimes A)x(t)-\sum_{k=1}^{r}((L_k+\Delta L_k)\otimes BK)x(t-\tau_k(t))+(I_N\otimes B_\varpi)\varpi(t),\ t\in[0,\infty), \\ x(t) = \phi(t), \quad t \in [-\bar{\tau}, 0], \end{cases} \tag{3}$$

where $\bar{\tau} = \max\{\bar{\tau}_1, \bar{\tau}_2, \cdots, \bar{\tau}_r\}$ and $\phi(t)$ is a continuous vector-valued function on $[-\bar{\tau}, 0]$.

The following two consensus problems are investigated: (i) The consensus analysis problems are addressed; that is, for a given K, under what conditions system (3) achieves consensus; (ii) How to determine the consensus function of system (3).

3. Main results

In this section, first a necessary and sufficient condition is presented for system (3) with $\varpi(t) \equiv 0$ to achieve consensus, and a method to determine the consensus function is shown. Then a sufficient condition is given for system (3) to achieve consensus with a desired $\mathcal{L}_2$ performance.

3.1. Consensus and consensus functions

In this subsection, by a state projection on CS and CCS, conditions for system (3) to achieve consensus are presented, and an explicit expression of the consensus function is given. We dealt with swarm systems with time delays, uncertainties and external disturbances in Xi et al. (2013), where it was assumed that the dynamics of each agent is a first-order integrator. In Xi et al (2010), we proposed a method of initial state decomposition to study consensus problems for high-order linear time-invariant swarm systems, but this method cannot be applied to deal with swarm systems with time delays and uncertainties.

Let $U^{-1} = \left[\upsilon^H, \tilde{U}^H\right]^H$, where H represents the Hermitian adjoint. Since $(L_k + \Delta L_k)\mathbf{1} = 0$, one has

$$U^{-1}(L_k + \Delta L_k)U = \begin{bmatrix} \upsilon \\ \tilde{U} \end{bmatrix} (L_k+\Delta L_k)[\mathbf{1}, \bar{U}] = \begin{bmatrix} 0 & \upsilon(L_k + \Delta L_k)\bar{U} \\ 0 & \tilde{U}(L_k + \Delta L_k)\bar{U} \end{bmatrix}, \tag{4}$$

where $k = 1, 2, \cdots, r$. Let $\tilde{x}(t) = (U^{-1} \otimes I_d)x(t) = \left[\tilde{x}_1^H(t), \tilde{x}_2^H(t), \cdots, \tilde{x}_N^H(t)\right]^H$. By (4), system (3) can be transformed into

$$\dot{\tilde{x}}(t) = (I_N \otimes A)\tilde{x}(t) - \sum_{k=1}^{r} \left(\begin{bmatrix} 0 & \upsilon(L_k + \Delta L_k)\bar{U} \\ 0 & \tilde{U}(L_k + \Delta L_k)\bar{U} \end{bmatrix} \otimes BK \right) \tilde{x}(t - \tau_k(t)) + \begin{bmatrix} \upsilon \otimes B_{\varpi} \\ \tilde{U} \otimes B_{\varpi} \end{bmatrix} \varpi(t). \quad (5)$$

Let $y(t) = \left[\tilde{x}_2^H(t), \cdots, \tilde{x}_N^H(t)\right]^H$, then system (5) can be rewritten as follows

$$\dot{\tilde{x}}_1(t) = A\tilde{x}_1(t) - \sum_{k=1}^{r} (\upsilon(L_k + \Delta L_k)\bar{U} \otimes BK)y(t - \tau_k(t)) + (\upsilon \otimes B_{\varpi})\varpi(t), \quad (6)$$

$$\dot{y}(t) = (I_{N-1} \otimes A)y(t) - \sum_{k=1}^{r} (\tilde{U}(L_k + \Delta L_k)\bar{U} \otimes BK)y(t - \tau_k(t)) + (\tilde{U} \otimes B_{\varpi})\varpi(t). \quad (7)$$

The following theorem presents a necessary and sufficient condition for system (3) with $\varpi(t) \equiv 0$ to achieve consensus.

THEOREM 1 *System (3) with $\varpi(t) \equiv 0$ achieves consensus if and only if subsystem (7) is asymptotically stable.*

PROOF By Lemma 3, the state $x(t)$ of system (3) with $\varpi(t) \equiv 0$ can be uniquely projected onto $\mathbb{C}(U)$ and $\bar{\mathbb{C}}(U)$, that is,

$$x(t) = x_C(t) + x_{\bar{C}}(t), \quad (8)$$

where $x_C(t) = \sum_{j=1}^{d} \alpha_j(t)p_j$ and $x_{\bar{C}}(t) = \sum_{j=d+1}^{Nd} \alpha_j(t)p_j$. Since $p_j = \mathbf{1} \otimes c_j$ $(j = 1, 2, \cdots, d)$, it follows that

$$x_C(t) = \mathbf{1} \otimes \sum_{j=1}^{d} \alpha_j(t)c_j. \quad (9)$$

By Definition 1, one has

$$(U^{-1} \otimes I_d)x_C(t) = \left[\sum_{j=1}^{d} \alpha_j^H(t)c_j^T, 0, \cdots, 0\right]^H, \quad (10)$$

$$(U^{-1} \otimes I_d)x_{\bar{C}}(t) = \left[0, \sum_{j=1}^{d} \alpha_{d+j}^H(t)c_j^T, \cdots, \sum_{j=1}^{d} \alpha_{(N-1)d+j}^H(t)c_j^T\right]^H. \quad (11)$$

