

Modelling of Non-ideal Signal Sampling via Averaging Operation and Spectrum of Sampled Signal Predicted by this Model

A. Borys

Gdynia Maritime University, Gdynia, Poland

ABSTRACT: In this paper, a novel model of a non-ideal signal sampling via a local, periodic averaging operation is present-ed. The spectrum of a sampled signal predicted by this model is also analysed as well as compared with a one following from another model.

1 INTRODUCTION

Because of physical and technological constraints the sampling of a signal value (that is its sample) at a zeroth time is not possible. In other words, the sampling time of a single signal sample is always greater than zero. Only in an abstract idealized case this time can be assumed to be equal to zero, and then the signal samples related with this case are considered as ideal ones. Moreover, we say that then the sampling operation is performed in an ideal way.

Let us denote here the sampling time of a single signal sample by τ and the sampling period by T , respectively. Obviously, the following relation: $\tau \leq T$ must hold between these quantities; but, as we know, practical reasons require that this inequality is much more sharper. That is we have $\tau \ll T$. And, this could suggest that we can assume approximately $\tau \cong 0$ (in the sense that the following three cases: $\tau \neq 0$ but very small with the sampling of signal values modelled as an operation of periodic cutting out a signal fragment, $\tau \rightarrow 0$ but always greater than zero ($\tau > 0$) with the sampled signal spectrum normalization with respect to τ , and $\tau = 0$ with a preceding normalization of the sampled signal spectrum with respect to τ – do not considerably

differ from each other) in the analyses involving signal samples.

The above-mentioned believing is however misleading; usually, we arrive at different outcomes in these three cases. This is strongly manifested in calculations of the sampled signal spectrum for these kinds of modelling, and analyzed in detail in (Borys A. 2020a).

In this paper, we discuss, from another perspective, the differences that occur in calculation of the sampled signal spectrum between the following two cases: the first one, in which signal samples are “smeared” on a time interval τ (the case of $\tau \neq 0$ but very small), and the second one, in which the signal sampling is performed in an ideal way (the case of $\tau = 0$). We consider them in the context of the Shannon’s proof of the reconstruction formula (Shannon C. E. 1949).

Modelling of the “smearing effect” occurring in a signal sampling process presented in this paper differs, however, considerably from the one used in (Borys A. Sept. 2020). Here, the “smeared” samples are not modelled as short signal impulses (as in (Borys A. Sept. 2020)), but as numbers obtained as a result of performing periodically a local averaging of these

impulses. So, in effect, we can expect receiving outcomes not identical with those we arrived at in (Borys A. Sept. 2020). And, really, it is so as we show in the next sections.

Moreover, in view of the above, a question arises as to which of the aforementioned approaches to modelling of the “sample smearing” effect describes in a better way a real, non-ideal signal sampling. This problem is, however, not addressed in this paper because of a lack in the literature of reliable data on behavior of real A/D converters in the “transient” interval τ . First, necessary measurements will have to be carried out and data collected; we hope that some researchers will be interested in performing this task.

The remainder of this paper is structured as follows. The next section presents a modelling of the “sample smearing” effect via a periodical local averaging – in the context of the proof (Shannon C. E. 1949) of the reconstruction formula that takes into account a non-ideal sampling. But, section 3 contains a comparison of the results which are achieved with the use of the aforementioned models. The paper ends with some conclusions.

2 SHANNON'S PROOF OF RECONSTRUCTION FORMULA TAKING INTO ACCOUNT NON-IDEAL SIGNAL SAMPLING AND POSSIBLE DEFINITIONS OF SAMPLED SIGNAL SPECTRUM

Let us take into account a bandlimited signal $x(t)$ of a continuous time t and denote a maximal frequency present in its spectrum by f_m . So, as well known (Marks II R. J. 1991), (Vetterli M., Kovacevic J., Goyal V. K. 2014), (Oppenheim A. V., Schaffer R. W., Buck J. R. 1998), (Bracewell R. N. 2000), (McClellan J. H., Schaffer R., Yoder M. 2015), (Brigola R. 2013), (So H. C. 2019), (Wang R. 2010), (Ingle V. K., Proakis J. G. 2012), it is possible to sample this signal and then reconstruct it perfectly if the sampling period T fulfils the following condition:

$$1/T = f_s \geq 2f_m, \quad (1)$$

where f_s stands for the sampling frequency (rate).

