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Complex Fibonacci (c, p)-numbers

Abstract In this paper a new complex Fibonacci $Q_{p,c}$ matrix for complex Fibonacci (c, p)-numbers, where p is a positive integer and c is a non zero complex number, is introduced. Thereby, we discuss various properties of $Q_{p,c}$ matrix, coding and decoding method followed from the $Q_{p,c}$ matrix.

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1. Introduction. The Fibonacci numbers are defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 2 \tag{1}$$

with initial terms $F_0 = 1$, $F_1 = 1$. The Fibonacci numbers, F_n and golden mean,

$$\tau = \lim_{n \longrightarrow \infty} \frac{F_n}{F_{n-1}} = \frac{1 + \sqrt{5}}{2}$$

have applications in cryptography (v. Taş et al. (2018), Stakhov (2007), Uçar et al. (2019)), information and coding theory (v. Basu and Prasad (2010, 2012)), sciences and arts Kocer et al. (2009), El Naschie (2007), Stakhov and Rozin (2006).

The complex Fibonacci numbers Jiang et al. (2014) are defined by the recurrence relation:

$$F^\star_n = F^\star_{n-1} + F^\star_{n-2} \quad \text{for} \quad n \geqslant 2 \tag{2}$$

with initial terms $F^\star_0 = i$, $F^\star_1 = 1 + i$ where i is the imaginary unit which satisfies $i^2 = -1$ and they are connected with Fibonacci numbers by the following way

$$F^\star_n = F_n + iF_{n+1}.$$

Prasad (2019) introduced complex Fibonacci p numbers by the following recurrence relation:

$$F_p^\star(n) = F_p^\star(n-1) + F_p^\star(n-p-1) \tag{3}$$

with $n > p+1$ and initial terms

$$F_p^\star(0) = i, F_p^\star(1) = F_p^\star(2) = \cdots = F_p^\star(p) = 1+i.$$

In this paper, we introduce complex Fibonacci (c,p)-numbers by the following recurrence relation:

$$F_{p,c}^\star(n) = cF_{p,c}^\star(n-1) + F_{p,c}^\star(n-p-1) \tag{4}$$

with initial terms

$$F_{p,c}^\star(1) = a_1, \quad F_{p,c}^\star(2) = a_2, \quad F_{p,c}^\star(3) = a_3, \quad \cdots, \quad F_{p,c}^\star(p+1) = a_{p+1} \tag{5}$$

where p is positive integer, c is a non zero complex number, $n > p+1$ and $a_1, a_2, a_3, \ldots, a_{p+1}$ are arbitrary complex numbers.

In this paper, we consider initial terms as

$$F_{p,c}^\star(n) = c^{n-1}, n = 1, 2, 3, 4, \cdots, p+1. \tag{6}$$

In this paper, we establish the relations among the code elements for all values of p where p is a positive integer. The relation among the code matrix elements for p where p is a positive integer and $c = 1$, coincides with the relation among the code matrix elements Prasad (2019) and correction ability of this method increases as p increases but it is independent of c.

2. Some properties of the complex Fibonacci (c,p)-numbers for a given initial terms We calculate the complex Fibonacci (c,p)-numbers for all values of n. Consider (6) as initial terms then from (4) we have,

$$F_{p,c}^\star(p+1) = cF_{p,c}^\star(p) + F_{p,c}^\star(0),$$

so $F_{p,c}^\star(0) = 0$. Continuing this process by substituting $n = p,\ p-1, \cdots, 2$ in (4) we have,

$$F_{p,c}^\star(-1) = F_{p,c}^\star(-2) = \ldots = F_{p,c}^\star(-p+1) = 0.$$

Now by substituting $n = 1$ in (4) we have,

$$F_{p,c}^\star(1) = cF_{p,c}^\star(0) + F_{p,c}^\star(-p)$$

Therefore, $F_{p,c}^\star(-p) = 1$. Continuing this process by substituting $n = 0,\ -1,\ -2, \cdots, -p+2$ in (4) we have,

$$F_{p,c}^\star(-p-1) = F_{p,c}^\star(-p-2) = \ldots = F_{p,c}^\star(-2p+1) = 0.$$