Since $\tilde{x}(t) = (U^{-1} \otimes I_d)x(t)$, one can obtain

$$\tilde{x}_1(t) = \sum_{j=1}^{d} \alpha_j(t)c_j, \tag{12}$$

$$y(t) = \left[\sum_{j=1}^{d} \alpha_{d+j}^H(t)c_j^T, \cdots, \sum_{j=1}^{d} \alpha_{(N-1)d+j}^H(t)c_j^T \right]^H. \tag{13}$$

Necessity: We prove the conclusion by contradiction. If subsystem (7) is not asymptotically stable, then the limit of $y(t)$ as $t \to \infty$ does not exist or is nonzero if $y(s)$ is not identical to 0 for $s \in [-\bar{\tau}, 0]$. By (11) and (13), the limit of $x_{\bar{C}}(t)$ as $t \to \infty$ does not exist or is nonzero. Since system (3) with $\varpi(t) \equiv 0$ attains consensus, by (8) and (9) there exists a vector-valued function $\bar{c}(t) \in \mathbb{R}^d$ such that $x_{\bar{C}}(t) \to \mathbf{1} \otimes \bar{c}(t)$ as $t \to \infty$. Because $c_j \in \mathbb{R}^d$ $(j = 1, 2, \cdots, d)$ are linearly independent, there exist $\beta_j(t) \in \mathbb{R}$ $(j = 1, 2, \cdots, d)$ such that $\bar{c}(t) = \sum_{j=1}^{d} \beta_j(t)c_j$. Based on the structure of p_j $(j = 1, 2, \cdots, d)$, one has $x_{\bar{C}}(t) \to \sum_{j=1}^{d} \beta_j(t)p_j \in \mathbb{C}(U)$ as $t \to \infty$. Since $x_{\bar{C}}(t) \in \bar{\mathbb{C}}(U)$ and $\mathbb{C}(U) \cap \bar{\mathbb{C}}(U) = 0$, one has $\lim_{t\to\infty} x_{\bar{C}}(t) = 0$. A contradiction is obtained. Therefore it is necessary for the subsystem (7) to be asymptotically stable.

Sufficiency: If subsystem (7) is asymptotically stable for any bounded initial condition, then $\lim_{t\to\infty} x_{\bar{C}}(t) = 0$ by (11) and (13). From (8) and (9), one knows that system (3) with $\varpi(t) \equiv 0$ attains consensus and the consensus function $c(t)$ satisfies $\lim_{t\to\infty}(\tilde{x}_1(t) - c(t)) = 0$. The proof of Theorem 1 is completed.

REMARK 1 *Two subsystems, with $x_{\bar{C}}(t)$ and $x_C(t)$ being the states, describe the disagreement dynamics and consensus dynamics of system (3) with $\varpi(t) \equiv 0$, respectively. Theorem 1 implies that the asymptotic stability of the subsystem with $x_{\bar{C}}(t)$ being the state is a necessary and sufficient condition for system (3) with $\varpi(t) \equiv 0$ to achieve consensus, and the consensus function is determined by the subsystem with $x_C(t)$ being the state.*

Let $P_{\mathbb{C}(U),\bar{\mathbb{C}}(U)} = [p_1, \cdots, p_d, 0, \cdots, 0]\, P^{-1}$ be an oblique projector onto $\mathbb{C}(U)$ along $\bar{\mathbb{C}}(U)$ where $P = [p_1, p_2, \cdots, p_{Nd}]$. The following theorem presents the structures of the consensus function of system (3) with $\varpi(t) \equiv 0$.

THEOREM 2 *If system (3) with $\varpi(t) \equiv 0$ attains consensus, then the consensus function satisfies $\lim_{t\to\infty}(c(t) - (c_0(t) + c_\tau(t) + c_\Delta(t))) = 0$, where*

$$c_0(t) = e^{At}\, [I_d, 0, \cdots, 0]\, P_{\mathbb{C}(U),\bar{\mathbb{C}}(U)} x(0) - \int_0^t e^{A(t-s)}(\upsilon L\bar{U} \otimes BK)y(s)ds,$$

$$c_\tau(t) = -\sum_{k=1}^{r} \int_0^t e^{A(t-s)}(\upsilon L_k\bar{U} \otimes BK)\, [y(s - \tau_k(s)) - y(s)]\, ds,$$

$$c_\Delta(t) = -\sum_{k=1}^{r}\int_0^t e^{A(t-s)}(\upsilon\Delta L_k\bar{U}\otimes BK)y(s-\tau_k(s))ds.$$

PROOF For any initial state $x(0)$, one has $x_C(0) = P_{\mathbb{C}(U),\bar{\mathbb{C}}(U)}x(0)$ by Lemma 3. By (9) and (12), one can obtain that $\tilde{x}_1(0) = [I_d, 0, \cdots, 0]\, x_C(0)$. By (6), one has

$$\tilde{x}_1(t) = e^{At}\tilde{x}_1(0) - \sum_{k=1}^{r}\int_0^t e^{A(t-s)}\left(\upsilon(L_k+\Delta L_k)\bar{U}\otimes BK\right)y(s-\tau_k(s))ds.$$

Since $L = \sum_{k=1}^{r} L_k$ and $\lim_{t\to\infty}(c(t)-\tilde{x}_1(t)) = 0$, one has $\lim_{t\to\infty}(c(t)-(c_0(t)+c_\tau(t)+c_\Delta(t))) = 0$. The proof of Theorem 2 is completed.

