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Influence of Temperature on Dynamic Properties of Plates with Viscoelastic Dampers

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Abstract

The free damped vibrations of thin (Kirchhoff-Love) plates equipped with viscoelastic dampers are considered in the paper. It is assumed that the dampers are described according to the generalized rheological model. Influence of temperature on the parameters of dampers is taken into account using the frequency-temperature correspondence principle. Isotropic and rectangular plates are analysed in numerical tests included in this study. The natural frequencies and non-dimensional damping ratios are determined for these plates by solving the non-linear eigenproblem using the continuation method. The Finite Element Method is used to determine the stiffness matrix and the mass matrix occurring in the considered eigenproblem. The results of exemplary numerical calculations are presented and discussed at the end of this paper.

Keywords: thin plates, free damped vibrations, viscoelastic dampers, finite element method, frequency-temperature correspondence principle, continuation method.

1. Introduction

Equipping a structure with vibration dampers is one of the ways of reducing its susceptibility to dynamic influences. In structures subjected to vibrations, it is often necessary to change their dynamic parameters, first of all, the natural frequencies. Devices that make it possible may be viscoelastic dampers discussed in this paper. The study of structures equipped with this type of dampers is the subject of many scientific works. In [1] the vibrations of shear frames are considered taking into account the influence of temperature on the properties of viscoelastic dampers installed on the structure. The authors of [2] investigate the behavior of a viscoelastic material e.g. under the influence of temperature increase. The method of solving the equation of motion of a structure with viscoelastic dampers is discussed in [3] where the continuation method is explained and used for numerical tests.

The free damped vibration analysis of rectangular thin (Kirchhoff-Love) plates with viscoelastic dampers is considered in the paper. It is assumed that the dampers are attached to the plate with one end in selected points of its surface and the other end to the rigid base. All dampers have the same properties within one plate. A generalized rheological model is used to describe the dampers. Their parameters depend on the temperature according to the frequency-temperature correspondence principle that is discussed e.g. in [1,2,4,5]. The Finite Element Method (FEM) is used to describe a plate deformation. The rectangular four-node finite elements described in [6] are used for discretization of the plate surface in accordance with FEM principles.

2. Description of the viscoelastic damper model

In this paper, a description of viscoelastic damper according to a generalized rheological model is assumed. This model is discussed, inter alia, in [1,3], while in [3] one can also find another, alternative model called the fractional.

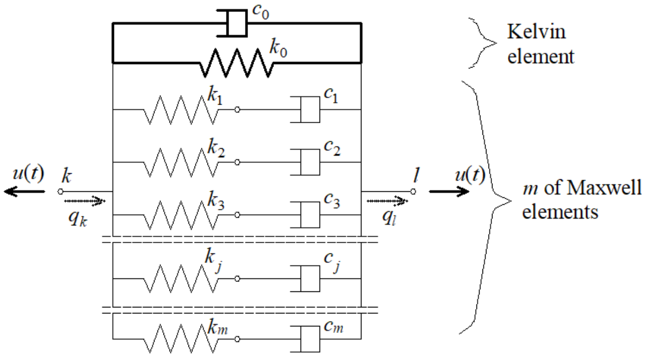

Figure 1. Viscoelastic damper according to the generalized rheological model

Using the classic description, the viscoelastic damper can be shown graphically as in Fig. 1. It can be seen from the figure that the damper consists of $m+1$ elements. Each of these elements contains a viscous part with the constant c_j (dashpot) and an elastic part (spring) with the constant k_j where $j = 0,1,2,\dots,m$. Element number zero is called a Kelvin element and the remaining m elements are Maxwell elements.

The time-dependent force in the damper, denoted as $u(t)$, is the sum of the forces occurring in the individual elements, i.e.

$$u(t) = \sum_{j=0}^{m} u_j(t). \tag{1}$$

For $j = 0$, the force in the Kelvin element is expressed as the following formula:

$$u_0(t) = k_0 \Delta q(t) + c_0 \Delta \dot{q}(t), \tag{2}$$

where k_0, c_0 are respectively the stiffness and the damping parameters of the Kelvin element and $\Delta q(t) = q_l(t) - q_k(t)$ is the relative displacement of the damper (i.e. the difference of the displacements of the ends l and k of the damper). For $j = 1,2,\dots,m$, the force in the j-th Maxwell element satisfies the following formula:

$$\nu_j u_j(t) + \dot{u}_j(t) = k_j \Delta \dot{q}(t) \tag{3}$$

where $\nu_j = \frac{k_j}{c_j}$ is the quotient of the stiffness and damping coefficients of the j-th Maxwell element.

