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The norming sets of $\mathcal{P}(^2d_*(1,w)^2)$

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Summary. Let $n \in \mathbb{N}$. An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$, where $\mathcal{P}(^nE)$ denotes the space of all continuous n-homogeneous polynomials on E. For $P \in \mathcal{P}(^nE)$, we define

$$\mathrm{Norm}(P) = \{x \in E : x \text{ is a norming point of } P\}.$$

$\mathrm{Norm}(P)$ is called the *norming set* of P. We classify $\mathrm{Norm}(P)$ for every $P \in \mathcal{P}(^2d_*(1,w)^2)$, where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm of weight $0 < w < 1$.

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1. Introduction

Let us introduce a brief history of norm attaining multilinear forms and polynomials on Banach spaces. In 1961 Bishop and Phelps [3] showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon–Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [4] showed that the Radon–Nikodym property is also sufficient for the denseness of norm attaining polyno-

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mials. Jimenez-Sevilla and Paya [6] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}, n \geqslant 2$. We write S_E for the unit sphere of a real Banach space E. A mapping $P: E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^n E)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [5].

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^n E)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}(^n E)$, we define

$$\text{Norm}(P) = \left\{x \in E : x \text{ is a norming point of } P\right\}.$$

$Norm(P)$ is called the *norming set* of P. Notice that $x \in \text{Norm}(P)$ if and only if $-x \in \text{Norm}(P)$. Indeed, if $x \in \text{Norm}(P)$, then

$$\left|P(-x)\right| = \left|(-1)^n P(x)\right| = \left|P(x)\right| = \|P\|,$$

which shows that $-x \in \text{Norm}(P)$. If $-x \in \text{Norm}(P)$, then $x = -(-x) \in \text{Norm}(P)$. The following examples show that it is possible that $\text{Norm}(P)$ be empty, a finite or an infinite set.

1.1. Examples.

(i) Let

$$P\big((x_i)_{i\in\mathbb{N}}\big) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i^2 \in \mathcal{P}(^2 c_0).$$

Then, $\text{Norm}(P) = \varnothing$.

(ii) Let

$$P\big((x_i)_{i\in\mathbb{N}}\big) = x_1^2 - \sum_{i=2}^{\infty} \frac{1}{2^i} x_i^2 \in \mathcal{P}(^2 c_0).$$

Then,

$$\text{Norm}(P) = \{\pm e_1\}.$$

(iii) Let

$$P\big((x_i)_{i\in\mathbb{N}}\big) = x_1^2 \in \mathcal{P}(^2 c_0).$$

Then,

$$\text{Norm}(P) = \left\{(\pm 1, x_2, x_3, \ldots) \in c_0 : |x_j| \leqslant 1,\ j = 2, 3, \ldots\right\}.$$

If $\text{Norm}(P) \neq \varnothing$, $P \in \mathcal{P}(^n E)$ is called a *norm attaining* polynomial (see [4]).

It seems to be natural and interesting to study $\text{Norm}(P)$ for $P \in \mathcal{P}(^nE)$. For $m \in \mathbb{N}$, let $l_\infty^m := \mathbb{R}^m$ with the supremum norm. Notice that for every $P \in \mathcal{P}(^nl_\infty^m)$, $\text{Norm}(P) \neq \emptyset$ since $S_{l_\infty^m}$ is compact. Kim [8] classified $\text{Norm}(P)$ for every $P \in \mathcal{P}(^2l_\infty^2)$.

Let $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm of weight $0 < w < 1$ endowed with $\|(x,y)\|_{d_*(1,w)} = \max\left\{|x|, |y|, \frac{|x|+|y|}{1+w}\right\}$.

In this paper, we classify $\text{Norm}(P)$ for every $P \in \mathcal{P}(^2d_*(1,w)^2)$.

2. Results

Throughout this paper, we let $0 < w < 1$.

Aron and Klimek [2] defined the following two norms on $\mathbb{R}^3$:

$$\|(a,b,c)\|_{\mathbb{R}} = \sup\left\{|ax^2 + bx + c| : x \in [-1,1]\right\} \text{ and}$$
$$\|(a,b,c)\|_{\mathbb{C}} = \sup\left\{|az^2 + bz + c| : z \in \mathbb{C}, |z| \leqslant 1\right\}.$$

Explicit formulas for computing these norm were given and the extreme points of the corresponding unit balls were characterized.

