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Topological sensitivity analysis for a coupled nonlinear problem with an obstacle*

by

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Abstract: The Topological Derivative has been recognized as a powerful tool in obtaining the optimal topology for several kinds of engineering problems. This derivative provides the sensitivity of the cost functional for a boundary value problem for nucleation of a small hole or a small inclusion at a given point of the domain of integration. In this paper, we present a topological asymptotic analysis with respect to the size of singular domain perturbation for a coupled nonlinear PDEs system with an obstacle on the boundary. The domain decomposition method, referring to the Steklov-Poincaré pseudo-differential operator, is employed for the asymptotic study of boundary value problem with respect to the size of singular domain perturbation. The method is based on the observation that the known expansion of the energy functional in the ring coincides with the expansion of the Steklov-Poincaré operator on the boundary of the truncated domain with respect to the small parameter, which measures the size of perturbation. In this way, the singular perturbation of the domain is reduced to the regular perturbation of the Steklov-Poincaré mapping for the ring. The topological derivative for a tracking type shape functional is evaluated so as to obtain the useful formula for application in the numerical methods of shape and topology optimization.

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1. Introduction

The topological sensitivity analysis is a powerful framework of numerical methods for obtaining the optimal topology for several engineering problems. The aim of the topological sensitivity analysis is to determine an asymptotic expansion of a shape functional with respect to the variation of the topology of the domain. The notion of topological derivative of a shape functional was introduced in Sokolowski and Zochowski (1999, 2001). Namely, let us consider a cost function $\mathcal{J}(\Omega) := J_\Omega(u_\Omega)$, where u_Ω is the solution of a partial differential equation, defined in the domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3. Denote by $B_\varepsilon(x_0)$ a ball of the centre at $x_0 \in \Omega$ with small radius $\varepsilon > 0$.

The *topological asymptotic expansion* is an expression of the form

$$\mathcal{J}(\Omega_\varepsilon) = \mathcal{J}(\Omega) + f(\varepsilon)\,\mathcal{T}(x_0) + o(f(\varepsilon))$$

where $\Omega_\varepsilon := \Omega \backslash \overline{B_\varepsilon(x_0)}$ is singular geometrical perturbation of Ω including the small hole, $f(\varepsilon)$ is a positive function tending to zero with ε such that $\dfrac{o(f(\varepsilon))}{f(\varepsilon)} \to 0$ as $\varepsilon \to 0$.

Therefore, to minimize the criterion, we are interested in removing the matter where the topological derivative of the cost $\mathcal{T}$, called also in the literature the *topological gradient*, is negative. Hence the topological derivative measures the sensitivity of the cost for the problem under consideration for nucleation of a small hole at each point of the domain.

A general framework, enabling to calculate the topological asymptotics for a large class of shape functionals, has been worked out in several studies, see, for example, Feijóo et al. (2003), Jleli, Samet and Vial (2015), Nazarov and Sokolowski (2003), Novotny and Sokolowski (2013), Masmoudi (2001), Iguernane et al. (2009), Laurain (2006), or Sokolowski and Zochowski (2005), for linear or semilinear problems with arbitrary functionals and variational inequalities with the energy functional. For linear problems, those studies are based on an adaptation of the adjoint state method and a domain truncation technique providing an equivalent formulation of the PDE in the fixed functional space.

In our study, the problem under consideration is a coupled system of an elliptic equation with an obstacle on the boundary and a modified Helmholtz equation. The decomposition method and asymptotic analysis are used in derivation of the topological derivatives. This technique has been used widely giving rise to various interesting results (see, for example, Novotny and Sokolowski, 2013, or Feijóo et al., 2003). In our case, the topological sensitivity is performed for the Helmholtz equation, which is defined in the domain given by the support of a characteristic function χ. The Helmholtz equation is coupled with a variational inequality defined in the complement of Supp(χ).