Further, assume that, as well known, the sampling operation of $x(t)$ cannot be performed in an ideal way; simply, A/D converters that provide perfect signal samples by sampling a signal “pointwise on the time axis” do not exist. All the real A/D converters need some time, denoted here as τ , to carry out the sampling operation. And, a value of the parameter τ depends, obviously, upon the design principles and electronics that are used in construction of a given A/D converter. Therefore, the values that appear at outputs of A/D converters must be treated as “deformed” or “smeared” ones – in comparison to the wished perfect samples.

Further, note also that the aforementioned behavior is usually modelled in the literature by a local convolution (with the signal being sampled) (Marks II R. J. 1991) or as a local signal averaging; see,

for example, (Vetterli M., Kovacevic J., Goyal V. K. 2014). (By the way, note that the latter can be also expressed as a convolution, see, for instance, (Borys A. 2020b).)

In this paper, we use a description of the non-ideal signal sampling that follows from modelling it by a local signal averaging. And, this seems to be a reasonable approach, as shown in the literature, using many convincing arguments, see, for example, (Borys A. 2020b), (Borys A. 2020c), (Strichartz R. 1994). Here, we exploit a slightly modified version of that presented in (Borys A. 2020b). This modification regards the instant of “delivering” a “smeared” value of a sample. Namely, in modelling of a measuring process, this instant must be the one at which a result of a local signal averaging process “departs” this process. But, unlike this, in a non-ideal sampling, the result of a local signal averaging has to be “glued” to the instant of beginning the averaging process.

For illustration, consider Fig. 1.

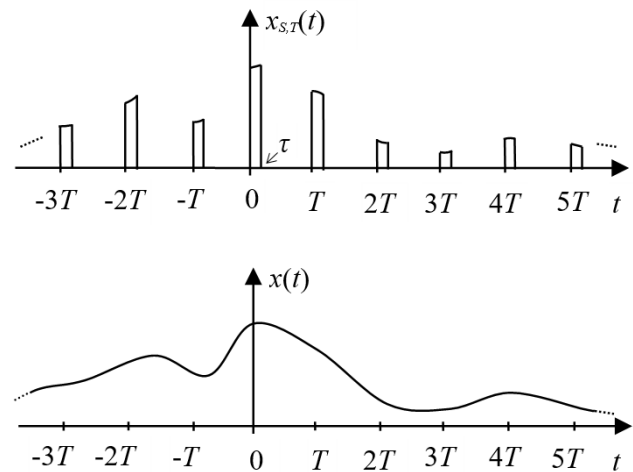


Figure 1. Illustration to non-ideal sampling; representation of an example, not ideally sampled signal (upper curve) in form of a series of smeared samples of the signal (forming narrow impulses) shown below it (lower curve). Figure taken from (Borys A. 2020a).

In Fig. 1, $x_{s,T}(t)$ means a „smeared” sampled signal in which every sample does not represent a single value, but it builds up an impulse of width τ . So, it can be viewed as “a kind of smearing of a discrete sample value on an interval τ ”. And, this way of modelling of the non-ideal signal sampling has been used in (Borys A. 2020a). But, as opposed to the approach applied in (Borys A. 2020a), we assume here that the non-ideal signal sampling delivers not impulses at an A/D output but discrete sample values – as in an ideal case. However, now, they are modified by the smearing process of the ones that would have been obtained in an ideal sampling. And, they are “glued” to the instants of virtual appearances of the latter ones. Furthermore, the aforementioned signal sample smearing process is modelled here as a local signal averaging.