Also, by substituting $n = -p+1, -p, -p-1$ in (4) we have,

$$F^{\star}_{p,c}(-2p) = -c,$$
$$F^{\star}_{p,c}(-2p-1) = 1,$$
$$F^{\star}_{p,c}(-2p-2) = 0.$$

So, we summarized above in the table 1:

Table 1: Complex Fibonacci (c,p)-numbers, $F^{\star}_{p,c}(n)$

$n \rightarrow$	0	-1	...	...	-p+1	-p	-p-1	...	...	-2p+1	-2p	-2p-1	-2p-2
$F^{\star}_{p,c}(n)$	0	0	...	...	0	1	0	...	...	0	-c	1	0

Thus, we get complex Fibonacci (c,p)-numbers,

$$F^{\star}_{p,c}(n) = cF^{\star}_{p,c}(n-1) + F^{\star}_{p,c}(n-p-1) \tag{7}$$

for any integer n and $p = 0, 1, 2, 3, \ldots$ where $F^{\star}_{p,c}(n) = c^{n-1}$, $n = 1, 2, 3, \ldots, p+1$. $F^{\star}_{p,c}(n)$ numbers are of theoretical interest for discrete mathematics and open new perspectives for the development of complex number theory and information sciences.

3. Complex Fibonacci $Q_{p,c}$ matrix In this paper, we define a new matrix called complex Fibonacci $Q_{p,c}$ matrix (8) of order $(p+1)$ on the complex Fibonacci (c,p)-numbers where p is a positive integer and c is a non zero complex number.

$$Q_{p,c} = \begin{pmatrix} c & 1 & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & 1 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & . & 0 & 1 \\ 1 & 0 & 0 & . & . & . & . & 0 & 0 \end{pmatrix} \tag{8}$$

Using (6) we can write

$$Q_{p,c} = \begin{pmatrix} F^{\star}_{p,c}(2) & F^{\star}_{p,c}(1) & . & . & . & . & F^{\star}_{p,c}(3-p) & F^{\star}_{p,c}(2-p) \\ F^{\star}_{p,c}(2-p) & F^{\star}_{p,c}(1-p) & . & . & . & . & F^{\star}_{p,c}(3-2p) & F^{\star}_{p,c}(2-2p) \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ F^{\star}_{p,c}(0) & F^{\star}_{p,c}(-1) & . & . & . & . & F^{\star}_{p,c}(1-p) & F^{\star}_{p,c}(-p) \\ F^{\star}_{p,c}(1) & F^{\star}_{p,c}(0) & . & . & . & . & F^{\star}_{p,c}(2-p) & F^{\star}_{p,c}(1-p) \end{pmatrix} \tag{9}$$

Theorem 3.1 For any integer n, the nth power of the $Q_{p,c}$ matrix is equal

$$\begin{pmatrix} F^\star_{p,c}(n+1) & F^\star_{p,c}(n) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{p,c}(n-p+2) & F^\star_{p,c}(n-p+1) \\ F^\star_{p,c}(n-p+1) & F^\star_{p,c}(n-p) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{p,c}(n-2p+2) & F^\star_{p,c}(n-2p+1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ F^\star_{p,c}(n-1) & F^\star_{p,c}(n-2) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{p,c}(n-p) & F^\star_{p,c}(n-p-1) \\ F^\star_{p,c}(n) & F^\star_{p,c}(n-1) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{p,c}(n-p+1) & F^\star_{p,c}(n-p) \end{pmatrix}$$

where $F^\star_{p,c}(n)$ is the nth complex Fibonacci (c,p)-numbers.

Proof When $p=1$, we have to prove

$$Q^n_{1,c} = \begin{pmatrix} F^\star_{1,c}(n+1) & F^\star_{1,c}(n) \\ F^\star_{1,c}(n) & F^\star_{1,c}(n-1) \end{pmatrix}. \tag{10}$$