The variational formulation of the coupled system in the singularly perturbed domain Ω_ε is given by an equivalent problem in form of the variational inequality in the appropriate Sobolev space on Ω_ε. It turns out that for the purposes of asymptotic analysis a regular pertubed variational inequality in the fixed domain Ω can be considered following Sokolowski and Zochowski (2005). Such a variational inequality is obtained by replacing the exact energy of the given system by its appropriate asymptotic approximation, obtained from the asymptotic analysis of Steklov-Poincaré operator in the ring (see Sokolowski and Zochowski, 2005). In such a way, the singularly perturbed geometrical domain can be replaced by a regular perturbation of bilinear form. It turns out that such a replacement does not change the first order asymptotic expansion for the energy functional associated with the coupled boundary value problem. For the modified energy functional, we can apply the abstract theorem (from Sokolowski and Zolesio, 1992) on the conical differentiability of the solution to the regularly perturbed variational inequality. The conical derivative of the solution is given by a unique solution of an auxiliary variational inequality over the convex subset of the energy space. In this way, we obtain an asymptotic expansion of the solution with respect to ε and we get the same approximation for the original variational inequality but far from the singularity. Finally, we use this result to derive the formula of the topological derivative of a tracking type shape functional. Such formula is useful in numerical methods for shape optimization.

The paper is organized as follows. In Section 2, the model problem considered is given. The different steps followed in the asymptotic analysis are described in Section 3. The asymptotic expansion of the Steklov-Poincaré is studied in Section 4. The conical derivatives of solutions in perturbed and fixed domains are presented and the main result on the topological asymptotic analysis of the problem under consideration is given in Section 5.

2. Problem formulation

Let Ω be an open and bounded domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega := \Gamma_0 \cup \Gamma_s$ such that $\Gamma_0 \cap \Gamma_s = \emptyset$. The system under consideration is a coupled system defined by an Helmholtz equation in $\Omega_u := \text{Supp}\,(\chi)$, the support of a characteristic function χ with $\Omega_u \subset \Omega$ and a variational inequality in Ω_w the outer domain of Supp (χ) ; in Ω, with transmission conditions on the interface Σ and Signorini conditions on the boundary Γ_s which is assumed to be C^2.

The strong form of boundary value problem is formulated as follows: Find a function $z := (u, w) : \Omega \to \mathbb{R}^2$ such that

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega_u \\ u = w & \text{on } \Sigma \\ \partial_n u = \partial_n w & \text{on } \Sigma \\ -\Delta w = f & \Omega_w \\ w = 0 & \text{on } \Gamma_0 \\ w \geq 0, \ \partial_n w \geq 0, \ w\partial_n w = 0 & \text{on } \Gamma_s \end{cases} \tag{2.1}$$

where $f \in L^2(\Omega_w)$, n is the unit outward normal vector to $\partial\Omega$ and ∂_n stands for the derivative along the outward normal.

The coupled problem (2.1) admits a unique weak solution $z := z(\Omega)$ in $K(\Omega) \subset H^1(\Omega)$ characterized by the variational inequality

$$a(\Omega; z, v - z) - \mathcal{L}(\Omega; v - z) \geq 0, \quad \forall v \in K(\Omega) \tag{2.2}$$

with $a(\Omega; v, v) := \int_\Omega |\nabla v|^2 + \chi v^2$, $\mathcal{L}(\Omega; v) := \int_\Omega (1 - \chi) f v$ and the closed, convex cone

$$K(\Omega) := \left\{ v \in H^1_{\Gamma_0}(\Omega) : v \geq 0 \text{ a.e on } \Gamma_s \right\},$$

where $H^1_{\Gamma_0}(\Omega)$ stands for the classical Sobolev space of functions, which belong to $H^1(\Omega)$ and with null traces on the boundary Γ_0. Observe that the solution $z \in K(\Omega)$ is given by the unique minimizer of quadratic energy functional

$$\mathcal{E}(\Omega) := \frac{1}{2} a(\Omega; v, v) - \mathcal{L}(\Omega; v) \tag{2.3}$$

over $K(\Omega)$.

Note also that the weak solution $z := z(\Omega)$ of problem (2.1) takes the form:

$$z := \begin{cases} u & \text{in } \Omega_u \\ w & \text{in } \Omega_w \end{cases}$$

where u is a solution of the linear Helmholtz equation in Ω_u and w is a solution of the nonlinear Signorini problem in Ω_w, the two problems are coupled by transmission conditions prescribed on the interface Σ.