Using Laplace transform ($\mathcal{L}$-transform) with zero initial conditions for formulas (2) and (3) causes them to transform into forms (4) and (5), respectively.

$$\bar{u}_0(s) = k_0 \Delta\bar{q}(s) + sc_0 \Delta\bar{q}(s) \tag{4}$$

$$\bar{u}_j(s) = \frac{k_j s}{\nu_j + s} \Delta\bar{q}(s) \tag{5}$$

In the above formulas, s is an $\mathcal{L}$-transform variable and $\bar{u}_j(s)$, $\Delta\bar{q}(s)$ are respectively $\mathcal{L}$-transforms of the time-dependent force function $u_j(t)$ in the j-th damper element and the relative displacement function $\Delta q(t)$ of the damper. Laplace transform of the total force $u(t)$ in the damper takes the following form:

$$\bar{u}(s) = \sum_{j=0}^{m} \bar{u}_j(s) = \left(k_0 + sc_0 + \sum_{j=1}^{m} \frac{k_j s}{\nu_j + s} \right) \Delta\bar{q}(s). \tag{6}$$

Formula (6) can be written in a shorter form as below:

$$\bar{u}(s) = (K_r + C_r(s) + G_r(s)) \Delta\bar{q}(s), \tag{7}$$

where

$$K_r = k_0;\ C_r(s) = sc_0;\ G_r(s) = \sum_{j=1}^{m} \frac{k_j s}{\nu_j + s}. \tag{8}$$

3. Description of the dependence of dampers parameters on temperature

The stiffness and damping parameters k_j, c_j of the individual elements constituting the viscoelastic damper attached to the structure depend on the temperature.

Let the damper parameters be known for a certain reference temperature T_0 and have values $\bar{k}_j, \bar{c}_j$. Then, for a different temperature T the dampers have parameters k_j, c_j that can be determined e.g. using the frequency-temperature correspondence principle. A more detailed explanation of this principle is contained e.g. in [1,4].

According to [1], the following relationships can be written for a damper performing excited harmonic vibrations:

$$\Delta q(t) = \Delta Q e^{i\lambda t}; u(t) = U e^{i\lambda t};\ u_j(t) = U_j e^{i\lambda t}; j = 0,1,2,\dots,m, \tag{9}$$

where λ is the frequency of excitation, ΔQ, U and U_j are the amplitudes of the relative displacement of the damper, the total force in the damper and the force in the j-th damper element, respectively. After substituting (9) to (1) – (3), the following relationship is obtained:

$$U = \left[k_0 + i\lambda c_0 + \sum_{j=1}^{m} \frac{i\lambda k_j}{\nu_j + i\lambda} \right] \Delta Q = K^*(\lambda)\Delta Q = [K'(\lambda) + iK''(\lambda)]\Delta Q. \tag{10}$$

In the formula above, $K^*(\lambda)$ is the complex modulus. Its real part $K'(\lambda)$ is called the storage modulus and the imaginary part $K''(\lambda)$ is called the loss modulus. From formula (10) it follows that:

$$K'(\lambda) = k_0 + \sum_{j=1}^{m} k_j \frac{\lambda^2}{\nu_j^2 + \lambda^2};\ K''(\lambda) = \lambda c_0 + \sum_{j=1}^{m} k_j \frac{\lambda \nu_j}{\nu_j^2 + \lambda^2}. \tag{11}$$