We will prove it by mathematical induction. For $n=1$

$$Q_{1,c} = \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F^\star_{1,c}(2) & F^\star_{1,c}(1) \\ F^\star_{1,c}(1) & F^\star_{1,c}(0) \end{pmatrix} \text{ by (7)}$$

which is true for $n=1$.
For $n=2$

$$Q^2_{1,c} = \begin{pmatrix} c^2+1 & c \\ c & 1 \end{pmatrix} = \begin{pmatrix} F^\star_{1,c}(3) & F^\star_{1,c}(2) \\ F^\star_{1,c}(2) & F^\star_{1,c}(1) \end{pmatrix} \text{ by (7)}$$

which is true for $n=2$.
Suppose (10) is true for integer $n=k$, then

$$Q^k_{1,c} = \begin{pmatrix} F^\star_{1,c}(k+1) & F^\star_{1,c}(k) \\ F^\star_{1,c}(k) & F^\star_{1,c}(k-1) \end{pmatrix}$$

Now, we can write

$$Q^{k+1}_{1,c} = (Q^k_{1,c})(Q_{1,c}) = \begin{pmatrix} F^\star_{1,c}(k+1) & F^\star_{1,c}(k) \\ F^\star_{1,c}(k) & F^\star_{1,c}(k-1) \end{pmatrix} \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} F^\star_{1,c}(k+2) & F^\star_{1,c}(k+1) \\ F^\star_{1,c}(k+1) & F^\star_{1,c}(k) \end{pmatrix} \text{ by (7)}$$

Hence by induction, we can write

$$Q^n_{1,c} = \begin{pmatrix} F^\star_{1,c}(n+1) & F^\star_{1,c}(n) \\ F^\star_{1,c}(n) & F^\star_{1,c}(n-1) \end{pmatrix}.$$

When $p = 2$, we have to prove

$$Q_{2,c}^{n} = \begin{pmatrix} F_{2,c}^{\star}(n+1) & F_{2,c}^{\star}(n) & F_{2,c}^{\star}(n-1) \\ F_{2,c}^{\star}(n-1) & F_{2,c}^{\star}(n-2) & F_{2,c}^{\star}(n-3) \\ F_{2,c}^{\star}(n) & F_{2,c}^{\star}(n-1) & F_{2,c}^{\star}(n-2) \end{pmatrix}. \tag{11}$$

We will prove it by mathematical induction. For $n = 1$

$$Q_{2,c} = \begin{pmatrix} c & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} F_{2,c}^{\star}(2) & F_{2,c}^{\star}(1) & F_{2,c}^{\star}(0) \\ F_{2,c}^{\star}(0) & F_{2,c}^{\star}(-1) & F_{2,c}^{\star}(-2) \\ F_{2,c}^{\star}(1) & F_{2,c}^{\star}(0) & F_{2,c}^{\star}(-1) \end{pmatrix} \text{ by (7)}$$

which is true for $n = 1$.
For $n = 2$

$$Q_{2,c}^{2} = \begin{pmatrix} c^2 & c & 1 \\ 1 & 0 & 0 \\ c & 1 & 0 \end{pmatrix} = \begin{pmatrix} F_{2,c}^{\star}(3) & F_{2,c}^{\star}(2) & F_{2,c}^{\star}(1) \\ F_{2,c}^{\star}(1) & F_{2,c}^{\star}(0) & F_{2,c}^{\star}(-1) \\ F_{2,c}^{\star}(2) & F_{2,c}^{\star}(1) & F_{2,c}^{\star}(0) \end{pmatrix} \text{ by (7)}$$

which is true for $n = 2$.
Suppose (11) is true for integer $n = k$, then

$$Q_{2,c}^{k} = \begin{pmatrix} F_{2,c}^{\star}(k+1) & F_{2,c}^{\star}(k) & F_{2,c}^{\star}(k-1) \\ F_{2,c}^{\star}(k-1) & F_{2,c}^{\star}(k-2) & F_{2,c}^{\star}(k-3) \\ F_{2,c}^{\star}(k) & F_{2,c}^{\star}(k-1) & F_{2,c}^{\star}(k-2) \end{pmatrix}$$