Solutions of problem (2.1) are used in order to evaluate the topological derivative of the energy given by (2.3) and of the following tracking type shape functional

$$\mathcal{J}(\Omega) = \frac{1}{2} \int_\Omega \chi (z - z_d)^2 = \int_{\Omega_w} (w - z_d)^2 \tag{2.4}$$

where $z_d \in L^2(\Omega)$ is given. Our purpose for (2.4) is to study the topological optimization problem in order to decrease the functional $\mathcal{J}$ by creating a hole in Ω_u to make the state $z(\Omega)$ as close as possible to a desired state z_d. The case of the shape functional defined in Ω_u can be considered in the same way.

3. Asymptotic analysis

We present here the different steps of asymptotic analysis, leading to the main result of the paper. For simplicity, all technical proofs are reported in Sections 4 and 5.1.

3.1. Domain decomposition method

For a small parameter $\varepsilon > 0$, consider the perforated domain $\Omega_\varepsilon := \Omega \backslash \overline{B_\varepsilon(x_0)}$ where $\overline{B_\varepsilon(x_0)}$ is the closed ball in Ω_u of radius ε with center at an arbitrary point x_0, and with the boundary Γ_ε. Let us assume for the sake of simplicity that the center $x_0 = \mathcal{O}$ is the origin of the coordinate system and denote $B_\varepsilon(\mathcal{O})$ by B_ε.

In order to obtain the topological derivative of the shape functional (2.4) (respectively, (2.3)) associated to the coupled problem (2.1), we need to establish the asymptotic expansion of the functional $\mathcal{J}(\Omega_\varepsilon)$ (respectively $\mathcal{E}(\Omega_\varepsilon)$) with $z_\varepsilon := z(\Omega_\varepsilon)$ being the solution of the perturbed problem: find a function $z_\varepsilon : (u_\varepsilon, w_\varepsilon) : \Omega_\varepsilon \to \mathbb{R}^2$ such that

$$\left\{\begin{array}{cc} -\Delta u_\varepsilon + u_\varepsilon = 0 & \text{in } \Omega_u^\varepsilon = \Omega_u \backslash \overline{B}_\varepsilon \\ \partial_n u_\varepsilon = 0 & \text{on } \Gamma_\varepsilon \\ u_\varepsilon = w_\varepsilon & \text{on } \Sigma \\ \partial_n u_\varepsilon = \partial_n w_\varepsilon & \text{on } \Sigma \\ -\Delta w_\varepsilon = f & \Omega_w \\ w_\varepsilon = 0 & \text{on } \Gamma_0 \\ w_\varepsilon \geq 0,\ \partial_n w_\varepsilon \geq 0,\ w_\varepsilon \partial_n w_\varepsilon = 0 & \text{on } \Gamma_s. \end{array}\right. \tag{3.5}$$

This problem admits a unique weak solution z_ε for ε small enough, which is the solution of the variational inequality in the domain Ω_ε, that is

$$a(\Omega_\varepsilon; z_\varepsilon, v - z_\varepsilon) - \mathcal{L}(\Omega_\varepsilon; v - z_\varepsilon) \geq 0, \quad \forall v \in K(\Omega_\varepsilon) \tag{3.6}$$

with $z_\varepsilon \in K(\Omega_\varepsilon) := \left\{ v \in H^1_{\Gamma_0}(\Omega_\varepsilon) : v \geq 0 \text{ a.e. on } \Gamma_s \right\}$.

Now, we shall make use of the domain decomposition technique and of the associated Steklov-Poincaré operator in order to justify (see Sokolowski and Zochowski, 2005) the so-called truncated domain technique, which is an important tool for variational inequalities. To this end, let us decompose the domain Ω into two subdomains, Ω_R and B_R, with the interface Γ_R.

Take $R > \varepsilon > 0$ such that $\overline{B}_\varepsilon \subset B_R$, $\overline{B}_R \subset \Omega_u$ and denote by Γ_R the boundary of $\overline{B}_R$. Define the truncated open subset Ω_R and the ring $C(\varepsilon, R)$ by setting:

$$\Omega_R := \Omega \backslash \overline{B}_R, \quad C(\varepsilon, R) := B_R \backslash \overline{B}_\varepsilon. \tag{3.7}$$

We claim that the variational inequality (3.6) can be modified in an appropriate way, such that Ω_ε can be replaced by the truncated domain Ω_R with $R > \varepsilon > 0$ sufficiently small.