In Theorem 2, $c_0(t)$ is said to be a nominal consensus function, which describes the consensus function of a swarm system without time delays, uncertainties and external disturbances. $c_\tau(t)$ and $c_\Delta(t)$ describe the impacts of time-delays and uncertainties respectively.

Let $U^{-1}LU = J_L$ where J_L is the Jordan canonical form of L, and λ_i $(i = 1, 2, \cdots, N)$ denote the eigenvalues of L with $\lambda_1 = 0$. If the communication graph G, associated with L, has a spanning tree, by Lemma 1 one has $\upsilon L\bar{U} = 0$ and $\tilde{U}L\bar{U} = \tilde{J}_L$, where $\tilde{J}_L$ consists of Jordan blocks associated with $\lambda_2, \cdots, \lambda_N$. If G does not have a spanning tree, by Lemma 2 one can set that $\lambda_2 = 0$ in $\tilde{J}_L$ and $\upsilon L\bar{U} = 0$. For this choice of U, the following two corollaries can be obtained.

COROLLARY 1 *If G has a spanning tree, then system (3) without time delays, uncertainties and external disturbances attains consensus if and only if $A-\lambda_i BK$ $(i = 2, \cdots, N)$ are Hurwitz. The nominal consensus function is* $\lim_{t\to\infty}(c_0(t) - e^{At}[I_d, 0, \cdots, 0]\, P_{\mathbb{C}(U),\bar{\mathbb{C}}(U)}x(0)) = 0$.

COROLLARY 2 *If G does not have a spanning tree, then system (3) without time delays, uncertainties and external disturbances attains consensus if and only if A and $A-\lambda_i BK$ $(\lambda_i \neq 0, i\in\{2, \cdots, N\})$ are Hurwitz. The nominal consensus function is* 0.

REMARK 2 *By the above analysis, the consensus property of high-order swarm systems is jointly determined by the consensus protocol, the dynamics of each agent, and the communication topology. The dynamics of each agent in swarm systems, discussed in Jadbabaie et al. (2003), Olfati-Saber and Murray (2004), Ren (2004), Porfiri et al. (2008), Cai and Zhong (2010), Sun et al. (2008), Xi et al. (2013), Lin et al. (2008) is a first-order integrator, which means that if an agent does not receive the information from other agents, then its state is time-invariant. In this case, the consensus property is completely determined by the communication topology. However, the state of an agent in high-order swarm systems may be time-varying even if the agent does not interact with other agents. This is the key difference between first-order and high-order swarm systems, and makes consensus problems of high-order swarm systems more challenging.*

Remark 3 *Olfati-Saber and Murray proposed the χ-consensus problem to determine consensus functions of swarm systems in Olfati-Saber and Murray (2004), where it was assumed that communication topologies are balanced and strongly connected. In this case, the consensus function is the average value of the initial states of all agents. We presented an explicit expression of the consensus functions of swarm systems with time-varying delays and uncertainties in Xi et al. (2013), where the communication topology is described by any directed graph. In Olfati-Saber and Murray (2004) and Xi et al. (2013), the dynamics of each agent is described by a first-order integrator. In Xi et al. (2010), based on the decomposition of the initial state, we presented an approach to determine consensus functions of high-order swarm systems. But when time delays and uncertainties are considered, the method in Xi et al. (2010) is no longer valid. Theorem 2 presents an explicit expression of the consensus function and describes the impacts of time delays and uncertainties.*

3.2. Consensus with a desired $\mathcal{L}_2$ performance

In this subsection, a sufficient condition for system (3) to achieve consensus with a desired $\mathcal{L}_2$ performance will be presented in terms of LMIs. From the proof of Theorem 1, one can see that it is not related to the choice of $\bar{U}$ for system (3) to achieve consensus. If U is a complex matrix, the calculation complexity will increase when solving LMIs. Hence it is assumed that $\bar{U} = [e_2, e_3, \cdots, e_N]$.

The following lemmas are useful to get the conditions of consensus with a desired $\mathcal{L}_2$ performance.

Lemma 4 *Let D_k be a 0-1 matrix with rows and columns indexed by the nodes and edges of G, and E_k be a 0-1 matrix with rows and columns indexed by the edges and nodes of G, defined as*

$$D_{kve} = \begin{cases} 1, & \text{if the node } v \text{ is the child node of the edge } e \text{ of } G_k, \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{kev} = \begin{cases} 1, & \text{if the node } v \text{ is the parent node of the edge } e \text{ of } G_k, \\ 0, & \text{otherwise.} \end{cases}$$

Let $D = \sum_{k=1}^{r} D_k$ and $\Lambda = diag\{\mu_1, \mu_2, \cdots, \mu_\kappa\}$, where μ_m $(m = 1, 2, \cdots, \kappa)$ are the weight of the mth edge of G and κ is the number of the edges of G. Then L_k can be denoted by $L_k = D\Lambda(D_k^T - E_k)$ $(k = 1, 2, \cdots, r)$.