So, as a result of performing these two operations described above, we get

$$x_{A,T}(t) = \text{AV}(x_{S,T}(t)) = \sum_{k=-\infty}^{\infty} \text{av}(\text{from } x_{S,T}(\lambda = kT) \quad (2)$$

$$\text{to } x_{S,T}(\lambda = kT + \tau) \delta_{k,t/T}(t) = \sum_{k=-\infty}^{\infty} \bar{x}(kT) \delta_{k,t/T}(t),$$

where the symbol AV stands for an operator that transforms an infinite train of impulses as in Fig. 1 into an infinite train of single values as shown in Fig. 2 – according to these two rules given in a descriptive form above. The result of this operation is the signal $x_{A,T}(t)$. And, the next symbol, “small av”, means performing a local averaging around an indicated “smeared” sample (that is on a given impulse of $x_{S,T}(t)$); the result of this operation is denoted here by $\bar{x}(kT)$. And finally, $\delta_{k,t/T}(t)$ in (2) means a time-shifted Kronecker time function (Borys A. 2020a), (Borys A. 2020d).

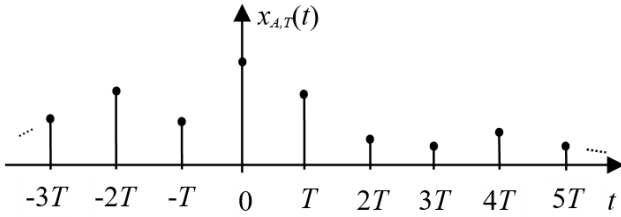


Figure 2. Illustration to transformation of the signal $x_{S,T}(t)$ shown in Fig. 1 to the signal $x_{A,T}(t)$. Note that the lengths of “posts” in Fig. 2 are not equal to the values of ideal signal samples $x(kT)$. They differ from them and equal the values of $\bar{x}(kT)$.

Here, to model the local signal averaging, we use its description presented in detail in (Borys A. 2020b) and (Strichartz R. 1994). So, along these lines, we write

$$\begin{aligned} \bar{x}(kT) &= \text{av}(\text{from } x(t = kT) \text{ to } x(t = kT + \tau)) = \\ &= \int_{kT}^{kT+\tau} x(\lambda) a(kT + \tau - \lambda) d\lambda, \end{aligned} \quad (3)$$

where the function $a(t)$ is assumed to have the following form (Borys A. 2020b):

$$a(t) = 1/\tau \text{ for } 0 < t < \tau \text{ and } 0 \text{ elsewhere.} \quad (4)$$

Note further that, because of a sifting character of the function $a(t)$ given by (4) – in the interval from 0 to τ , (3) can be rewritten as

$$\bar{x}(kT) = \int_{-\infty}^{\infty} x(\lambda) a(kT + \tau - \lambda) d\lambda. \quad (5)$$

So, we conclude from (5) that the averaged (smeared sample) value $\bar{x}(kT)$ can be expressed as a convolution of the signal $x(t)$ with an impulse response $a(t + \tau)$, and that is calculated for the instant kT . (For the needs of our further derivations, we assume that this convolution exists for all $t \in \mathbb{R}$, where \mathbb{R} denotes the set of real numbers.) Moreover, note that the above convolution can be equivalently calculated in the frequency domain as a

product of the following Fourier transforms: $X(f)$ and $\mathcal{F}(a(t + \tau)) = A(f) \exp(j2\pi f\tau)$ with $\mathcal{F}(\cdot)$ denoting a standard Fourier transform (of a given function), $A(f)$ – standing for this transform for $a(t)$, and $j = \sqrt{-1}$. Furthermore, by carrying out a few standard calculations involving properties of Fourier transformation, we obtain

$$\begin{aligned} Y(f) &= X(f) \cdot \mathcal{F}(a(t + \tau)) = X(f) \text{sinc}(f\tau) \cdot \\ &\cdot \exp(j\pi f\tau) = X(f) \frac{\sin(\pi f\tau)}{\pi f\tau} \exp(j\pi f\tau), \end{aligned} \quad (6)$$

where $Y(f)$ means the aforementioned product, but the definition of the function $\text{sinc}(f\tau)$ used here follows from a comparison of the first and the second line of (6).