Now, we can write

$$\begin{aligned} Q_{2,c}^{k+1} = (Q_{2,c}^{k})(Q_{2,c}) &= \begin{pmatrix} F_{2,c}^{\star}(k+1) & F_{2,c}^{\star}(k) & F_{2,c}^{\star}(k-1) \\ F_{2,c}^{\star}(k-1) & F_{2,c}^{\star}(k-2) & F_{2,c}^{\star}(k-3) \\ F_{2,c}^{\star}(k) & F_{2,c}^{\star}(k-1) & F_{2,c}^{\star}(k-2) \end{pmatrix} \begin{pmatrix} c & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} F_{2,c}^{\star}(k+2) & F_{2,c}^{\star}(k+1) & F_{2,c}^{\star}(k) \\ F_{2,c}^{\star}(k) & F_{2,c}^{\star}(k-1) & F_{2,c}^{\star}(k-2) \\ F_{2,c}^{\star}(k+1) & F_{2,c}^{\star}(k) & F_{2,c}^{\star}(k-1) \end{pmatrix} \text{ by (7)} \end{aligned}$$

Hence by induction, we can write

$$Q_{2,c}^{n} = \begin{pmatrix} F_{2,c}^{\star}(n+1) & F_{2,c}^{\star}(n) & F_{2,c}^{\star}(n-1) \\ F_{2,c}^{\star}(n-1) & F_{2,c}^{\star}(n-2) & F_{2,c}^{\star}(n-3) \\ F_{2,c}^{\star}(n) & F_{2,c}^{\star}(n-1) & F_{2,c}^{\star}(n-2) \end{pmatrix}.$$

Therefore, given theorem is true for $p = 2$.

Suppose that formula for $Q^n_{p,c}$ is true for integer $p=k$, then $Q^n_{k,c}$ is equal

$$\begin{pmatrix} F^\star_{k,c}(n+1) & F^\star_{k,c}(n) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{k,c}(n-k+2) & F^\star_{k,c}(n-k+1) \\ F^\star_{k,c}(n-k+1) & F^\star_{k,c}(n-k) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{k,c}(n-2k+2) & F^\star_{k,c}(n-2k+1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ F^\star_{k,c}(n-1) & F^\star_{k,c}(n-2) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{k,c}(n-k) & F^\star_{k,c}(n-k-1) \\ F^\star_{k,c}(n) & F^\star_{k,c}(n-1) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{k,c}(n-k+1) & F^\star_{k,c}(n-k) \end{pmatrix}$$

Therefore, $Q^n_{k+1,c}$ is equal

$$\begin{pmatrix} F^\star_{k,c}(n+1) & F^\star_{k,c}(n) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{k,c}(n-k+1) & F^\star_{k,c}(n-k) \\ F^\star_{k,c}(n-k) & F^\star_{k,c}(n-k-1) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{k,c}(n-2k) & F^\star_{k,c}(n-2k-1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ F^\star_{k,c}(n-1) & F^\star_{k,c}(n-2) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{k,c}(n-k-1) & F^\star_{k,c}(n-k-2) \\ F^\star_{k,c}(n) & F^\star_{k,c}(n-1) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{k,c}(n-k) & F^\star_{k,c}(n-k-1) \end{pmatrix}$$

This implies that $Q^n_{p,c}$ is also true for integer $p=k+1$, which ends the proof.∎

Theorem 3.2 $Q^n_{p,c} = cQ^{n-1}_{p,c} + Q^{n-(p+1)}_{p,c}$ *where p is a positive integer, c is a non zero complex number and n is any integer.*

Proof By Theorem 3.1 we have that $Q^n_{p,c}$ is equal

$$\begin{pmatrix} F^\star_{p,c}(n+1) & F^\star_{p,c}(n) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{p,c}(n-p+2) & F^\star_{p,c}(n-p+1) \\ F^\star_{p,c}(n-p+1) & F^\star_{p,c}(n-p) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{p,c}(n-2p+2) & F^\star_{p,c}(n-2p+1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ F^\star_{p,c}(n-1) & F^\star_{p,c}(n-2) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{p,c}(n-p) & F^\star_{p,c}(n-p-1) \\ F^\star_{p,c}(n) & F^\star_{p,c}(n-1) & \cdot & \cdot & \cdot & \cdot & \cdot & F^\star_{p,c}(n-p+1) & F^\star_{p,c}(n-p) \end{pmatrix}$$