Proof For any $l \in \{1, 2, \cdots, r\}$, the ijth element of $D\Lambda D_l^T$ can be written as $\sum_{m=1}^{\kappa} D_{im}\mu_m D_{ljm}$. Since $L = \sum_{k=1}^{r} L_k$, G has the same nodes as G_k, and the edges of G consist of the ones of G_k $(k = 1, 2, \cdots, r)$ without any superposition. Because any edge only has one child node, it follows that $D_{im}D_{ljm} = 0$ for any $i \neq j$. Thus, one has that $D\Lambda D_l^T$ is a diagonal matrix with the iith element equal to the in-degree of the node v_i of G_l. Hence, the degree matrix of G_l satisfies $\tilde{D}_l = D\Lambda D_l^T$. Similarly, the ijth element of $D\Lambda E_l$ can be written as $\sum_{m=1}^{\kappa} D_{im}\mu_m E_{lmj}$, which is equal to the weight of the edge (v_j, v_i) of G_l, therefore the adjacency matrix of G_l can be denoted by $\tilde{W}_l = D\Lambda E_l$. Since $L_l = \tilde{D}_l - \tilde{W}_l$ by the definition of the Laplacian matrix in Section 2, one has

$L_l = D\Lambda(D_l^T - E_l)$. By considering all subgraphs G_k ($k = 1, 2, \cdots, r$), the conclusion of Lemma 4 can be obtained.

By Lemma 1, the uncertainty matrix ΔL_k of L_k can be written as $\Delta L_k = DF(t)\bar{E}_k$ ($k = 1, 2, \cdots, r$), where $\bar{E}_k \in \mathbb{R}^{\kappa \times N}$ and $F(t)$ is a diagonal matrix whose diagonal elements are uncertainties of the edges. By assumption **(A2)**, one has $|\Delta w_{ij}(t)|/a_{ij} \leqslant 1$ ($i, j \in \{1, 2, \cdots, N\}$). Without loss of generality, it is assumed that $F^T(t)F(t) \leqslant I_\kappa$, $\forall t$.

LEMMA 5 *Wu et al. (2004) Given matrices $Q = Q^T$, H and Z, for $F(t)$ satisfying $F^T(t)F(t) \leqslant I$, $Q + HF(t)Z + Z^T F^T(t) H^T < 0$ if and only if there exists a $\rho > 0$ such that $Q + \rho HH^T + \rho^{-1} Z^T Z < 0$.*

Let $\int_0^T y^T(t)y(t)dt \leqslant \gamma^2 \int_0^T \varpi^T(t)\varpi(t)dt$ with $\gamma > 0$ denote $\|y\|_{T2} \leqslant \gamma \|\varpi\|_{T2}$. The following theorem presents a sufficient condition for system (3) to achieve consensus with a desired $\mathcal{L}_2$ performance.

THEOREM 3 *Suppose that assumptions **(A1)** and **(A2)** hold. Then system (3) attains consensus with $\|y\|_{T2} \leqslant \gamma \|\varpi\|_{T2}$ ($\forall T \geqslant 0$) for any $\tau_k(t) \in [0, \bar{\tau}_k]$ ($k = 1, 2, \cdots, r$) if there exist real symmetric matrices $R > 0$, $Q_k > 0$, $S_k > 0$, $M_k = \begin{bmatrix} M_{k,11} & M_{k,12} & M_{k,13} \\ * & M_{k,22} & M_{k,23} \\ * & * & M_{k,33} \end{bmatrix} \geqslant 0$, real matrices X_k and Y_k, and a constant $\rho > 0$, such that the following LMIs are feasible:*

$$\Xi = \begin{bmatrix} \Xi_{11} + \rho I & \Xi_{12} & \Xi_{\varpi 13} & \Xi_{14} & \Xi_{\Delta 15} \\ * & \Xi_{22} + \Xi_{\Delta 22} & \Xi_{\varpi 23} & \Xi_{24} & 0 \\ * & * & \Xi_{\varpi 33} & \Xi_{\varpi 34} & 0 \\ * & * & * & \Xi_{44} & \Xi_{\Delta 45} \\ * & * & * & * & -I \end{bmatrix} < 0, \tag{14}$$

$$\Theta_k = \begin{bmatrix} M_{k,11} & M_{k,12} & M_{k,13} & X_k \\ * & M_{k,22} & M_{k,23} & Y_k \\ * & * & M_{k,33} & 0 \\ * & * & * & S_k \end{bmatrix} \geqslant 0, \tag{15}$$

where $k = 1, 2, \cdots, r$, and

$\Xi_{11} = R(I_{N-1} \otimes A) + (I_{N-1} \otimes A)^T R + \sum\limits_{k=1}^r Q_k + \sum\limits_{k=1}^r (X_k + X_k^T) + \sum\limits_{k=1}^r \bar{\tau}_k M_{k,11}$,

$\Xi_{12} = [-R(\tilde{U}L_1\bar{U} \otimes BK) - X_1 + Y_1^T + \bar{\tau}_1 M_{1,12}, \cdots, -R(\tilde{U}L_r\bar{U} \otimes BK) - X_r + Y_r^T + \bar{\tau}_r M_{r,12}]$,

$\Xi_{\varpi 13} = R(\tilde{U} \otimes B_\varpi) + \sum\limits_{k=1}^r \bar{\tau}_k M_{k,13}$,

$\Xi_{14} = \left[\bar{\tau}_1 (I_{N-1} \otimes A)^T S_1, \cdots, \bar{\tau}_r (I_{N-1} \otimes A)^T S_r\right]$,