Proposition 1 *Actually, we can show that the restriction $z_\varepsilon^R \in K(\Omega_R)$ of $z_\varepsilon \in K(\Omega_\varepsilon)$ to the truncated domain is given by the solution $\tilde{z}_\varepsilon \in K(\Omega_R)$ to the following variational inequality*

$$a\left(\Omega_R; \tilde{z}_\varepsilon, v - \tilde{z}_\varepsilon\right) - \mathcal{L}\left(\Omega_R; v - \tilde{z}_\varepsilon\right) + \langle \mathcal{A}_\varepsilon\left(\tilde{z}_\varepsilon,\right), v - \tilde{z}_\varepsilon\rangle_{\Gamma_R} \geq 0 \ \forall v \in K(\Omega_R) \quad (3.8)$$

where $\mathcal{A}_\varepsilon$ is the Steklov-Poincaré operator, which will replace the portion of the bilinear form defined over the ring $C(\varepsilon, R)$.

The proof of Proposition 1 is left to the reader.

In this way, we obtain that

$$\begin{aligned} \mathcal{E}\left(\Omega_\varepsilon\right) &= \frac{1}{2} a\left(\Omega_R; z_\varepsilon^R, z_\varepsilon^R\right) - \mathcal{L}\left(\Omega_R; z_\varepsilon^R\right) + \langle \mathcal{A}_\varepsilon\left(z_\varepsilon^R\right), z_\varepsilon^R\rangle_{\Gamma_R} \\ &:= j_1(\varepsilon) + j_2(\varepsilon) \\ \text{(respectively } \mathcal{J}\left(\Omega_\varepsilon\right) &= \frac{1}{2}\int_{\Omega_w} \left(z_\varepsilon^R - z_d\right)^2 := j(\varepsilon)) \end{aligned}$$

and we can evaluate the topological derivative of the shape function by using the expansions of j_1, j_2 and j.

For this purpose, we need first to introduce the Steklov-Poincaré operator. Consider the following problems in B_R and $C(\varepsilon, R)$, respectively:
Given $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$ find $z_0^\varphi \in H^1(B_R)$ and $z_\varepsilon^\varphi \in H^1(C(\varepsilon, R))$, the solutions of

$$\begin{cases} -\Delta z_0^\varphi + z_0^\varphi = 0 & \text{in } B_R \\ z_0^\varphi = \varphi & \text{on } \Gamma_R \end{cases} \quad (3.9)$$

and

$$\begin{cases} -\Delta z_\varepsilon^\varphi + z_\varepsilon^\varphi = 0 & \text{in } C(\varepsilon, R) \\ z_\varepsilon^\varphi = \varphi & \text{on } \Gamma_R \\ \partial_n z_\varepsilon^\varphi = 0 & \text{on } \Gamma_\varepsilon. \end{cases} \quad (3.10)$$

We define the Steklov-Poincaré operator by the boundary value problem (3.10) in the following way:

$$\begin{array}{rcl} \mathcal{A}_\varepsilon : H^{\frac{1}{2}}(\Gamma_R) & \to & H^{-\frac{1}{2}}(\Gamma_R) \\ \varphi & \to & \mathcal{A}_\varepsilon(\varphi) = \partial_n \varphi. \end{array} \quad (3.11)$$

Taking into account the relation, which follows by integration by parts, we find that

$$\int_{C(\varepsilon,R)} \left(-\Delta z_\varepsilon^\varphi + z_\varepsilon^\varphi\right) z_\varepsilon^\varphi = \int_{C(\varepsilon,R)} |\nabla z_\varepsilon^\varphi|^2 + z_\varepsilon^{\varphi 2} - \int_{\Gamma_R} \left(\partial_n z_\varepsilon^\varphi\right) z_\varepsilon^\varphi - \int_{\Gamma_\varepsilon} \left(\partial_n z_\varepsilon^\varphi\right) z_\varepsilon^\varphi = 0$$

and since

$$\int_{\Gamma_\varepsilon} \left(\partial_n z_\varepsilon^\varphi\right) z_\varepsilon^\varphi = 0$$

we get

$$\langle \mathcal{A}_\varepsilon(\varphi), \varphi\rangle_{\Gamma_R} = \int\limits_{C(\varepsilon,R)} |\nabla z_\varepsilon^\varphi|^2 + z_\varepsilon^{\varphi 2}.$$

