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ON ASSOCIATIVE RATIONAL FUNCTIONS WITH MULTIPLICATIVE GENERATORS

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Abstract

We consider the class of rational functions defined by the formula

$$F(x,y) = \varphi^{-1}(\varphi(x)\varphi(y)),$$

where φ is a homographic function and we describe associative functions of the above form.

1. Motivation

The functional equation of the form

$$f(x+y) = F(f(x), f(y)), \quad x, y \in S$$

where F is an associative rational function and S is a group or a semigroup, is called an addition formula. For the rational two-place real-valued function F given by

$$F(x,y) = \varphi^{-1}(\varphi(x)\varphi(y)),$$

where φ is a homographic function (such F is called a function with a multiplicative generator), the addition formula has the form

$$h(x+y) = h(x)h(y), \quad x, y \in S$$

where $h := \varphi \circ f$ and it is a conditional functional equation if the domain of φ is not equal to $\mathbb{R}$.

It seems worth considering which homographic functions φ make F of the above form to be associative.

The following functions (with natural domains in question) are the only associative members of the class $\mathcal{F}$ of rational functions of the form

$$F(x,y) = \frac{a_1xy + a_2(x+y) + a_3}{a_4xy + a_5(x+y) + a_6},$$

where $a_i = 0$ for at last one of $i \in \{1, ..., 6\}$ (see [1]):

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$$F(x,y)=\frac{x+y+\beta}{\alpha xy+\alpha\beta(x+y)+\alpha\beta^2+1}\quad,\quad \alpha,\beta\neq 0;$$

$$F(x,y)=\frac{(1+\alpha\beta)xy+\alpha(x+y)+\frac{\alpha}{\beta}}{\beta xy+x+y}\quad,\quad \alpha\in\mathbb{R},\beta\neq 0;$$

$$F(x,y)=\frac{\alpha xy}{xy+\beta(x+y)+\beta(\beta-\alpha)}\quad,\quad \alpha,\beta\neq 0;$$

$$F(x,y)=\frac{xy+\alpha}{x+y+\beta}\quad,\quad \alpha,\beta\in\mathbb{R};$$

$$F(x,y)=\frac{\alpha xy+x+y}{\beta xy+1}\quad,\quad \alpha\in\mathbb{R},\beta\neq 0.$$

We examine which functions of the above form have a multiplicative generator.

The following lemma will be useful in the sequel.

Lemma. *Let $A,B,C,D\in\mathbb{R}$ be given and let $AD\neq BC$, $C\neq 0$. For φ given by*

$$\varphi(x)=\frac{Ax+B}{Cx+D},$$

it holds

$$\varphi^{-1}(\varphi(x)\varphi(y))=\frac{(BC^2-A^2D)xy-BD(A-C)(x+y)-BD(B-D)}{AC(A-C)xy+AC(B-D)(x+y)+B^2C-AD^2}.$$

Proof. We have

$$\varphi(x)\varphi(y)=\frac{A^2xy+AB(x+y)+B^2}{C^2xy+CD(x+y)+D^2}$$

and

$$\varphi^{-1}(x)=\frac{-Dx+B}{Cx-A}.$$

A simple calculation shows that the above equation holds true.

□

2. Results

We proceed with a description of the class $\mathcal{F}$ of rational functions of the form

$$F(x,y)=\frac{a_1xy+a_2(x+y)+a_3}{a_4xy+a_5(x+y)+a_6}$$

(where $a_i=0$ for at last one of $i\in\{1,...,6\}$) with multiplicative generators.

The case of F being a polynomial is trivial, then in the light of Lemma, we can consider only such homographies φ that $A,C\neq 0$ and $(B-D)^2+(C-A)^2\neq 0$.