When $p=1$

$$Q^n_{1,c} = \begin{pmatrix} F^\star_{1,c}(n+1) & F^\star_{1,c}(n) \\ F^\star_{1,c}(n) & F^\star_{1,c}(n-1) \end{pmatrix}$$

$$= \begin{pmatrix} cF^\star_{1,c}(n) + F^\star_{1,c}(n-1) & cF^\star_{1,c}(n-1) + F^\star_{1,c}(n-2) \\ cF^\star_{1,c}(n-1) + F^\star_{1,c}(n-2) & cF^\star_{1,c}(n-2) + F^\star_{1,c}(n-3) \end{pmatrix}$$

$$= \begin{pmatrix} cF^{\star}_{1,c}(n) & cF^{\star}_{1,c}(n-1) \\ cF^{\star}_{1,c}(n-1) & cF^{\star}_{1,c}(n-2) \end{pmatrix}$$

$$+ \begin{pmatrix} F^{\star}_{1,c}(n-1) & F^{\star}_{1,c}(n-2) \\ F^{\star}_{1,c}(n-2) & F^{\star}_{1,c}(n-3) \end{pmatrix}$$

$$= cQ^{n-1}_{1,c} + Q^{n-2}_{1,c}.$$

When $p = 2$

$$Q^n_{2,c} = \begin{pmatrix} F^{\star}_{2,c}(n+1) & F^{\star}_{2,c}(n) & F^{\star}_{2,c}(n-1) \\ F^{\star}_{2,c}(n-1) & F^{\star}_{2,c}(n-2) & F^{\star}_{2,c}(n-3) \\ F^{\star}_{2,c}(n) & F^{\star}_{2,c}(n-1) & F^{\star}_{2,c}(n-2) \end{pmatrix}$$

$$= \begin{pmatrix} cF^{\star}_{2,c}(n) + F^{\star}_{2,c}(n-2) & cF^{\star}_{2,c}(n-1) + F^{\star}_{2,c}(n-3) & cF^{\star}_{2,c}(n-2) + F^{\star}_{2,c}(n-4) \\ cF^{\star}_{2,c}(n-2) + F^{\star}_{2,c}(n-4) & cF^{\star}_{2,c}(n-3) + F^{\star}_{2,c}(n-5) & cF^{\star}_{2,c}(n-4) + F^{\star}_{2,c}(n-6) \\ cF^{\star}_{2,c}(n-1) + F^{\star}_{2,c}(n-3) & cF^{\star}_{2,c}(n-2) + F^{\star}_{2,c}(n-4) & cF^{\star}_{2,c}(n-3) + F^{\star}_{2,c}(n-5) \end{pmatrix}$$

$$= \begin{pmatrix} cF^{\star}_{2,c}(n) & cF^{\star}_{2,c}(n-1) & cF^{\star}_{2,c}(n-2) \\ cF^{\star}_{2,c}(n-2) & cF^{\star}_{2,c}(n-3) & cF^{\star}_{2,c}(n-4) \\ cF^{\star}_{2,c}(n-1) & cF^{\star}_{2,c}(n-2) & cF^{\star}_{2,c}(n-3) \end{pmatrix}$$

$$+ \begin{pmatrix} F^{\star}_{2,c}(n-2) & F^{\star}_{2,c}(n-3) & F^{\star}_{2,c}(n-4) \\ F^{\star}_{2,c}(n-4) & F^{\star}_{2,c}(n-5) & F^{\star}_{2,c}(n-6) \\ F^{\star}_{2,c}(n-3) & F^{\star}_{2,c}(n-4) & F^{\star}_{2,c}(n-5) \end{pmatrix}$$

$$= cQ^{n-1}_{2,c} + Q^{n-3}_{2,c}.$$

Suppose that $Q^n_{p,c} = cQ^{n-1}_{p,c} + Q^{n-(p+1)}_{p,c}$ is true for $p = k$.
Therefore, $Q^n_{k,c} = cQ^{n-1}_{k,c} + Q^{n-(k+1)}_{k,c}$. We will show that it is also true for $p = k+1$.
Now, $cQ^{n-1}_{k+1,c}$ is equal