Kim [7] presented an explicit formulae for the norm of $P \in \mathcal{P}(^2d_*(1,w)^2)$ and classified the extreme points of the unit ball of $\mathcal{P}(^2d_*(1,w)^2)$.

2.1. Theorem ([7]). *Let $P \in \mathcal{P}(^2d_*(1,w)^2)$ with $P(x,y) = ax^2 + by^2 + cxy$ for $a \geqslant |b| \geqslant 0, c \geqslant 0$. Then:*

(i) *If $0 \leqslant c < 2|b|$, then*

(a) $b < 0$

i. *If $\frac{c}{2|b|} \leqslant w$, then*

$$\|P\| = a + \frac{c^2}{4|b|}.$$

ii. *If $\frac{c}{2|b|} > w$, then*

$$\|P\| = bw^2 + cw + a.$$

(b) *If $b > 0$, then*

$$\|P\| = bw^2 + cw + a.$$

(ii) *If $2|b| \leqslant c \leqslant 2a$, then*

$$\|P\| = bw^2 + cw + a.$$

(iii) *If $2a < c$, then*

(a) *If $\frac{c-2a}{c-2b} < w$, then*

$$\|P\| = bw^2 + cw + a.$$

(b) *If If* $\frac{c-2a}{c-2b} \geqslant w$, *then*

$$\|P\| = \frac{(c^2-4ab)(1+w)^2}{4(c-a-b)}.$$

Notice that if $P(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2d_*(1,w)^2)$ for some $a \geqslant |b|, c \geqslant 0$ with $\|P\| = 1$, then $|a| \leqslant 1, |b| \leqslant 1$ and $|c| \leqslant \frac{4}{(1+w)^2}$.

Let

$$\begin{aligned} L_1 &= \Big\{(t,1) : 0 \leqslant t \leqslant w\Big\}, \\ L_2 &= \Big\{(t,-t+(1+w)) : w < t \leqslant 1\Big\}, \\ L_3 &= \Big\{(1,t) : -w < t < w\Big\}, \\ L_4 &= \Big\{(t,t-(1+w)) : w < t \leqslant 1\Big\}, \\ L_5 &= \Big\{(t,-1) : 0 \leqslant t < w\Big\}. \end{aligned}$$

2.2. Lemma. *Let* $P(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2d_*(1,w)^2)$ *for some* $a \geqslant |b|$ *and* $c \geqslant 0$ *with* $\|P\| = 1$. *Then* $\mathrm{Norm}(P) = \{\pm(x,y) : (x,y) \in \bigcup_{1\leqslant j\leqslant 5} L_j, |P(x,y)| = 1\}$.

Proof. It is obvious. □

2.3. Lemma. *Let* $P(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2d_*(1,w)^2)$ *for some* $a \geqslant |b|$ *and* $c \geqslant 0$ *with* $\|P\| = 1$ *and* $abc \neq 0$. *Then the following assertions hold:*

(i) *If* $b > 0, bw^2 + cw + a = 1$, *then*

$$\mathrm{Norm}(P) = \Big\{\pm(1,w)\Big\}.$$

(ii) *If* $b < 0$ *and* $a + \frac{c^2}{4|b|} = 1$, *then*

$$\mathrm{Norm}(P) = \Big\{\pm\Big(1, -\frac{c}{2b}\Big)\Big\}.$$

(iii) *If* $2a < c$, $\frac{c-2a}{c-2b} \geqslant w$ *and* $\frac{(c^2-4ab)(1+w)^2}{4(c-a-b)} = 1$, *then*

$$\mathrm{Norm}(P) = \Big\{\pm\Big(\frac{(c-2b)(1+w)}{2(c-a-b)}, \frac{(c-2a)(1+w)}{2(c-a-b)}\Big)\Big\}.$$

Proof. Notice that

$$\begin{aligned} &\Big|P(1,w)\Big| = bw^2 + cw + a, \quad \Big|P\Big(1, -\frac{c}{2b}\Big)\Big| = a + \frac{c^2}{4|b|}, \\ &\Big|P\Big(\frac{(c-2b)(1+w)}{2(c-a-b)}, \frac{(c-2a)(1+w)}{2(c-a-b)}\Big)\Big| = \frac{(c^2-4ab)(1+w)^2}{4(c-a-b)}. \end{aligned}$$