Theorem 1. *The following functions (with natural domains in question) are the only associative members with multiplicative generators of the class* $\mathcal{F}$:

$$F(x,y)=\frac{xy}{(a-1)bxy+a(x+y)+\frac{a}{b}}\quad,\quad a,b\neq 0;$$

$$F(x,y)=\frac{xy-ab}{x+y+a+b}\quad,\quad a,b\in\mathbb{R},a\neq b;$$

$$F(x,y)=\frac{(a+b)xy+x+y}{1-abxy}\quad,\quad a,b\neq 0,a\neq b;$$

$$F(x,y)=-\frac{(a^2+ab+b^2)xy+(a+b)(x+y)+1}{(a^2b+ab^2)xy+ab(x+y)}\quad,\quad a,b\neq 0,a\neq b;$$

$$F(x,y)=-\frac{ab(x+y)+a^2b+ab^2}{xy+(a+b)(x+y)+a^2+ab+b^2}\quad,\quad a,b\neq 0,a\neq b.$$

Proof. Assume that F is associative and that it has a multiplicative generator

$$\varphi(x)=\frac{Ax+B}{Cx+D},$$

where $A,B,C,D\in\mathbb{R}$ and $AD\neq BC$.

From Lemma, we get that

$$F(x,y)=\frac{(BC^2-A^2D)xy-BD(A-C)(x+y)-BD(B-D)}{AC(A-C)xy+AC(B-D)(x+y)+B^2C-AD^2}\qquad(\star)$$

First, assume that $B=0$. If $D=0$, then $AD=BC$, which contradicts the assumption. Hence $D\neq 0$. Taking $B=0$ in $(\star)$, we obtain

$$\begin{aligned}F(x,y)&=\frac{-A^2Dxy}{-AC(C-A)xy-ACD(x+y)-AD^2}\\&=\frac{xy}{\frac{C}{D}(\frac{C}{A}-1)xy+\frac{C}{A}(x+y)+\frac{D}{A}}\end{aligned}$$

Consequently, taking

$$a = \frac{C}{A}, \quad b = \frac{C}{D},$$

we infer that

$$F(x,y) = \frac{xy}{(a-1)bxy + a(x+y) + \frac{a}{b}} \quad , \quad a, b \neq 0.$$

Now, let $D = 0$. If $B = 0$ then $AD = BC$, which contradicts the assumption. Thus, $B \neq 0$. Taking $D = 0$ in $(\star)$, we have

$$\begin{aligned} F(x,y) &= \frac{BC^2xy}{AC(A-C)xy + ABC(x+y) + B^2C} \\ &= \frac{xy}{\frac{A}{BC}(A-C)xy + \frac{A}{C}(x+y) + \frac{B}{C}} \end{aligned}$$

Consequently, taking

$$a = \frac{A}{C}, \quad b = \frac{A}{B},$$

we infer that

$$F(x,y) = \frac{xy}{(a-1)bxy + a(x+y) + \frac{a}{b}} \quad , \quad a, b \neq 0.$$

In case $A = C$, we have $B \neq D$ and, by means of $(\star)$, we obtain

$$\begin{aligned} F(x,y) &= \frac{(BC^2 - A^2D)xy + BD(D-B)}{AC(B-D)(x+y) + B^2C - AD^2} \\ &= \frac{(B-D)xy + \frac{BD}{A^2}(D-B)}{(B-D)(x+y) + (B^2-D^2)\frac{1}{A}} = \frac{xy - ab}{x+y+a+b} \end{aligned}$$

with $a \neq b, a = \frac{B}{A}, b = \frac{D}{A}$.

In case $B = D$, we have $A \neq C$ and again, by means of $(\star)$, we infer that

$$\begin{aligned} F(x,y) &= \frac{(BC^2 - A^2D)xy + BD(C-A)(x+y)}{AC(A-C)xy + B^2C - AD^2} \\ &= \frac{(C^2 - A^2)\frac{1}{B}xy + (C-A)(x+y)}{-\frac{A}{B}\cdot\frac{C}{B}(C-A)(x+y) + C - A} = \frac{(a+b)xy + x + y}{1 - abxy} \end{aligned}$$

with $a \neq b, a = \frac{A}{B}, b = \frac{C}{B}$.