Let us now come back to the assumed band-limitedness of the signal $x(t)$. It means that its Fourier transform $X(f)$ has nonzero values only in the range $\langle -f_m, f_m \rangle$ (that is this is a support of this function). Further, because of this reason the function $Y(f)$ given by (6) is also bandlimited to the interval $\langle -f_m, f_m \rangle$. So, it allows its periodization (i.e. obtaining a repetition of this function in form of a Fourier series). But, because of the reasons explained in (Borys A. 2020e), we extend here the support of $Y(f)$ to the interval $\langle -f_s/2, f_s/2 \rangle$, and consequently construct a corresponding Fourier series with a fundamental frequency $f_0 = f_s$ (not $2f_m \leq f_s$).

In what follows now, we proceed similarly as in (Borys A. 2020e). That is we perform first periodization of the function $Y(f)$ to a periodic one, say, $Y_p(f)$, and expand it in a Fourier series. Next, we express coefficients of this series through the samples $y(kT)$ of the function $y(t) = \mathcal{F}^{-1}(Y(f))$, where $\mathcal{F}^{-1}(\cdot)$ stands for the inverse Fourier transform. And, in the next step, we introduce them into the aforementioned Fourier series. In this way, we get the discrete time Fourier transform (DTFT) (McClellan J. H., Schafer R., Yoder M. 2015), (Vetterli M., Kovacevic J., Goyal V. K. 2014), (Wang R. 2010), (Ingle V. K., Proakis J. G. 2012) of the sampled (discrete) signal $y(kT)$, which is equal to $Y_p(f)$. And, we call it a spectrum of this signal, say, SPECT1 (it forms a first of its possible definitions). Next, we recall at this point that a different definition of the spectrum of the aforementioned sampled signal is also possible (Borys A. 2020f); it is named the modified DTFT in (Borys A. 2020f) (in short, DTFTm). Furthermore, we know from (Borys A. 2020e) that the following: SPECT2 = DTFTm = $Y(f)$ holds, where SPECT2 stands (in short) for the second possible spectrum definition – of the sampled signal $y(kT)$.

Both these spectra occur in the “inverse part” of the Shannon’s proof; that is in

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} Y(f) \exp(j2\pi ft) df = \\
&= \int_{-f_s/2}^{f_s/2} Y_p(f) \exp(j2\pi ft) df = \\
&= \int_{-f_s/2}^{f_s/2} \text{DTFT}(y(kT)) \exp(j2\pi ft) df = \\
&= \int_{-\infty}^{\infty} \text{DTFTm}(y(kT)) \cdot \\
&\cdot \exp(j2\pi ft) df = \text{and so on.}
\end{aligned} \tag{7}$$

And, we see that the way they occur in (7) is exactly the same as in its counterpart in (Borys A. 2020e). So, we conclude, similarly as in (Borys A. 2020e), that the “clever” Shannon’s proof of the reconstruction formula does not provide, also here, any tool for resolving the question of which of them:

$$\text{SPECT1} = \text{DTFT} = Y_p(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} Y(f - k/T)$$

or $\text{SPECT2} = \text{DTFTm} = Y(f)$

is a correct one? This is so, as already found in (Borys A. 2020e), because the Shannon’s proof does not need, in fact, to use such a notion as the sampled signal spectrum.

Further details and explanations concerning the above, the interested reader finds in (Borys A. 2020e).

3 COMPARISON OF RESULTS PROVIDED BY TWO MODELS THAT TAKE INTO ACCOUNT FINITE DURATION OF GETTING SIGNAL SAMPLE

As already said, the method presented here of taking into account a finite duration of getting a sample – in a model of the non-ideal signal sampling – is not the only one possible. One can, for example, model also a train of non-ideal samples of a signal as a train of impulses, as illustrated in Fig. 1 (upper curve). That is as a signal $x_{s,T}(t)$ denoted there. And, there is no problem with calculation of its spectrum, as shown in (Borys A. 2020a). Moreover, it has been shown in (Borys A. 2020a) that

$$X_{s,T}(f) = \sum_{k=-\infty}^{\infty} a_k X(f - k/T), \tag{8}$$

where $X(\cdot)$ and $X_{s,T}(f)$ stand for the Fourier transforms (spectra) of the signals $x(t)$ and $x_{s,T}(t)$, respectively. And, the coefficients a_k in (8) are given by

$$a_k = \frac{\tau}{T} \exp(-j\pi k \tau/T) \cdot \text{sinc}(\pi k \tau/T). \tag{9}$$