By this setting, one can write the problem in the truncated domain as:

$$\left\{\begin{array}{cc} -\Delta u_\varepsilon^R + u_\varepsilon^R = 0 & \text{in } \Omega_u^R := \Omega_u \backslash B_R \\ \mathcal{A}_\varepsilon u_\varepsilon^R = \partial_n u_\varepsilon^R & \text{on } \Gamma_R \\ u_\varepsilon^R = w_\varepsilon^R & \text{on } \Sigma \\ \partial_n u_\varepsilon^R = \partial_n w_\varepsilon^R & \text{on } \Sigma \\ -\Delta w_\varepsilon^R = f & \text{in } \Omega_w \\ w_\varepsilon^R = 0 & \text{on } \Gamma_0 \\ w_\varepsilon^R \geq 0,\ \partial_n w_\varepsilon^R \geq 0,\ w_\varepsilon^R \partial_n w_\varepsilon^R = 0 & \text{on } \Gamma_s. \end{array}\right. \tag{3.12}$$

Let z_ε^R be the unique solution of this system. By construction, the following Lemma can be easily proved.

LEMMA 1 *For $\varepsilon > 0$ the restriction of the solution z_ε to Ω_R of variational inequality (3.8) coincides with the solution z_ε^R to variational inequality (3.6) in the sense of the Sobolev space $H^1(\Omega_R)$, that is*

$$z_\varepsilon(x) = z_\varepsilon^R(x) \quad \textit{quasi everywhere} \quad \textit{on} \ \ \Omega_R.$$

Furthermore, the restriction to Ω_R of the solution to variational inequality (3.8) for $\varepsilon = 0$ coincides with the restriction to Ω_R of the solution to problem (2.2).

3.2. Approximation of the problem in Ω_R

The restriction of z_ε to the truncated domain solves the boundary value problem with the nonlocal boundary conditions on Γ_R, defined by the Steklov-Poincaré operator. Indeed, since by domain decomposition, one has

$$a(\Omega_\varepsilon; z_\varepsilon, v) = a\left(\Omega_R; z_\varepsilon^R, v\right) + a\left(C(\varepsilon, R); z_\varepsilon^R, v\right)$$

and by construction

$$a\left(C(\varepsilon,R); z_\varepsilon^R, v\right) = \left\langle \mathcal{A}_\varepsilon\left(z_\varepsilon^R\right), v\right\rangle_{\Gamma_R} = \int_{\Gamma_R} \partial_n z_\varepsilon^R v,$$

where $\int_{\Gamma_R} \partial_n z_\varepsilon^R v$ stands for the duality pairing between the fractional Sobolev spaces $H^{-1/2}(\Gamma_R) \ni \partial_n z_\varepsilon^R = \mathcal{A}_\varepsilon\left(z_\varepsilon^R\right)$ and $H^{1/2}(\Gamma_R) \ni v$. Hence the bilinear form in the topologically perturbed domain Ω_ε can be replaced by the bilinear form in the unperturbed domain with a nonlocal pseudo-differential operator. That is,

$$a(\Omega_\varepsilon; z_\varepsilon, v) = a\left(\Omega_R; z_\varepsilon^R, v\right) + \left\langle \mathcal{A}_\varepsilon\left(z_\varepsilon^R\right), v\right\rangle_{\Gamma_R}$$

where $\mathcal{A}_\varepsilon$ is the Steklov-Poincaré operator defined by (3.11).

This replacement is important for purposes of evaluation of the topological derivative of the shape functional considered for the coupled system. Moreover, the knowledge of the asymptotic expansion of the Steklov-Poincaré operator as (see Section 4):

$$\mathcal{A}_\varepsilon = \mathcal{A}_0 - \varepsilon^2 \mathcal{B} + \mathcal{R}_\varepsilon \tag{3.13}$$

with a remainder $\mathcal{R}_\varepsilon$ of order $o\left(\varepsilon^2\right)$ in $\mathcal{L}(H^{\frac{1}{2}}(\Gamma_R), H^{-\frac{1}{2}}(\Gamma_R))$, leads to the following asymptotic expansion of the solution in the truncated domain Ω_R

$$z^\varepsilon = z + \varepsilon^2 q + o\left(\varepsilon^2\right)$$

where $z^\varepsilon := z^\varepsilon(\Omega_R)$ is the solution of the auxiliary variational inequality problem:

Find $z \in K(\Omega_R)$ such that

$$a(\Omega_R; z, v - z) - \varepsilon^2 \langle \mathcal{B}(z), v - z\rangle_{\Gamma_R} - \mathcal{L}(\Omega_R; v - z) \geq 0, \quad \forall v \in K(\Omega_R) \tag{3.14}$$

where $\varepsilon > 0$ is small enough. The solution $z^\varepsilon \in H^1(\Omega_R)$ is an outer approximation of the solution $z_\varepsilon \in H^1(\Omega_\varepsilon)$.