Now, assume that $A, B, C, D \neq 0, A \neq C, B \neq D$. Let $B^2C - AD^2 = 0$. Applying this to $(\star)$, we have

$$F(x,y) = \frac{\left(BC^2 - \frac{B^4C^2}{D^4}D\right)xy + BD\left(C - \frac{B^2C}{D^2}\right)(x+y) + BD(D-B)}{\frac{B^2C^2}{D^2}\left(\frac{B^2C}{D^2} - C\right)xy + \frac{B^2C^2}{D^2}(B-D)(x+y)} =$$

$$\frac{\frac{C^2(D^3-B^3)}{D^3}xy + CD\frac{D^2-B^2}{D^2}(x+y) + D(D-B)}{\frac{BC^3}{D^2}\frac{B^2-D^2}{D^2}xy + \frac{BC^2}{D^2}(B-D)(x+y)} =$$

$$-\frac{\frac{C^2}{D^3}(D^2+BD+B^2)xy + \frac{C}{D}(C+D)(x+y) + D}{\frac{BC^3}{D^2}\cdot\frac{B+D}{D^2}xy + \frac{BC^2}{D^2}(x+y)} =$$

$$-\frac{\left(\frac{C^2}{D^2} + \frac{BC^2}{D^3} + \frac{B^2C^2}{D^4}\right)xy + \left(\frac{C}{D} + \frac{BC}{D^2}\right)(x+y) + 1}{\left(\frac{B^2C^3}{D^5} + \frac{BC^3}{D^4}\right)xy + \frac{BC^2}{D^3}(x+y)} =$$

$$-\frac{(a^2+ab+b^2)xy + (a+b)(x+y) + 1}{(a^2b+ab^2)xy + ab(x+y)},$$

where $a = \frac{BC}{D^2}, b = \frac{C}{D}$. Obviously $a \neq b$.

At last, assume that $BC^2 - A^2D = 0$ and $A, B, C, D \neq 0, A \neq C, B \neq D$. Applying this in $(\star)$, we obtain

$$F(x,y) = \frac{\frac{B^2C^2}{A^2}(C-A)(x+y) + \frac{B^2C^2}{A^2}B\left(\frac{C^2}{A^2} - 1\right)}{AC(A-C)xy + ABC(1 - \frac{C^2}{A^2})(x+y) + B^2C - AB^2\frac{C^4}{A^4}} =$$

$$-\frac{\frac{B^2C}{A^2}(C-A)(x+y) + \frac{B^3C}{A^4}(C^2-A^2)}{A(C-A)xy + \frac{B}{A}(C^2-A^2)(x+y) + \frac{B^2}{A^3}(C^3-A^3)} =$$

$$-\frac{\frac{B^2C}{A^2}(x+y) + \frac{B^3C}{A^4(A+C)}}{Axy + \frac{B}{A}(A+C)(x+y) + \frac{B^2}{A^3}(A^2+AC+C^2)} =$$

$$-\frac{\frac{B^2C}{A^3}(x+y) + \frac{B^3C^2}{A^5} + \frac{B^3C}{A^4}}{xy + \left(\frac{BC}{A^2} + \frac{B}{A}\right)(x+y) + \frac{B^2}{A^2} + \frac{B^2C}{A^3} + \frac{B^2C^2}{A^4}} =$$

$$-\frac{ab(x+y) + a^2b + ab^2}{xy + (a+b)(x+y) + a^2 + ab + b^2}$$

with $a = \frac{BC}{A^2}, b = \frac{B}{A}$. It is clear that $a \neq b$.

It is easy to check (see Theorem 2 or Theorem 1 in [1]) that each of the functions above yields to a rational associative function. Thus, the proof has been completed. □

Now, we determine homographic functions φ which by means of the formula

$$F(x,y) = \varphi^{-1}(\varphi(x)\varphi(y)) \qquad (\star\star)$$

lead to associative functions F.