We remark at this point that the detailed derivations and explanations concerning (8) and (9)

have been provided in (Borys A. 2020a). They are not repeated here because of a lack of space as well as to avoid accusation of auto-plagiarism. Moreover, the reference (Borys A. 2020a) is well available.

For performing comparisons between the sampled signal spectra foreseen in the discussed models – in a clear way, let us denote now by $\text{SPECT} = (1/T) \sum X(f - k/T)$ the spectrum which one obtains in the highly celebrated and commonly used model (see, for example, (Marks II R. J. 1991), (Vetterli M., Kovacevic J., Goyal V. K. 2014), (Oppenheim A. V., Schafer R. W., Buck J. R. 1998), (Bracewell R. N. 2000), (McClellan J. H., Schafer R., Yoder M. 2015), (Brigola R. 2013), (So H. C. 2019), (Wang R. 2010), (Ingle V. K., Proakis J. G. 2012)) that uses Dirac deltas in description of the sampled signal in the continuous time domain. Further, let us denote by $\text{SPECT0} = X_{s,T}(f)$ given by (8).

In what follows, we remark that:

1. The form of SPECT0 for positive values of the parameter τ is identical with the one of SPECT , except the coefficients which multiply the shifted spectra $X(f - k/T)$. They are given by (9) in the first case and are identically equal to $1/T$ for all the indices k in the second one.
2. Because of the oscillatory-damping character of the magnitude of the coefficients a_k (see (9)) the same character has also the magnitude of SPECT0 . So, this is also the character of the aliasing and folding effects in the sampled signal spectrum via this model. Obviously, it differs substantially from the case of SPECT .
3. An expectation that the problem of modelling properly a non-ideal behavior of getting samples in the signal sampling process can be uniquely resolved by describing it through local signal averaging operations on short time intervals τ , as discussed in this paper, turned out to be only a vain hope. At least with regard to the spectrum of the sampled signal. Furthermore, note that the averaging operation does not also provide a description of the sampled signal in the time domain in an idealized case, in which $\tau \rightarrow 0$, as a train of sample values multiplied by Dirac deltas (as a highly celebrated and commonly used model (Marks II R. J. 1991), (Vetterli M., Kovacevic J., Goyal V. K. 2014), (Oppenheim A. V., Schafer R. W., Buck J. R. 1998), (Bracewell R. N. 2000), (McClellan J. H., Schafer R., Yoder M. 2015), (Brigola R. 2013), (So H. C. 2019), (Wang R. 2010), (Ingle V. K., Proakis J. G. 2012) foresees). To see this, consider (4) with $\tau \rightarrow 0$ in it. And, using arguments presented, for example, in (Strichartz R. 1994) for this case, we can write, then, $a(t) \rightarrow \delta(t)$, where $\delta(t)$ means a Dirac delta. So, applying this result in (5) allows us to express the latter as

$$\bar{x}(kT) \rightarrow x(kT) = \int_{-\infty}^{\infty} x(\lambda) \delta(kT - \lambda) d\lambda = \int_{-\infty}^{\infty} x(kT) \cdot \delta(kT - \lambda) d\lambda$$

1 But, it does not mean that an object $x(kT)\delta(\cdot)$ occurring under the symbol of integration in the latter equation represents the sample value $x(kT)$. As seen, it is a result of performing the above operation of integration. In other words, simply, $x(kT)\delta(\cdot) \neq x(kT)$.