$$\begin{pmatrix} cF^{\star}_{k+1,c}(n) & cF^{\star}_{k+1,c}(n-1) & \cdots & cF^{\star}_{k+1,c}(n-k) & cF^{\star}_{k+1,c}(n-k-1) \\ cF^{\star}_{k+1,c}(n-k-1) & cF^{\star}_{k+1,c}(n-k-2) & \cdots & cF^{\star}_{k+1,c}(n-2k-1) & cF^{\star}_{k+1,c}(n-2k-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ cF^{\star}_{k+1,c}(n-2) & cF^{\star}_{k+1,c}(n-3) & \cdots & cF^{\star}_{k+1,c}(n-k-2) & cF^{\star}_{k+1,c}(n-k-3) \\ cF^{\star}_{k+1,c}(n-1) & cF^{\star}_{k+1,c}(n-2) & \cdots & cF^{\star}_{k+1,c}(n-k-1) & cF^{\star}_{k+1,c}(n-k-2) \end{pmatrix} \tag{12}$$

and $Q_{k+1,c}^{n-(k+2)}$ is equal

$$\begin{pmatrix} F^{\star}_{k+1,c}(n-k-1) & F^{\star}_{k+1,c}(n-k-2) & \cdots & F^{\star}_{k+1,c}(n-2k-1) & F^{\star}_{k+1,c}(n-2k-2) \\ F^{\star}_{k+1,c}(n-2k-2) & F^{\star}_{k+1,c}(n-2k-3) & \cdots & F^{\star}_{k+1,c}(n-3k-2) & F^{\star}_{k+1,c}(n-3k-3) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F^{\star}_{k+1,c}(n-k-3) & F^{\star}_{k+1,c}(n-k-4) & \cdots & F^{\star}_{k+1,c}(n-2k-3) & F^{\star}_{k+1,c}(n-2k-4) \\ F^{\star}_{k+1,c}(n-k-2) & F^{\star}_{k+1,c}(n-k-3) & \cdots & F^{\star}_{k+1,c}(n-2k-2) & F^{\star}_{k+1,c}(n-2k-3) \end{pmatrix} \quad (13)$$

Adding equations (12) and (11), we get $cQ_{k+1,c}^{n-1} + Q_{k+1,c}^{n-(k+1)} =$

$$\begin{pmatrix} F^{\star}_{k,c}(n+1) & F^{\star}_{k,c}(n) & \cdots & F^{\star}_{k,c}(n-k+1) & F^{\star}_{k,c}(n-k) \\ F^{\star}_{k,c}(n-k) & F^{\star}_{k,c}(n-k-1) & \cdots & F^{\star}_{k,c}(n-2k) & F^{\star}_{k,c}(n-2k-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F^{\star}_{k,c}(n-1) & F^{\star}_{k,c}(n-2) & \cdots & F^{\star}_{k,c}(n-k-1) & F^{\star}_{k,c}(n-k-2) \\ F^{\star}_{k,c}(n) & F^{\star}_{k,c}(n-1) & \cdots & F^{\star}_{k,c}(n-k) & F^{\star}_{k,c}(n-k-1) \end{pmatrix} = Q_{k+1,c}^{n}$$

This end the proof of Theorem 3.2. ■

THEOREM 3.3 For a given positive integer p and for any integer n the determinant of the matrix $Q_{p,c}^{n}$ is given by Det $Q_{p,c}^{n} = (-1)^{pn}$.

PROOF We have

$$\mathbf{Det}Q_{1,c} = -1.$$

Therefore, by matrix theory we have,

$$\begin{aligned} \mathbf{Det}Q_{2,c} &= (-1)^5\mathbf{Det}Q_{1,c} = 1 = (-1)^2 \\ \mathbf{Det}Q_{3,c} &= (-1)^7\mathbf{Det}\ Q_{2,c} = -1 = (-1)^3 \\ \mathbf{Det}Q_{4,c} &= (-1)^9\mathbf{Det}Q_{3,c} = 1 = (-1)^4 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathbf{Det}Q_{p,c} &= (-1)^p \end{aligned}$$

and $\mathbf{Det}Q_{p,c}^{n} = (-1)^{pn}$.

This end of Theorem 3.3. ■

THEOREM 3.4 For a given positive integer p and c is a non zero complex number the determinant of the matrix $Q_{p,c}^{-1}$ is given by Det $Q_{p,c}^{-1}$=Det $Q_{p,c}$.