Theorem 2. *For the following homographic functions (with natural domains in question) we obtain by* $(\star\star)$ *rational associative functions with a multiplicative generators:*

$$\varphi(x) = \frac{1}{d} \cdot \frac{cx}{cx+1}$$

$$\varphi(x) = \frac{x+a}{x+b}$$

$$\varphi(x) = \frac{ax+1}{bx+1}$$

$$\varphi(x) = \frac{a}{b} \cdot \frac{ax+1}{bx+1}$$

$$\varphi(x) = \frac{a}{b} \cdot \frac{x+a}{x+b},$$

where $a \neq b$ *and* $a, b, c, d \in \mathbb{R} \setminus \{0\}$ *are arbitrary constants.*

Proof. It is easy to check that each of the functions above is a generator of the rational associative function. Moreover, they generate

$$F(x,y) = \frac{xy}{(d-1)cxy + d(x+y) + \frac{d}{c}} \quad , \quad a, b \neq 0;$$

$$F(x,y) = \frac{xy - ab}{x+y+a+b} \quad , \quad a, b \in \mathbb{R}, a \neq b;$$

$$F(x,y) = \frac{(a+b)xy + x + y}{1 - abxy} \quad , \quad a, b \neq 0, a \neq b;$$

$$F(x,y) = -\frac{(a^2+ab+b^2)xy + (a+b)(x+y) + 1}{(a^2b+ab^2)xy + ab(x+y)} \quad , \quad a, b \neq 0;$$

$$F(x,y) - \frac{ab(x+y) + a^2b + ab^2}{xy + (a+b)(x+y) + a^2 + ab + b^2} \quad , \quad a, b \neq 0,$$

respectively. Thus, according to Theorem 1, the proof is completed. □

Notice that for any homography φ the following equality is fullfiled:

$$\varphi^{-1}(\varphi(x)\varphi(y)) = \breve{\varphi}^{-1}(\breve{\varphi}(x)\breve{\varphi}(y)),$$

where $\breve{\varphi} = \frac{1}{\varphi}$.

At last, let us observe that

$$F(x,y) = \frac{xy}{(a-1)bxy + a(x+y) + \frac{a}{b}} \quad , \quad a,b \neq 0$$

is of the form

$$F(x,y) = \frac{\alpha xy}{xy + \beta(x+y) + \beta(\beta - \alpha)}$$

with $\alpha = \frac{1}{(a-1)b}, \beta = \frac{a}{(a-1)b}$ in case $a \neq 1$ and of the form

$$F(x,y) = \frac{xy + \alpha}{x + y + \beta}$$

with $\alpha = 0, \beta = \frac{1}{b}$, otherwise, i.e. $a = 1$.

The rational function

$$F(x,y) = \frac{xy - ab}{x + y + a + b} \quad , \quad a,b \in \mathbb{R}, a \neq b$$

can be written in the form

$$F(x,y) = \frac{xy + \alpha}{x + y + \beta}.$$

It is clear that

$$F(x,y) = \frac{(a+b)xy + x + y}{1 - abxy} \quad , \quad a,b \neq 0, a \neq b$$

is of the form

$$F(x,y) = \frac{\alpha xy + x + y}{\beta xy + 1}.$$

Further,

$$F(x,y) = -\frac{(a^2 + ab + b^2)xy + (a+b)(x+y) + 1}{(a^2 b + ab^2)xy + ab(x+y)} \quad , \quad a,b \neq 0, a \neq b$$

is of the form

$$F(x,y) = \frac{(1 + \alpha\beta)xy + \alpha(x+y) + \frac{\alpha}{\beta}}{\beta xy + x + y}$$

with $\alpha = -\frac{a+b}{ab}, \beta = a + b$ if $a + b \neq 0$ and of the form

$$F(x,y) = \frac{xy + \alpha}{x + y + \beta}$$

with $\alpha = \frac{1}{a^2}, \beta = 0$, otherwise, i.e. $a + b = 0$.

Finally,

$$F(x,y) = -\frac{ab(x+y)+a^2b+ab^2}{xy+(a+b)(x+y)+a^2+ab+b^2} \quad , \quad a,b \neq 0, a \neq b$$

is of the form

$$F(x,y) = \frac{x+y+\beta}{\alpha xy+\alpha\beta(x+y)+\alpha\beta^2+1}$$

with $\alpha = -\frac{1}{ab}, \beta = a+b$ if $a+b \neq 0$ and of the form

$$F(x,y) = \frac{\alpha xy+x+y}{\beta xy+1}$$

with $\alpha = 0, \beta = \frac{1}{a^2}$, otherwise, i.e. $a+b=0$.

Associative rational functions with an additive generator are described in [2].

References

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