$\Xi_{\Delta 15} = R(\tilde{U}D \otimes I_d)$,

$\Xi_{22} = diag\{(d_1 - 1)Q_1 - Y_1 - Y_1^T + \bar{\tau}_1 M_{1,22}, \cdots, (d_r - 1)Q_r - Y_r - Y_r^T + \bar{\tau}_r M_{r,22}\}$,

$$\Xi_{\Delta 22}=\begin{bmatrix}(\bar{E}_1\bar{U})^T\bar{E}_1\bar{U} & \cdots & (\bar{E}_1\bar{U})^T\bar{E}_r\bar{U}\\ \vdots & \ddots & \vdots\\ (\bar{E}_r\bar{U})^T\bar{E}_1\bar{U} & \cdots & (\bar{E}_r\bar{U})^T\bar{E}_r\bar{U}\end{bmatrix}\otimes(BK)^TBK,$$
$$\Xi_{\varpi 23}=\left[\bar{\tau}_1M_{1,23}^T,\cdots,\bar{\tau}_rM_{r,23}^T\right]^T,$$
$$\Xi_{24}=\begin{bmatrix}-\bar{\tau}_1(\tilde{U}L_1\bar{U}\otimes BK)^TS_1 & \cdots & -\bar{\tau}_r(\tilde{U}L_1\bar{U}\otimes BK)^TS_r\\ \vdots & \ddots & \vdots\\ -\bar{\tau}_1(\tilde{U}L_r\bar{U}\otimes BK)^TS_1 & \cdots & -\bar{\tau}_r(\tilde{U}L_r\bar{U}\otimes BK)^TS_r\end{bmatrix},$$
$$\Xi_{\varpi 33}=\sum_{k=1}^{r}\bar{\tau}_kM_{k,33}-\gamma^2\rho I,$$
$$\Xi_{\varpi 34}=\left[\bar{\tau}_1(\tilde{U}\otimes B_{\varpi})^TS_1,\cdots,\bar{\tau}_r(\tilde{U}\otimes B_{\varpi})^TS_r\right],$$
$$\Xi_{44}=diag\{-\bar{\tau}_1S_1,\cdots,-\bar{\tau}_rS_r\},$$
$$\Xi_{\Delta 45}=\left[\bar{\tau}_1(\tilde{U}D\otimes I_d)^TS_1,\cdots,\bar{\tau}_r(\tilde{U}D\otimes I_d)^TS_r\right]^T.$$

PROOF First, discuss the stability of subsystem (7) without external disturbances, i.e. $\varpi(t)\equiv 0$, and consider the following Lyapunov-Krasovskii functional candidate:

$$V(y(t))=V_1+V_2+V_3, \tag{16}$$

where $V_1=y^T(t)Ry(t)$, $V_2=\sum_{k=1}^{r}\int_{t-\tau_k(t)}^{t}y^T(\theta)Q_ky(\theta)d\theta$,
$V_3=\sum_{k=1}^{r}\int_{-\bar{\tau}_k}^{0}\int_{t+\theta}^{t}\dot{y}^T(s)S_k\dot{y}(s)dsd\theta$. By taking the derivative of these functionals with respect to the time t along the solution to subsystem (7) with $\varpi(t)\equiv 0$, one obtains

$$\begin{aligned}\dot{V}_1=&y^T(t)(R(I_{N-1}\otimes A)+(I_{N-1}\otimes A)^TR)y(t)\\ -&\sum_{k=1}^{r}2y^T(t)R(\tilde{U}(L_k+\Delta L_k)\bar{U}\otimes BK)y(t-\tau_k(t)),\end{aligned} \tag{17}$$

$$\dot{V}_2\leqslant\sum_{k=1}^{r}y^T(t)Q_ky(t)-\sum_{k=1}^{r}(1-d_k)y^T(t-\tau_k(t))Q_ky(t-\tau_k(t)), \tag{18}$$

$$\dot{V}_3\leqslant\sum_{k=1}^{r}\bar{\tau}_k\dot{y}^T(t)S_k\dot{y}(t)-\sum_{k=1}^{r}\int_{t-\tau_k(t)}^{t}\dot{y}^T(\theta)S_k\dot{y}(\theta)d\theta. \tag{19}$$

Due to $\int_{t-\tau_k(t)}^{t}\dot{y}(\theta)d\theta=y(t)-y(t-\tau_k(t))$, for any appropriately dimensioned real matrices X_k and Y_k $(k=1,2,\cdots,r)$, one has

$$\begin{aligned}\Omega_1\ =&\ \sum_{k=1}^{r}2\left[y^T(t),y^T(t-\tau_k(t))\right]\begin{bmatrix}X_k\\ Y_k\end{bmatrix}\left(y(t)-y(t-\tau_k(t))-\int_{t-\tau_k(t)}^{t}\dot{y}(\theta)d\theta\right)\\ =&\ 0.\end{aligned} \tag{20}$$

In addition, for real symmetric matrices $\tilde{M}_k = \begin{bmatrix} M_{k,11} & M_{k,12} \\ * & M_{k,22} \end{bmatrix} \geqslant 0 \ (k=1,2,\cdots,r)$, the following holds,

$$\Omega_2 = \sum_{k=1}^{r} \bar{\tau}_k \left[y^T(t), y^T(t-\tau_k(t))\right] \tilde{M}_k \begin{bmatrix} y(t) \\ y(t-\tau_k(t)) \end{bmatrix} - \sum_{k=1}^{r} \int_{t-\tau_k(t)}^{t} \left[y^T(t), y^T(t-\tau_k(t))\right] \tilde{M}_k \begin{bmatrix} y(t) \\ y(t-\tau_k(t)) \end{bmatrix} d\theta \geqslant 0. \tag{21}$$