In accordance with the frequency-temperature correspondence principle, it is possible to link $K'(\lambda,T)$ and $K''(\lambda,T)$ modules, determined for different temperatures, using the following relationships [1]:

$$K'(\lambda,T) = K'(\alpha_T\lambda,T_0);\; K''(\lambda,T) = K''(\alpha_T\lambda,T_0). \tag{12}$$

where α_T is the shift factor relating T to T_0.
Using (11) and (12) it can be written that

$$k_0 + \sum_{j=1}^{m} k_j \frac{\lambda^2}{\nu_j^2+\lambda^2} = \bar{k}_0 + \sum_{j=1}^{m} \bar{k}_j \frac{\alpha_T^2\lambda^2}{\bar{\nu}_j^2+\alpha_T^2\lambda^2}; \tag{13a}$$

$$\lambda c_0 + \sum_{j=1}^{m} k_j \frac{\lambda\nu_j}{\nu_j^2+\lambda^2} = \alpha_T\lambda\bar{c}_0 + \sum_{j=1}^{m} \bar{k}_j \frac{\alpha_T\lambda\bar{\nu}_j}{\bar{\nu}_j^2+\alpha_T^2\lambda^2}, \tag{13b}$$

It follows from the above formulas that the damper model constants at temperature T can be expressed as below:

$$k_j = \bar{k}_j;\; c_j = \bar{c}_j\alpha_T;\; j = 0,1,2,\dots,m. \tag{14}$$

The shift factor α_T is the function of the temperature T and can be expressed by the William-Landel-Ferry formula given e.g. in [1,4]:

$$\log_{10}\alpha_T = \frac{-C_1(T-T_0)}{C_2+T-T_0}, \tag{15}$$

where C_1, C_2 are constants that are determined experimentally.

4. Description of the plate model according to the Finite Element Method

In the Finite Element Method, the center plane of the plate is divided into a finite number of elements. In this study, rectangular plate finite elements of plQ4 type, described in [6], are used. Each such finite element is characterized by four nodes with three degrees of freedom at each node. Thus, the deformation vector $\boldsymbol{w}_i^e$ of the i-th node in the finite element e is defined by three quantities: deflection w_i and two angles of rotation φ_{ix} and φ_{iy}, so it can be written that

$$\boldsymbol{w}_i^e = \begin{bmatrix} w_i & \varphi_{ix} & \varphi_{iy} \end{bmatrix}^T = \begin{bmatrix} w_i & \frac{\partial w_i}{\partial y} & -\frac{\partial w_i}{\partial x} \end{bmatrix}^T; i = 1,2,3,4. \tag{16}$$

The displacement field within each finite element is approximated with a fourth order polynomial $w^e(x,y)$ of two variables x and y, the formula of which is given in [6]. This polynomial has twelve unknown coefficients α_k $(k = 1,2,3,\dots,12)$ due to the number of degrees of freedom in one finite element.

For each finite element e, twelve shape functions $N_k^e(x,y)$ $(k = 1,2,3,\dots,12)$ are determined. Each shape function $N_k^e(x,y)$ corresponds to the k-th degree of freedom of the element and is determined based on the displacement field $w^e(x,y)$ formula. For this, an appropriate system of twelve equations is solved. Knowing all the shape functions of an element, the displacement field within the element e can be expressed as a linear combination of the shape functions $N_k^e(x,y)$ with coefficients being nodal displacements, i.e.

$$w^e(x,y) = \boldsymbol{N}_e\boldsymbol{w}_e, \tag{17}$$

where $\boldsymbol{w}_e = [\boldsymbol{w}_1^e\;\boldsymbol{w}_2^e\;\boldsymbol{w}_3^e\;\boldsymbol{w}_4^e]^T$ and $\boldsymbol{N}_e = [N_1^e\;N_2^e\;N_3^e\;\dots\;N_{12}^e]$.

On the basis of the knowledge of the shape functions, it is possible to determine the element stiffness matrix $\boldsymbol{K}_e$ and the consistent inertia matrix $\boldsymbol{M_e}$ of the element. The formulas for determining these matrices are given in [6] and [7], respectively. The dimension of the stiffness and inertia matrices $\boldsymbol{K_e}$ and $\boldsymbol{M_e}$ is 12×12 due to the twelve degrees of freedom of a single finite element.

5. The equation of motion of the plate with dampers and its solution

The equation of motion of a structure with viscoelastic dampers can be written in the following form [1,3]:

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \boldsymbol{C}\dot{\boldsymbol{q}}(t) + \boldsymbol{K}\boldsymbol{q}(t) = \boldsymbol{f}(t). \tag{18}$$

In the above equation, $\boldsymbol{K}$, $\boldsymbol{M}$ and $\boldsymbol{C}$ denote the global plate stiffness, inertia and damping matrices, respectively. The dimension of these matrices is $3n \times 3n$, where n is the total number of nodes of all finite elements making up the plate. There are also two vectors in the equation: $\boldsymbol{q}(t)$ is the $3n$-dimensional plate nodal displacement vector and $\boldsymbol{f}(t)$ is the vector of the forces acting on the plate from dampers. It is assumed in the equation that the structural plate is not loaded with additional excitation forces.