4. Also, we draw the reader's attention here to the fact that the averaging procedure in the time domain (applied in this paper to the smeared sample impulses and connected then with the spectrum definition SPECT1) provides a different result compared to the normalization of the spectrum $X_{s,T}(f)$ with respect to the parameter τ (Borys A. 2020a) (to avoid its vanishing with $\tau \rightarrow 0$). The resulting expressions that describe the spectra of the sampled signal in both cases are similar in form but not identical. However, because of a lack of space we do not discuss here this interesting observation in more detail.
5. Note once again that the scheme of the Shannon's proof applied in this paper to the signal sampled not in an ideal way differs from the one discussed in (Borys A. 2020e) only in one aspect, namely $Y(f)$ (in this paper) is not a Fourier transform of the signal to be sampled. It is a "deformed" spectrum of this signal. And, it follows from (6) that a level of its deformation can be expressed by, say, a "deformation" factor $\beta(f)$ defined as

$$\beta(f) = \frac{X_d(f)}{X(f)} = \frac{\sin(\pi f \tau)}{\pi f \tau} \exp(j\pi f \tau), \quad (10)$$

1 where $X_d(f) = Y(f)$ stands for the spectrum $X(f)$ that is deformed by the local averaging operator av . Moreover, note that because of the band-limitedness of $X(f)$ (and also of $Y(f)$) it has only sense in the frequency interval $\langle -f_m, f_m \rangle$ (outside this range, it should be assumed to be equal to zero). Further, see from (10) that both the magnitude and phase of the spectrum $X(f)$ get deformed. Here, for illustration, let us consider only a deformation in the magnitude. And, we make a few observations:

1. See first that $|\beta(f)| \leq 1$ for all possible values of frequency and parameter τ .
2. For $\tau = 0$, $|\beta(f)| = 1$. That is there occurs no sampled signal deformation in this case (as it should be for $\tau = 0$).
3. For illustration, let us assume that we wish to have the deformation of the sampled signal spectrum magnitude less than 10% in the worst case. To determine a condition for the parameter τ that satisfies the above requirement, we consider the magnitude of $\beta(f)$ given by (10) for positive frequencies f . And, note that the most critical here is the frequency f_m . Further, assume that the sampling rate is so chosen that $f_s = 2f_m \rightarrow f_m = 1/(2T)$ holds. So, for this value of $f = f_m$, we have $|\beta(f_m)| = \text{sinc}(\pi\tau/(2T))$. And, we require to satisfy the following: $|\beta(f_m)| \geq 0,9$; while the latter is satisfied approximately for $\tau/T < 1/2$. Obviously, at the same time, this is a condition we looked for.

4 CONCLUSIONS

The problem of modelling the sample "smearing" behavior of real A/D converters used in signal sampling has not received much attention in the literature. It seems to have been assumed that this effect is irrelevant – compared to, for example, (amplitude) quantization errors produced by A/D converters. As we show in this paper and in a previous one (Borys A. 2020a), such reasoning is rather not correct. This is so because the aforementioned effect has a significant influence on the sampled signal spectrum – and, this has been already proven. What remains to be done yet should concentrate, in our opinion, on finding a detailed model and adjusting it to the sample "smearing" behavior of real A/D converters.

Two relevant models has been already proposed, a one in this paper and the second in (Borys A. 2020a) (perhaps, there will be also others).

Note that a modelling principle of the first one is based on performing periodically a local averaging of impulses of short duration, starting at sampling instants, and delivering its averages at the ends of the aforementioned impulses (which, however, are "glued" to their beginning instants). So, in this case, the outcomes of the "sample smearing" operations are numbers. In contrast to this, in the model presented in (Borys A. 2020a) the impulses mentioned above are taken to constitute the "smeared" samples (that is electric "spikes" of duration τ).

There is a variety of design principles, techniques, and circuit schemes for A/D converters. Therefore, probably, more than only one model for describing correctly their "sample smearing" behavior will be needed. And, for checking practical usefulness of these models many investigations will be also needed. Moreover, note that there are still open questions of more general nature as, for example, the one considered in (Borys A. 2020d). So, we are still far from a satisfactory solution to the problem of the sampled signal spectrum.

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