Proof: We know that $Q_{p,c}Q_{p,c}^{-1} = I$ where I is the identity matrix. Therefore, Det $(Q_{p,c}Q_{p,c}^{-1})$=Det I
$\Rightarrow$ Det $Q_{p,c}$ Det $Q_{p,c}^{-1}$=Det I
$\Rightarrow$ Det $Q_{p,c}^{-1}$=Det $Q_{p,c}$, since the Det $Q_{p,c}$ and Det $Q_{p,c}^{-1}$ are 1 or -1 by the previous theorem 3.3.

4. Complex Fibonacci $Q_{p,c}$ coding and decoding method The complex Fibonacci $Q_{p,c}$ matrix allows to develop the applications to the coding theory. Let us represent the initial message in the form of the nonsingular square matrix M of order $(p+1)$ where p is positive integer. We take $Q_{p,c}^n$ matrix of order $(p+1)$ as a coding matrix and its inverse matrix $(Q_{p,c}^n)^{-1}$ as a decoding matrix . We name a transformation $MQ_{p,c}^n = E$ as coding and a transformation $E(Q_{p,c}^n)^{-1} = M$ as decoding and E is known as code matrix Stakhov (2006).

For example, consider the case for $p = 1$, we represent the initial message M in the form of nonsingular square matrix of order 2

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \tag{14}$$

where all elements of the matrix are positive integers. i.e. $m_1, m_2, m_3, m_4 > 0$. Let us select for any value of n, the $Q_{1,c}^n$ matrix treated as the coding matrix. For $n = 4$ we have

$$Q_{1,c}^4 = \begin{pmatrix} c^4+3c^2+1 & c^3+2c \\ c^3+2c & c^2+1 \end{pmatrix} \tag{15}$$

Then the coding of the message (14) consists of the multiplication by the initial matrix (15) that is

$$\begin{aligned} MQ_{1,c}^4 &= \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} c^4+3c^2+1 & c^3+2c \\ c^3+2c & c^2+1 \end{pmatrix} \\ &= \begin{pmatrix} m_1c^4+3m_1c^2+m_1+m_2c^3+2m_2c & m_1c^3+2m_1c+m_2c^2+m_2 \\ m_3c^4+3m_3c^2+m_3+m_4c^3+2m_4c & m_3c^3+2m_3c+m_4c^2+m_4 \end{pmatrix} \\ &= \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} = E, \end{aligned} \tag{16}$$

where $e_1 = m_1c^4+3m_1c^2+m_1+m_2c^3+2m_2c$, $e_2 = m_1c^3+2m_1c+m_2c^2+m_2$, $e_3 = m_3c^4+3m_3c^2+m_3+m_4c^3+2m_4c$, $e_4 = m_3c^3+2m_3c+m_4c^2+m_4$. The inverse matrix of (15) is given by

$$(Q_{1,c}^4)^{-1} = \begin{pmatrix} c^2+1 & -c^3-2c \\ -c^3-2c & c^4+3c^2+1 \end{pmatrix} \tag{17}$$

The decoding of the code message E by (16) is

$$E(Q_{p,c}^4)^{-1} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \begin{pmatrix} c^2+1 & -c^3-2c \\ -c^3-2c & c^4+3c^2+1 \end{pmatrix}$$
$$= \begin{pmatrix} e_1c^2+e_1-e_2c^3-2e_2c & -e_1c^3-2e_1c+e_2c^4+3e_2c^2+e_2 \\ e_3c^2+e_3-e_4c^3-2e_4c & -e_3c^3-2e_3c+e_4c^4+3e_4c^2+e_4 \end{pmatrix}$$
$$= \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = M.$$

The code matrix E is defined by the following formula $E = MQ_{p,c}^n$. According to the matrix theory, we have

$$Det\ E = Det\ (MQ_{p,c}^n) = Det\ MDet\ Q_{p,c}^n = Det\ M(-1)^{pn} = (-1)^{pn}Det\ M$$