The matrices $\boldsymbol{K}$ and $\boldsymbol{M}$ appearing in equation (18) arise as a result of the aggregation process of element matrices $\boldsymbol{K}_e$ and $\boldsymbol{M_e}$ respectively. In numerical tests included in this paper, the damping matrix $\boldsymbol{C}$ is omitted.

After applying the Laplace transform with zero initial conditions, equation (18) takes the following algebraic form:

$$(s^2\boldsymbol{M} + s\boldsymbol{C} + \boldsymbol{K})\overline{\boldsymbol{q}}(s) = \overline{\boldsymbol{f}}(s), \tag{19}$$

where $\overline{\boldsymbol{q}}(s)$ is the $\mathcal{L}$-transform of $\boldsymbol{q}(t)$ and $\overline{\boldsymbol{f}}(s)$ can be expressed as follows:

$$\overline{\boldsymbol{f}}(s) = -\sum_{r=1}^{n_d}(K_r + C_r(s) + G_r(s))\mathbf{L}_\mathrm{r}\overline{\boldsymbol{q}}(s). \tag{20}$$

In the formula above, n_d is the total number of dampers attached to the plate at selected nodes of a finite element mesh and $\mathbf{L}_\mathrm{r}$ is a global matrix indicating the location of the r-th damper within the plate. It is a diagonal matrix with one in the row representing the translational degree of freedom along which the r-th damper works. Formulas for determining K_r, $C_r(s)$ and $G_r(s)$ for the r-th damper are given in chapter 2.

After substituting (20) to (19), the $\mathcal{L}$-transform of the equation (18) of motion of a plate with viscoelastic dampers takes the form:

$$(s^2\boldsymbol{M} + s\boldsymbol{C} + \boldsymbol{C}_d(s) + \boldsymbol{K} + \boldsymbol{K}_d + \boldsymbol{G}_d(s))\overline{\boldsymbol{q}}(s) = \mathbf{0}, \tag{21}$$

where

$$\boldsymbol{K}_d = \sum_{r=1}^{n_d} K_r\mathbf{L}_\mathrm{r}\,;\ \boldsymbol{C}_d(s) = \sum_{r=1}^{n_d} C_r(s)\mathbf{L}_\mathrm{r}\,;\ \boldsymbol{G}_d(s) = \sum_{r=1}^{n_d} G_r(s)\mathbf{L}_\mathrm{r}. \tag{22}$$

Equation (21) is a nonlinear eigenproblem which is solved by eigenvalues s and eigenvectors $\overline{\boldsymbol{q}}(s)$. This problem can be solved e.g. according to the algorithm of the continuation method described in more detail in [3] and used in this paper. Other methods of solving this problem are described e.g. in [5].

In the case of equation (21), the components containing the variable s in the first power are multiplied by the parameter $\kappa \in [0;1]$. The equation can then be written as

$$\boldsymbol{h}_1(\overline{\boldsymbol{q}}, s) = \boldsymbol{D}(s)\overline{\boldsymbol{q}}(s) = \boldsymbol{0}, \tag{23}$$

where

$$\boldsymbol{D}(s) = s^2\boldsymbol{M} + \kappa s\boldsymbol{C} + \kappa\boldsymbol{C}_d(s) + \boldsymbol{K} + \boldsymbol{K}_d + \kappa\boldsymbol{G}_d(s). \tag{24}$$

In order for the elements of the eigenvector $\overline{\boldsymbol{q}}$ corresponding to the eigenvalue s to be determined unambiguously, an additional normalizing equation of the following form is introduced into the matrix equation (23):

$$h_2(\overline{\boldsymbol{q}}, s) = \frac{1}{2}\overline{\boldsymbol{q}}(s)^T\frac{\partial\boldsymbol{D}(s)}{\partial s}\overline{\boldsymbol{q}}(s) - a = \boldsymbol{0}, \tag{25}$$

where a has a given value.