From (16) to (21), one obtains

$$\dot{V}(y(t)) \leqslant \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \Omega_1 + \Omega_2 \leqslant \xi_0^T(t)\tilde{\Xi}\xi_0(t) - \sum_{k=1}^{r} \int_{t-\tau_k(t)}^{t} \xi_k^T(t,\theta)\Psi_k\xi_k(t,\theta)d\theta, \tag{22}$$

where $\xi_0(t) = \left[y^T(t), y^T(t-\tau_1(t)), \cdots, y^T(t-\tau_r(t))\right]^T$,
$\xi_k(t,\theta) = \left[y^T(t), y^T(t-\tau_k(t)), \dot{y}^T(\theta)\right]^T$, $\tilde{\Xi} = \begin{bmatrix} \Xi_{11} & \Xi_{12}+\tilde{\Xi}_{12} \\ * & \Xi_{22} \end{bmatrix} + \sum_{k=1}^{r} \bar{\tau}_k [I_{N-1} \otimes A, -\tilde{U}(L_1+\Delta L_1)\bar{U} \otimes BK, \cdots, -\tilde{U}(L_r+\Delta L_r)\bar{U} \otimes BK]^T S_k \times [I_{N-1} \otimes A, -\tilde{U}(L_1+\Delta L_1)\bar{U} \otimes BK, \cdots, -\tilde{U}(L_r+\Delta L_r)\bar{U} \otimes BK]$
with $\tilde{\Xi}_{12} = [-R(\tilde{U}\Delta L_1\bar{U} \otimes BK), \cdots, -R(\tilde{U}\Delta L_r\bar{U} \otimes BK)]$, and

$$\Psi_k = \begin{bmatrix} M_{k,11} & M_{k,12} & X_k \\ * & M_{k,22} & Y_k \\ * & * & S_k \end{bmatrix} \ (k = 1, 2, \cdots, r). \tag{23}$$

Since $\Delta L_k = DF(t)\bar{E}_k$ $(k = 1, 2, \cdots, r)$, by properties of Kronecker products one can obtain

$$\tilde{U}\Delta L_k\bar{U} \otimes BK = (\tilde{U}D \otimes I_d)(F(t) \otimes I_d)(\bar{E}_k\bar{U} \otimes BK), \tag{24}$$

where $k = 1, 2, \cdots, r$. By using Schur complement in Boyd et al. (1994) and (24), $\tilde{\Xi} < 0$ is equivalent to

$$\Phi + H(F(t) \otimes I_d)Z + Z^T(F(t) \otimes I_d)^T H^T < 0, \tag{25}$$

where
$H = [(\tilde{U}D \otimes I_d)^T R, 0, \cdots, 0, \bar{\tau}_1(\tilde{U}D \otimes I_d)^T S_1, \cdots, \bar{\tau}_r(\tilde{U}D \otimes I_d)^T S_r]^T$,
$Z = [0, -\bar{E}_1\bar{U}, \cdots, -\bar{E}_r\bar{U}, 0, \cdots, 0] \otimes BK$,
$\Phi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Phi_{13} \\ * & \Xi_{22} & \Phi_{23} \\ * & * & \Phi_{33} \end{bmatrix}$
with $\Phi_{13} = \Xi_{14}$, $\Phi_{23} = \Xi_{24}$ and $\Phi_{33} = \Xi_{44}$. Due to $F^T(t)F(t) \leqslant I_\kappa$, one can see that $(F(t) \otimes I_d)^T(F(t) \otimes I_d) \leqslant I_{\kappa d}$. By Lemma 5, (25) holds if and only if there exists a $\rho > 0$ such that

$$\Phi + \rho HH^T + \rho^{-1}Z^TZ < 0. \tag{26}$$

Replacing ρR, ρQ_k, ρS_k, ρM_k, ρX_k and ρY_k with R, Q_k, S_k, M_k, X_k and Y_k $(k = 1, 2, \cdots, r)$, respectively, and using Schur complement, (26) is equivalent to

$$\Phi_\Delta = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Phi_{13} & \Phi_{\Delta 14} \\ * & \Xi_{22} + \Phi_{\Delta 22} & \Phi_{23} & 0 \\ * & * & \Phi_{33} & \Phi_{\Delta 34} \\ * & * & * & -I \end{bmatrix} < 0, \tag{27}$$

where $\Phi_{\Delta 14} = \Xi_{\Delta 15}$, $\Phi_{\Delta 22} = \Xi_{\Delta 22}$ and $\Phi_{\Delta 34} = \Xi_{\Delta 45}$.

From (22) to (27), one knows that if $\Phi_\Delta < 0$ and $\Psi_k \geqslant 0$ $(k = 1, 2, \cdots, r)$, then one has $\dot{V}(y(t)) < -\varepsilon \|y(t)\|^2$ for a positive constant ε where $\|\cdot\|$ refers to the Euclidean norm for vectors, which implies that subsystem (7) with $\varpi(t) \equiv 0$ is asymptotically stable.