5. Relations among the code matrix elements **Case 1:** For $p = 1$, Similar to [10], we obtain $\frac{e_1}{e_2} \approx \mu_1,_c$; $\frac{e_3}{e_4} \approx \mu_1,_c$
where $\mu_1,_c = \frac{c+\sqrt{c^2+4}}{2}$, e_1, e_2, e_3, e_4 are given in (16).
Case 2: For $p = 2$, In this case, let the message
$M = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}$ then the $Q_{2,c}^n$ coding of the message M is equal

$$MQ_{2,c}^n = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} = E.$$

Similar to Prasad (2019), we obtain $\frac{e_1}{e_2} \approx \mu_{2,c}$; $\frac{e_2}{e_3} \approx \mu_{2,c}$ and $\frac{e_1}{e_3} \approx \mu_{2,c}^2$
$\frac{e_4}{e_5} \approx \mu_{2,c}$; $\frac{e_5}{e_6} \approx \mu_{2,c}$ and $\frac{e_4}{e_6} \approx \mu_{2,c}^2$
$\frac{e_7}{e_8} \approx \mu_{2,c}$; $\frac{e_8}{e_9} \approx \mu_{2,c}$ and $\frac{e_7}{e_9} \approx \mu_{2,c}^2$
where $\mu_{2,c} = \frac{h^2+2hc+4c^2}{6h}$ and $h = \sqrt{108+8c^3+12\sqrt{81+12c^3}}$.
In general, like [10], when $p = t$ and $n > p+1 = t+1$, The generalized relations among the code matrix elements are
$\frac{e_1}{e_2} \approx \mu_{t,c}$; $\frac{e_2}{e_3} \approx \mu_{t,c}$; $\cdots$; $\frac{e_t}{e_{t+1}} \approx \mu_{t,c}$
$\frac{e_1}{e_3} \approx \mu_{t,c}^2$; $\frac{e_2}{e_4} \approx \mu_{t,c}^2$; $\cdots$; $\frac{e_{t-1}}{e_{t+1}} \approx \mu_{t,c}^2$
$\cdots$
$\cdots$
$\frac{e_1}{e_{t+1}} \approx \mu_{t,c}^t$
where e_1, e_2, e_3, $\cdots$, e_t, e_{t+1} are the first row elements of the code matrix E and $\mu_{t,c}$ is golden (t,c)-proportion. We also obtain similar type of relations among the elements of the second row, third row, $\cdots$, $(t+1)$th row of the code matrix E.

6. Error detection and correction For the simplest case $p = 1$ the correction ability of the method is 93.33% [10] which exceeds the essentially all well known correcting codes. The correction ability of the method for $p = 2$ is 99.80% [10]. In general, for $p = t$ and $n > p + 1 = t + 1$ the correction ability of the method is $\frac{2^{(t+1)^2}-2}{2^{(t+1)^2}-1}$ which depends on p but not on c. Hence, for large value of p the correction ability of the method is $\frac{2^{(p+1)^2}-2}{2^{(p+1)^2}-1} \approx 1 = 100\%$.

7. Conclusion The complex Fibonacci coding and decoding method is the main application of the complex Fibonacci $Q_{p,c}$ matrix. There lies a difference between the classical algebraic coding and complex Fibonacci $Q_{p,c}$ coding method. The accuracy of complex Fibonacci $Q_{p,c}$ coding method is given below:

(1) This coding and decoding method converts to matrix multiplication. Nowadays it can be done very quickly by computer for large values of p.

(2) The correction ability of the method increases as p increases and it is independent of c.

(3) Complex Fibonacci $Q_{p,c}$ matrix coincides with Golden Matrix for $p = 1$, $c = 1$ which develops a new kind of cryptography (cf. Stakhov (2007).

(4) In future, based on the works of Flaut (2019), El Naschie (2006, 2007, 1995, 2004, 2006), Stakhov (2006), etc. we hope that the complex Fibonacci $Q_{p,c}$ matrix can also have wide applications in matrix theory, complex number theory, cryptography and information and coding theory.

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Złożone (c, p)-liczby Fibonacci'ego.
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Streszczenie W artykule przedstawiono nową macierz zespoloną Fibonacciego oznaczaną $Q_{p,c}$ dla liczb zespolonych Fibonacciego (c, p), gdzie p jest liczbą całkowitą dodatnią, a c jest niezerową liczbą zespoloną. Omówiono różne własności macierzy $Q_{p,c}$, oraz sposób kodowania i dekodowania wynikający z macierzy $Q_{p,c}$.

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