In the first step of the continuation method, in equation (23) the parameter $\kappa_1 = 0$ is assumed and the generalized eigenproblem is solved. As a result of solving this problem, the first approximations of eigenvalues $s_1^{(1)}, s_2^{(1)}, \dots, s_{3n}^{(1)}$ and eigenvectors $\overline{\boldsymbol{q}}_1^{(1)}, \overline{\boldsymbol{q}}_2^{(1)}, \dots, \overline{\boldsymbol{q}}_{3n}^{(1)}$ are obtained. On their basis, the parameter $a_j^{(1)} = s_j^{(1)}\left(\overline{\boldsymbol{q}}_j^{(1)}\right)^T\boldsymbol{M}\overline{\boldsymbol{q}}_j^{(1)}$, where $j = 1,2,\dots,3n$, is determined.

In the l-th step ($l = 2,3,4,\dots$) the increment $\Delta\kappa_l$ is assumed and the Newton method is used to solve the system of equations (23) with the additional equation (25). For this purpose, the system of incremental equations of the Newton method is solved using $\kappa_l = \kappa_{l-1} + \Delta\kappa_l$, $s_j^{(k-1)}$, $\overline{\boldsymbol{q}}_j^{(k-1)}$ and $a_j^{(k-1)}$. Increments $\delta\overline{\boldsymbol{q}}$ and δs are obtained from these equations and the following are calculated:

$$s_j^{(k)} = s_j^{(k-1)} + \delta s;\ \overline{\boldsymbol{q}}_j^{(k)} = \overline{\boldsymbol{q}}_j^{(k-1)} + \delta\overline{\boldsymbol{q}};\ a_j^{(k)} = \frac{1}{2}\left(\overline{\boldsymbol{q}}_j^{(k)}\right)^T\frac{\partial\boldsymbol{D}(s)}{\partial s}\overline{\boldsymbol{q}}_j^{(k)}. \tag{26}$$

Successive approximations of the j-th eigenvalue and j-th eigenvector in the l-th step of the algorithm are calculated until the desired accuracy of the final result is achieved.

The final values of $s_j^{(k)}$, $\overline{\boldsymbol{q}}_j^{(k)}$ and $a_j^{(k)}$ obtained in the l-th step are taken as starting values for the step $l+1$ and the new parameter $\kappa_{l+1} = \kappa_l + \Delta\kappa_{l+1}$.

The procedure described above is carried out up to the value of the parameter $\kappa = 1$, when the final eigenvalues and eigenvectors for the nonlinear eigenproblem (21) are obtained.

The obtained eigenvalues of the problem (21) are complex numbers of the form $s_j = \mu_j + i\eta_j$. On this basis, the j-th natural frequency ω_j of the structure and the non-dimensional damping ratio γ_j of the j-th mode of vibration are determined from the formulas:

$$\omega_j^2 = \mu_j^2 + \eta_j^2;\ \gamma_j = -\frac{\mu_j}{\omega_j}. \tag{27}$$

6. Numerical example

The square isotropic plate fixed on one edge is considered. The material properties of the plate are: E = 205 GPa, ν_p = 0.3 and $\rho_\text{p} = 7850$ kg/m^3. The plate dimensions are $l_x \times l_y \times h = (2.0 \times 2.0 \times 0.01)$m. Three viscoelastic dampers are mounted in the middle and at both ends of the free edge of the plate. The damper contains one Kelvin element and one Maxwell element with the following parameters determined at the reference temperature $T_0 = 0{,}2$°C based on [2]:

$$k_0 = 108{,}56\ \mathrm{N/m};\ c_0 = 0\ \mathrm{Ns/m};\ k_1 = 19968{,}09\ \mathrm{N/m};\ c_1 = 229{,}63\ \mathrm{Ns/m}. \quad (28)$$

The influence of temperature on the values of the above-mentioned parameters is taken into account by applying the frequency-temperature correspondence principle. In order to calculate the value of the shift function from the formula (15), the values of the constants $C_1 = 19{,}5$ and $C_2 = 80{,}2$ are adopted according to [2].

Obtained natural frequencies and non-dimensional damping ratios for the plate are presented in Table 1 and Table 2 for different temperatures. In Table 3, the dynamic characteristics of the plate are given for $T = 2$°C and for its various discretizations.