Let $(x,y) \in \mathrm{Norm}(P)$. By Lemma 2.2, we may assume that $(x,y) \in \bigcup_{1\leqslant j\leqslant 5} L_j$. Notice that $P(x,y) = \pm 1$ are quadratic equations of the variable $0 \leqslant t \leqslant 1$. Solving the quadratic equations $P(x,y) = \pm 1$, we complete the proof. □

2.4. Lemma. *Let* $P(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2d_*(1,w)^2)$ *for some* $a \geqslant |b|$ *and* $c \geqslant 0$ *with* $\|P\| = 1$. *Then the following assertions hold:*

(i) *Suppose that*

$$a + \frac{c^2}{4|b|} = bw^2 + cw + a = 1.$$

Then

$$\left(1, -\frac{c}{2b}\right) = (1, w).$$

(ii) *Suppose that*

$$bw^2 + cw + a = \frac{(c^2 - 4ab)(1+w)^2}{4(c-a-b)} = 1.$$

Then

$$(1, w) = \left(\frac{(c-2b)(1+w)}{2(c-a-b)}, \frac{(c-2a)(1+w)}{2(c-a-b)}\right).$$

Proof. (i). By Theorem 2.1, $b < 0$, $\frac{c}{2|b|} = w$. Thus,

$$\left(1, -\frac{c}{2b}\right) = (1, w).$$

(ii). By Theorem 2.1, $\frac{c-2a}{c-2b} = w$. This shows (ii).

□

We are in position to prove the main result of this paper.

2.5. Theorem. *Let* $P(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2d_*(1,w)^2)$ *for some* $a \geqslant |b|$ *and* $c \geqslant 0$ *with* $\|P\| = 1$. *Then the following assertions hold:*

(i) *Let* $abc = 0$. *If* $a = 1, b = c = 0$, *then*

$$\text{Norm}(P) = \left\{\pm(1,t) : -w \leqslant t \leqslant w\right\}.$$

If $b = 1, a = c = 0$, *then*

$$\text{Norm}(P) = \left\{\pm(t, \pm 1) : 0 \leqslant t \leqslant w\right\}.$$

If $a = b = 0, c = \frac{4}{(1+w)^2}$, *then*

$$\text{Norm}(P) = \left\{\pm\left(\frac{1+w}{2}, \pm\frac{1+w}{2}\right)\right\}.$$

If $c = 0, ab \neq 0, -1 < b < 0$, *then*

$$\text{Norm}(P) = \{\pm(1,0)\}.$$

If $c = 0, ab \neq 0, b = -1$, *then*

$$\mathrm{Norm}(P) = \Big\{\pm(1,0), \pm(0,1)\Big\}.$$

If $c = 0, ab \neq 0, b > 0$, *then*

$$\mathrm{Norm}(P) = \Big\{\pm(1,w)\Big\}.$$

If $b = 0, ac \neq 0$ *and* $(c < 2a$ *or* $c \geqslant 2a, \frac{c-2a}{c} < w)$, *then*

$$\mathrm{Norm}(P) = \Big\{\pm(1,w)\Big\}.$$

If $b = 0, ac \neq 0, c \geqslant 2a$ *and* $\frac{c-2a}{c} \geqslant w$, *then*

$$\mathrm{Norm}(P) = \Big\{\pm\Big(\frac{(c-2b)(1+w)}{2(c-a-b)}, \frac{(c-2a)(1+w)}{2(c-a-b)}\Big)\Big\}.$$

(ii) *Let* $abc \neq 0$. *If* $0 < c < 2|b|, b < 0$ *and* $\frac{c}{2|b|} \leqslant w$, *then*

$$\mathrm{Norm}(P) = \Big\{\pm\Big(1, -\frac{c}{2b}\Big)\Big\}.$$

If $(0 < c < 2|b|, b < 0, \frac{c}{2|b|} > w)$, $(0 < c < 2|b|, b > 0)$, $(0 < 2|b| \leqslant c \leqslant 2a)$ *or* $(0 < 2a < c, \frac{c-2a}{c-2b} < w)$, *then*

$$\mathrm{Norm}(P) = \Big\{\pm(1,w)\Big\}.$$

If $0 < 2a < c$ *and* $\frac{c-2a}{c-2b} \geqslant w$, *then*

$$\mathrm{Norm}(P) = \Big\{\pm\Big(\frac{(c-2b)(1+w)}{2(c-a-b)}, \frac{(c-2a)(1+w)}{2(c-a-b)}\Big)\Big\}.$$