Hence, the singular perturbation of the geomerical domain can be replaced by a regular perturbation of the bilineair form without losing the precision necessary for the evaluation of the topological derivatives. For such regular perturbations, one can use the standard sensitivity analysis of variational inequalities over polyhedric sets in Dirichlet spaces (see Sokolowski and Zolesio, 1992).

4. Expansion of Steklov-Poincaré operator in Γ_R

4.1. Explicit solutions

In order to establish the exact formula for the Steklov-Poincaré operator, we need the analytic form of solutions to the Helmholtz equation in the circular and the ring domains B_R and $C(\varepsilon, R)$, respectively.

Proposition 2 *For every $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$ the solution z_0^φ of (3.9) is given by*

$$z_0^\varphi(r, \theta) = \sum_{n\in\mathbb{Z}} \frac{I_n(r)}{I_n(R)} \varphi_n e^{in\theta}$$

where (r, θ) are the polar coordinates, φ_n are the Fourier coefficients in the expansion of φ, and symbols $I_n(r)$ denote the modified Bessel functions, given by

$$I_n(r) = \sum_{k\in\mathbb{N}} \frac{(r/2)^{n+2k}}{k!\,(n+k)!} \quad \forall n \in \mathbb{N} \tag{4.15}$$

with $I_{-n} = I_n$.

Proof. The solution z_0^φ of (3.9) is in $H^1(B_R)$ so we can write z_0^φ in the polar coordinate around the origin by the following form

$$z_0^\varphi(r,\theta) = \sum_{n\in\mathbb{Z}} p_n(r)\, e^{in\theta}.$$

By replacing in the Helmholtz equation, this leads to the modified Bessel equations (see Watson, 1922; Whittaker and Watson, 1963):

$$r^2 p_n''(r) + r p_n'(r) - \left(r^2+n^2\right) p_n(r) = 0 \ \forall n\in\mathbb{Z}$$

whose solutions are given by the sum of the first and second kind of Bessel functions I_n and K_n, respectively. Namely

$$p_n(r) = C_n I_n(r) + D_n K_n(r) \quad \forall n\in\mathbb{Z}$$

for some constants C_n and D_n. With I_n given by (4.15) and

$$K_n(r) := -\ln(\frac{r}{2}) I_n(r) + \frac{1}{2}\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}(\frac{r}{2})^{2k-n} \tag{4.16}$$

$$+\frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^k}{k!(n+k)!}[\psi(k)+\psi(n+k)](\frac{r}{2})^{2k+n}$$

for any $n>0$ and

$$K_0(r) := -\ln(\frac{r}{2}) I_0(r) + \sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}\psi(k)(\frac{r}{2})^{2k}; \tag{4.17}$$

here ψ is the logarithmic derivative of the Gamma function Γ, that is,

$$\psi(x) := \frac{\partial}{\partial h}\ln\Gamma(x+h).$$

Furthermore, as $z_0^\varphi \in H^1(B_R)$ $D_n = 0$, for all $n\in\mathbb{Z}$ and thus

$$z_0^\varphi(r,\theta) = \sum_{n\in\mathbb{Z}} C_n I_n(r)\, e^{in\theta} \tag{4.18}$$

and the boundary condition leads to

$$C_n I_n(R) = \varphi_n.$$

Substituting into (4.18), one gets the desired result. ∎

The next proposition gives the analytic form of the solution in the ring and a comparison between solutions in the circular and the ring domains.