Let us now discuss the performance of subsystem (7) with the disturbance $\varpi(t)$, and consider the following cost performance, for any $T \geqslant 0$,

$$J_T = \int_0^T (y^T(t)y(t) - \gamma^2 \varpi^T(t)\varpi(t))dt.$$

For a zero-valued initial condition, i.e., $\phi(t) \equiv 0$ on $[-\bar{\tau}, 0]$, one has

$$\begin{aligned} J_T &= \int_0^T (y^T(t)y(t) - \gamma^2 \varpi^T(t)\varpi(t) + \dot{V}(t))dt - V(T) + V(0) \\ &\leqslant \int_0^T (\eta_0^T(t)\Xi\eta_0(t) - \sum_{k=1}^r \int_{t-\tau_k(t)}^t \eta_k^T(t,\theta)\Theta_k\eta_k(t,\theta)d\theta)dt - V(T), \end{aligned}$$

where $\eta_0(t) = [\xi_0^T(t), \varpi^T(t)]^T$, $\quad \eta_k(t,\theta) = [y^T(t), y^T(t-\tau_k(t)), \varpi^T(t), \dot{y}^T(\theta)]^T$, and Ξ and Θ_k $(k = 1, 2, \cdots, r)$ are given in (14) and (15), respectively, which can be easily obtained by a similar analysis as the stability of subsystem (7) with $\varpi(t) \equiv 0$. If $\Xi < 0$ and $\Theta_k \geqslant 0$ $(k = 1, 2, \cdots, r)$, then $J_T \leqslant 0$, that is,

$$\int_0^T y^T(t)y(t)dt \leqslant \gamma^2 \int_0^T \varpi^T(t)\varpi(t)dt,$$

which means $\|y\|_{T2} \leqslant \gamma \|\varpi\|_{T2}$.

If LMIs (14) and (15) are feasible, then $\Phi_\Delta < 0$ and $\Psi_k \geqslant 0$ $(k = 1, 2, \cdots, r)$. Hence, system (3) attains consensus with $\|y\|_{T2} \leqslant \gamma \|\varpi\|_{T2}$ $(\forall T \geqslant 0)$ by Theorem 1. The proof of Theorem 3 is completed.

REMARK 4 *Since* $x_{\bar{C}}(t) = (U \otimes I_d)\left[0, y^T(t)\right]^T$*, it follows that*

$$\int_0^T x_{\bar{C}}^T(t)x_{\bar{C}}(t)dt = \int_0^T \left[0, y^T(t)\right](U^TU \otimes I_d)\begin{bmatrix} 0 \\ y(t) \end{bmatrix}dt \leqslant \lambda_{\max}\int_0^T y^T(t)y(t)dt,$$

where $\lambda_{\max}$ *is the maximum eigenvalue of* $U^TU \otimes I_d$*. If* $\|y\|_{T2} \leqslant \gamma \|\varpi\|_{T2}$ $(\forall T \geqslant 0)$*, then one has* $\|x_{\bar{C}}\|_{T2} \leqslant \gamma\sqrt{\lambda_{\max}}\|\varpi\|_{T2}$ $(\forall T \geqslant 0)$*. One can see that the disagreement dynamics of system (3) also satisfies a certain* $\mathcal{L}_2$ *performance. If* $\varpi(t) \in \mathcal{L}_2[0,\infty)$ $(t \geqslant 0)$*, then* $\|y\|_2$ *and* $\|\dot{y}\|_2$ *are bounded, so* $\lim_{t\to\infty} y(t) = 0$ *which implies that* $\lim_{t\to\infty} x_{\bar{C}}(t) = 0$*.*

Remark 5 *Based on LMI techniques, consensus problems of swarm systems were studied in Sun et al. (2008), Xi et al. (2013), Lin et al. (2008), Liu and Jia (2009). Swarm systems with multiple time delays were dealt with in Sun et al.(2008), where uncertainties and external disturbances were not considered, and the consensus function, which is important in the analysis of a swarm system, is difficult to be determined by the method there proposed. In Xi et al. (2013), we addressed swarm systems with multiple time delays and uncertainties, and presented an explicit expression of the consensus function. In Sun et al. (2008), Xi et al. (2013), Lin et al. (2008), it is assumed that the dynamics of each agent is a first-order integrator. By the H_∞ control method, consensus problems for high-order swarm systems with $\mathcal{L}_2$ external disturbances were investigated in Liu and Jia (2009), where a controlled output function was defined based on the average of states of all agents, and the norm of the closed-loop transfer function matrix from external disturbances to the controlled output was used to evaluate the influence of external disturbances. When consensus functions are not the average of states of all agents, the methods in Liu and Jia (2009) are no longer valid. In the current paper, high-order swarm systems with multiple time-varying delays, uncertainties and external disturbances are studied, and an explicit expression of the consensus function is presented based on the impacts of time delays and uncertainties.*

4. Numerical simulations

In this section, a numerical example is given to illustrate the effectiveness of theoretical results shown in the previous section.

Suppose that a swarm system consists of five agents with the dynamics described by (1) with

$$A = \begin{bmatrix} 0.5 & -1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, B_\varpi = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

A directed communication graph G of the system is shown in Fig. 1. The edges (v_4, v_2), (v_1, v_2), (v_1, v_3), (v_5, v_3), (v_3, v_5), (v_4, v_1), (v_1, v_4), (v_5, v_1), (v_1, v_5), (v_5, v_4) and (v_4, v_5) are labeled from 1 to 11 respectively. For simplicity, the adjacency matrix of G is set to be a 0-1 matrix. Consider the case where there exist two time-varying delays in different channels as shown in Fig. 2, and uncertainties are given by $\Delta L_1 = DF(t)\bar{E}_1$ and $\Delta L_2 = DF(t)\bar{E}_2$, where $\bar{E}_i = 0.07(D_i^T - E_i)$ $(i = 1, 2)$, D_i and E_i $(i = 1, 2)$ can be obtained according to Lemma 4, and $F(t) = diag\{\ 0.2\sin(t), 0.3, 0.4\sin(t), \cos(t), 0.4, 0.35\sin(t), 0.6\cos(t), 0.02, 0.01, 0.15\sin(t), 0.35\cos(t)\}$. The performance index γ is chosen as 1. Let $K = \begin{bmatrix} 1 & 4 \\ 0.5 & 12 \end{bmatrix}$, $d_1 = 0.02$, $d_2 = 0.01$, $\bar{\tau}_1 = 0.05s$ and $\bar{\tau}_2 = 0.03s$, then a feasible solution of LMIs (14) and (15) in Theorem 3 can be obtained by using FEASP solver in Matlab LMI Toolbox, Gahinet et al. (1995). It is assumed that the initial condition is $\phi(t) \equiv x(0) = [6, -6, 4, 3, -4, 0, -4, -2, -1, 5]^T$ on $[-\bar{\tau}, 0]$ and $T = 1$ for simplicity of simulation.