The dependence of the first plate natural frequency and non-dimensional damping ratio of the first mode of vibration on temperature is presented graphically in the Fig. 2.

Table 1. Natural frequencies of the plate for different temperatures

Mode	Natural frequencies ω [rad/s] for the FEM mesh 14 × 14.							
	$T = 0$°C	$T = 2$°C	$T = 4$°C	$T = 6$°C	$T = 8$°C	$T = 10$°C	$T = 12$°C	$T = 14$°C
1	14,877	13,687	13,586	13,574	13,572	13,572	13,572	13,572
2	36,577	33,365	33,027	32,986	32,980	32,980	32,980	32,979
3	84,186	82,753	82,442	82,400	82,395	82,394	82,394	82,394
4	110,496	106,621	105,338	105,155	105,129	105,126	105,125	105,125
5	123,193	120,756	119,868	119,736	119,717	119,715	119,714	119,714

Table 2. Non-dimensional damping ratios of the plate for different temperatures

Mode	Non-dimensional damping ratios γ [-] for the FEM mesh 14 × 14.							
	$T = 0$°C	$T = 2$°C	$T = 4$°C	$T = 6$°C	$T = 8$°C	$T = 10$°C	$T = 12$°C	$T = 14$°C
1	0,371343	0,120328	0,042116	0,015551	0,006004	0,002418	0,001013	0,000441
2	0,170925	0,069103	0,024636	0,009105	0,003516	0,001416	0,000594	0,000258
3	0,020028	0,012620	0,004872	0,001819	0,000703	0,000283	0,000119	0,000052
4	0,031183	0,028617	0,012004	0,004530	0,001754	0,000707	0,000296	0,000129
5	0,017372	0,016512	0,007103	0,002692	0,001043	0,000420	0,000176	0,000077

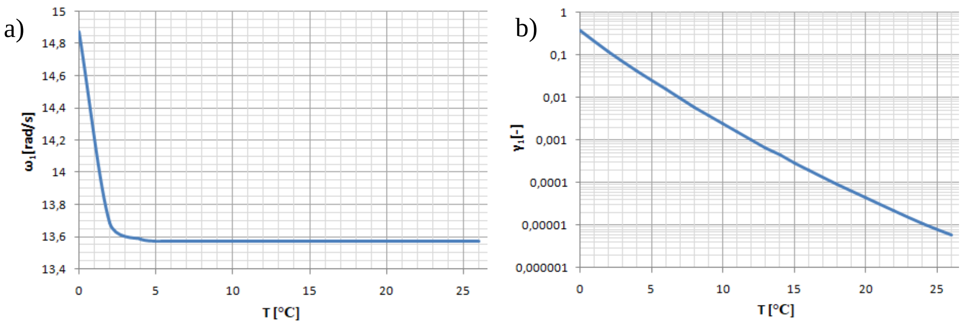

Figure 2. The first natural frequency (a) and the non-dimensional damping ratio (b) of the first mode of vibration versus temperature (FEM mesh 14 × 14)

Table 3. Dynamic characteristics of the plate for $T = 2℃$ and its different discretizations

Mode	Solution FEM for $T = 2℃$ and various FEM meshes					
	Natural frequencies ω [rad/s]			Non-dimensional damping ratios γ [-]		
	10×10	14×14	20×20	10×10	14×14	20×20
1	13,686	13,687	13,687	0,120300	0,120328	0,120343
2	33,370	33,365	33,362	0,069145	0,069103	0,069079
3	82,841	82,753	82,706	0,012607	0,012620	0,012628
4	106,547	106,621	106,660	0,028478	0,028617	0,028690
5	120,770	120,756	120,745	0,016559	0,016512	0,016483

7. Conclusions

The analysis of the influence of temperature on the dynamic characteristics of the isotropic plate with viscoelastic dampers was carried out in the paper. The numerical tests included there showed that a significant decrease in the plate natural frequencies occurs with a slight increase in temperature in relation to the reference temperature. At significant temperatures, the frequencies hardly change and reach a certain constant value. The non-dimensional damping ratios at temperatures higher than $\mathbf{10}℃$ tend to zero and thus the dampers lose their damping properties. The convergence tests carried out in the study showed that the use of a $\mathbf{10 \times 10}$ mesh is sufficient and too high mesh density extends the computation time considerably.

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