PROPOSITION 3 *For every $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$ the solution z_ε^φ of (3.10) is given by*

$$z_\varepsilon^\varphi(r,\theta) = \sum_{n\in\mathbb{Z}} \frac{K_n'(\varepsilon) I_n(r) - I_n'(\varepsilon) K_n(r)}{I_n(R) K_n'(\varepsilon) - I_n'(\varepsilon) K_n(R)} \varphi_n \; e^{in\theta}; \tag{4.19}$$

we have also that $z_\varepsilon^\varphi = z_0^\varphi + y_\varepsilon^\varphi$ with

$$y_\varepsilon^\varphi(r,\theta) = -\sum_{n\in\mathbb{Z}} \frac{\dfrac{I_n'(\varepsilon)}{K_n'(\varepsilon)}}{\dfrac{I_n(R)}{K_n(R)} - \dfrac{I_n'(\varepsilon)}{K_n'(\varepsilon)}} \left(\frac{K_n(r)}{K_n(R)} - \frac{I_n(r)}{I_n(R)} \right) \varphi_n \; e^{in\theta} \tag{4.20}$$

where I_n and K_n are given by (4.15) and (4.16-4.17), respectively.

Proof. We get expression (4.19) by following the same technique as the one used in the proof of Proposition 2. To obtain the decomposition $z_\varepsilon^\varphi = z_0^\varphi + y_\varepsilon^\varphi$ with y_ε^φ given by (4.20), it suffices to check that

$$\frac{K_n'(\varepsilon) I_n(r) - I_n'(\varepsilon) K_n(r)}{I_n(R) K_n'(\varepsilon) - I_n'(\varepsilon) K_n(R)} = \frac{I_n(r)}{I_n(R)} - \frac{\dfrac{I_n'(\varepsilon)}{K_n'(\varepsilon)}}{\dfrac{I_n(R)}{K_n(R)} - \dfrac{I_n'(\varepsilon)}{K_n'(\varepsilon)}} \left(\frac{K_n(r)}{K_n(R)} - \frac{I_n(r)}{I_n(R)} \right).$$

■

REMARK 1 *Since $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$, we can write φ as the Fourier series expansion in terms of θ :*

$$\varphi(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)) \tag{4.21}$$

with the Fourier coefficients satisfying

$$\sum_{n=1}^{\infty} \sqrt{1+n^2}(a_n^2 + b_n^2) \leq C,$$

where C is a constant depending only on R.

Whence, one gets the following expression for z_0^φ

$$z_0^\varphi(r,\theta) = \frac{1}{2}a_0 \frac{I_0(r)}{I_0(R)} + \sum_{n=1}^{\infty} \frac{I_n(r)}{I_n(R)} (a_n \cos(n\theta) + b_n \sin(n\theta)). \tag{4.22}$$

Similarly,

$$y_\varepsilon^\varphi(r,\theta) = -\frac{1}{2}a_0 \frac{\dfrac{I_0'(\varepsilon)}{K_0'(\varepsilon)}}{\dfrac{I_0(R)}{K_0(R)} - \dfrac{I_0'(\varepsilon)}{K_0'(\varepsilon)}} \left(\frac{K_0(r)}{K_0(R)} - \frac{I_0(r)}{I_0(R)} \right) \tag{4.23}$$

$$- \sum_{n=1}^{\infty} \frac{\dfrac{I_n'(\varepsilon)}{K_n'(\varepsilon)}}{\dfrac{I_n(R)}{K_n(R)} - \dfrac{I_n'(\varepsilon)}{K_n'(\varepsilon)}} \left(\frac{K_n(r)}{K_n(R)} - \frac{I_n(r)}{I_n(R)} \right) (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

4.2. Asymptotic expansion of Steklov-Poincaré operator

Now, by explicit solutions in B_R and $C(\varepsilon, R)$ we can obtain an expansion to Steklov-Poincaré operator (see Sokolowski and Zochowski, 2013). Let us first give the following useful Lemma. It gives asymptotic expansions for Bessel functions.

Lemma 2 *(Abramowitz and Stegun, 1964; Watson, 1922) For $\varepsilon > 0$ small enough, we have for $n \geq 1$,*

$$I_n'(\varepsilon) = \frac{\varepsilon^{n-1}}{2^n (n-1)!} + o\left(\varepsilon^{n+1}\right)$$

$$K_n'(\varepsilon) = \frac{n! 2^{n-1}}{\varepsilon^{n+1}} + o\left(\varepsilon^{-n-1}\right)$$

$$\frac{I_n'(\varepsilon)}{K_n'(\varepsilon)} = -\frac{\varepsilon^{2n}}{n!(n-1)! 2^{2n-1}} + o\left(\varepsilon^{2n}\right)$$

and for $n = 0$,

$$I_0'(\varepsilon) = \frac{\varepsilon}{2} + o\left(\varepsilon^3\right)$$

$$K_0'(\varepsilon) = -\frac{1}{\varepsilon} + o(\varepsilon)$$

$$\frac{I_0'(\varepsilon)}{K_0'(\varepsilon)} = -\frac{\varepsilon^2}{2} + o\left(\varepsilon^2\right).$$