Proof. Let $(x,y) \in \mathrm{Norm}(P)$. By Lemma 2.2, we may assume that $(x,y) \in \bigcup_{1\leqslant j\leqslant 5} L_j$. Let $abc = 0$. Let $P(x,y) = x^2$. It is obvious that

$$\mathrm{Norm}(P) = \Big\{\pm(1,t) : -w \leqslant t \leqslant w\Big\}.$$

Let $P(x,y) = \pm y^2$. It is obvious that

$$\mathrm{Norm}(P) = \Big\{\pm(t,\pm1) : 0 \leqslant t \leqslant w\Big\}.$$

Let $P(x,y) = \frac{4}{(1+w)^2}xy$. It is obvious that

$$\mathrm{Norm}(P) = \Big\{\pm\Big(\frac{1+w}{2}, \pm\frac{1+w}{2}\Big)\Big\}.$$

Let $P(x,y) = ax^2 + by^2$ for $ab \neq 0$ and $-1 < b < 0$. By Theorem 2.1, $P(x,y) = x^2 - |b|y^2$ and

$$\operatorname{Norm}(P) = \{\pm(1,0)\}.$$

Let $P(x,y) = ax^2 + by^2$ for $ab \neq 0$ and $b = -1$. By Theorem 2.1, $P(x,y) = x^2 - y^2$ and

$$\operatorname{Norm}(P) = \{\pm(1,0), \pm(0,1)\}.$$

Let $P(x,y) = ax^2 + by^2$ for $ab \neq 0$ and $b > 0$. By Theorem 2.1, $1 = bw^2 + a$. By Lemma 2.3,

$$\operatorname{Norm}(P) = \{\pm(1,w)\}.$$

Let $P(x,y) = ax^2 + cxy$ for $ac \neq 0$ and ($c < 2a$ or $c \geqslant 2a$, $\frac{c-2a}{c} < w$). By Theorem 2.1, $1 = cw + a$. By Lemma 2.3,

$$\operatorname{Norm}(P) = \{\pm(1,w)\}.$$

Let $P(x,y) = ax^2 + cxy$ for $ac \neq 0$, $c \geqslant 2a$ and $\frac{c-2a}{c} \geqslant w$. By Theorem 2.1, $\frac{c^2(1+w)^2}{4(c-a)} = 1$. By Lemma 2.3,

$$\operatorname{Norm}(P) = \left\{ \pm\left(\frac{c(1+w)}{2(c-a)}, \frac{(c-2a)(1+w)}{2(c-a)}\right)\right\}.$$

Let $abc \neq 0$. By Lemmas 2.3 and 2.4, the proof follows. This completes the proof. □

References

[1] R. M. Aron, C. Finet, and E. Werner, *Some remarks on norm-attaining n-linear forms*, Function spaces (Edwardsville, IL, 1994) 172 (1995), 19–28.

[2] R. M. Aron and M. Klimek, *Supremum norms for quadratic polynomials*, Arch. Math. 76 (2001), 73–80, DOI 10.1007/s000130050544.

[3] E. Bishop and R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. 67 (1961), 97–98, DOI 10.1090/S0002-9904-1961-10514-4.

[4] Y. S. Choi and S. G. Kim, *Norm or numerical radius attaining multilinear mappings and polynomials*, J. London Math. Soc. 54 (1996), 135–147, DOI 10.1112/jlms/54.1.135.

[5] S. Dineen, *Complex analysis on infinite-dimensional spaces*, Springer-Verlag, London 1999, DOI 10.1007/978-1-4471-0869-6.

[6] M. Jimenez Sevilla and R. Paya, *Norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces*, Studia Math. 127 (1998), 99–112, DOI 10.4064/sm-127-2-99-112.

[7] S. G. Kim, *The unit ball of $\mathcal{P}(^2d_*(1,w)^2)$*, Math. Proc. R. Ir. Acad. 111 (2011), no. 2, 79–94, DOI 10.3318/pria.2011.111.1.9.

[8] S. G. Kim, *The norming set of a polynomial in $\mathcal{P}(^2l^2_\infty)$*, Honam Math. J. 42 (2020), no. 3, 569–576, DOI 10.5831/HMJ.2020.42.3.569.