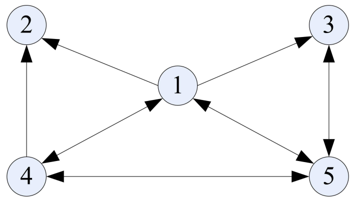

Figure 1. Directed communication graph G

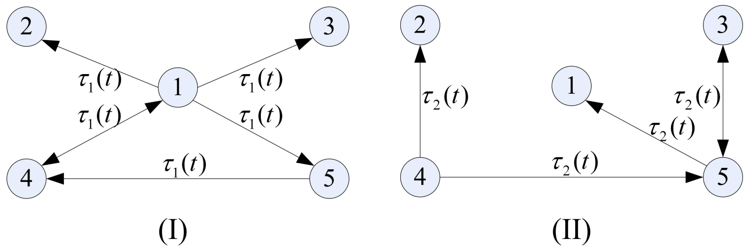

Figure 2. Communication channels with two time-varying delays in G

Fig. 3 shows the state trajectories of the swarm system with $\tau_1(t) = 0.03 + 0.02\sin(t)$, $\tau_2(t) = 0.02 + 0.01\cos(t)$ and $\varpi(t) \equiv [0, 0.6, 0.8, 0.5, 1]^T$, and Fig. 4 depicts the corresponding energy trajectories under the zero-valued initial condition. By Corollary 1, the nominal consensus function is $c_0(t) = \left[0.9722e^{0.5t}, 0\right]^T$ which is denoted by circle markers in Fig. 3. One can see that the system attains consensus with $\|y\|_{T2} \leqslant \|\varpi\|_{T2}$ $(T = 1)$. The state trajectories deviate from the one formed by circle markers, which means that the consensus function is impacted by time-varying delays, uncertainties and disturbances.

5. Conclusions

Consensus problems for high-order continuous-time swarm systems with time delays, uncertainties and external disturbances were studied. A swarm system with time delays and uncertainties was decomposed into two subsystems associated with the state projection on the complement consensus subspace (CCS) and the consensus subspace (CS), respectively. It was proven that the asymptotic stability of the subsystem associated with the state projection on CCS is a necessary and sufficient condition for the system to achieve consensus, and the subsystem associated with the state projection on CS determines the consensus function. An explicit expression of the consensus function was given according to different impacts of time delays and uncertainties. For the case with external disturbances, a sufficient condition for the system to achieve consensus with a desired $\mathcal{L}_2$ performance was presented. Numerical simulations were given to illustrate the effectiveness of theoretical results.

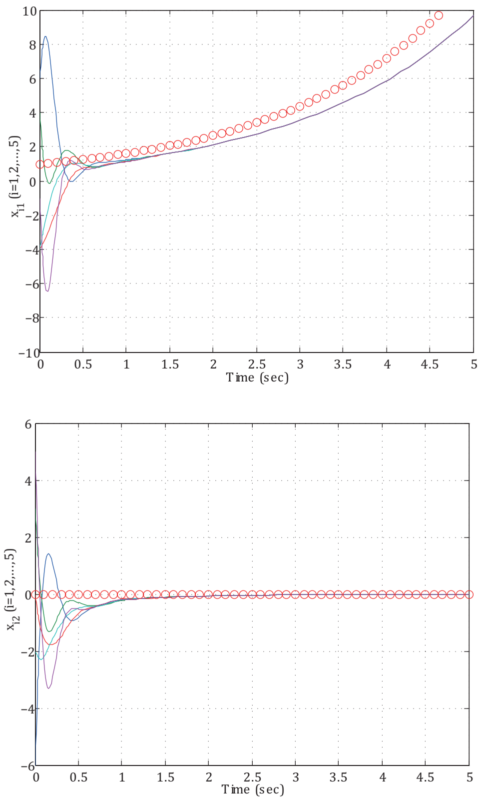

Figure 3. State trajectories of the swarm system

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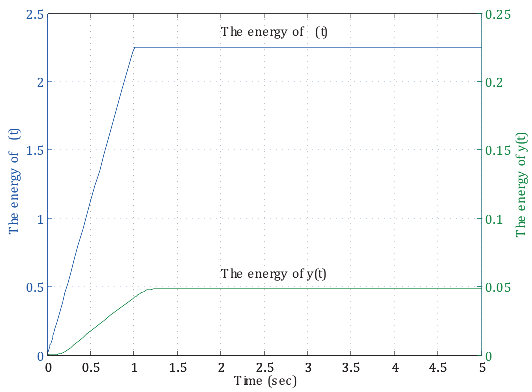

Figure 4. Energy trajectories of $y(t)$ and $\varpi(t)$

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