Consequently, $\dfrac{I_n'(\varepsilon)}{K_n'(\varepsilon)} = O\left(\varepsilon^{2n}\right)$ for any $n \geq 1$ and $\dfrac{I_0'(\varepsilon)}{K_0'(\varepsilon)} = O\left(\varepsilon^2\right)$. Moreover,

$$\int_\varepsilon^R r K_n'(r)^2 = O\left(\varepsilon^{-2n}\right) \quad \forall n \geq 1, \qquad \int_\varepsilon^R r K_0'(r)^2 = O(\ln \varepsilon) \tag{4.24}$$

$$\int_\varepsilon^R r I_n'(r)^2 = O(1) \quad \forall n \geq 0. \tag{4.25}$$

Here is the result on the asymptotic expansion the Steklov-Poincaré operator.

Proposition 4 *The Steklov-Poincaré operator admits the following expansion with respect to $\varepsilon \to 0^+$*

$$\langle \mathcal{A}_\varepsilon(\varphi)\ ,\varphi\rangle_{\Gamma_R} = \langle \mathcal{A}_0(\varphi),\varphi\rangle_{\Gamma_R} - \langle \mathcal{B}(\varphi),\varphi\rangle_{\Gamma_R}\varepsilon^2 + o\left(\varepsilon^2\right) \tag{4.26}$$

with

$$\langle \mathcal{B}(\varphi)\ ,\varphi\rangle_{\Gamma_R} = \frac{a_0^2\pi}{4I_0(R)^2} + \frac{\left(a_1^2+b_1^2\right)\pi}{2I_1(R)^2} \tag{4.27}$$

and a_0, a_1, b_1 are Fourier coefficients of $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$, given by

$$\begin{aligned}
a_0 &= \frac{1}{2\pi R}\int_{\Gamma_R} z_0^\varphi ds;\\
a_1 &= \frac{1}{\pi R^2}\int_{\Gamma_R} z_0^\varphi x_1 ds;\\
b_1 &= \frac{1}{\pi R^2}\int_{\Gamma_R} z_0^\varphi x_2 ds.
\end{aligned}$$

Proof. Let $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$ be as in (4.21), the solution of (3.10) is decomposed as $z_\varepsilon^\varphi = z_0^\varphi + y_\varepsilon^\varphi$;

$$\begin{aligned}
\langle \mathcal{A}_\varepsilon(\varphi),\varphi\rangle_{C(\varepsilon,R)} &= \int\limits_{C(\varepsilon,R)} |\nabla z_\varepsilon^\varphi|^2 + (z_\varepsilon^\varphi)^2 = \int\limits_{C(\varepsilon,R)} |\nabla(z_0^\varphi + y_\varepsilon^\varphi)|^2 + (z_0^\varphi + y_\varepsilon^\varphi)^2\\
&= \int\limits_{C(\varepsilon,R)} (|\nabla z_0^\varphi|^2 + (z_0^\varphi)^2) + 2\int\limits_{C(\varepsilon,R)} (\nabla z_0^\varphi.\nabla y_\varepsilon^\varphi + z_0^\varphi.y_\varepsilon^\varphi)\\
&+ \int\limits_{C(\varepsilon,R)} (|\nabla y_\varepsilon^\varphi|^2 + (y_\varepsilon^\varphi)^2)\\
&\pm \int\limits_{B_\varepsilon(x_0)} (|\nabla z_0^\varphi|^2 + (z_0^\varphi)^2)\\
&= \langle \mathcal{A}_0(\varphi),\varphi\rangle_{\Gamma_R} + R_1^\varphi(\varepsilon) + R_2^\varphi(\varepsilon) + R_3^\varphi(\varepsilon)
\end{aligned}$$

with

$$\begin{aligned}
R_1^\varphi(\varepsilon) &:= \int\limits_{C(\varepsilon,R)} |\nabla y_\varepsilon^\varphi|^2 + (y_\varepsilon^\varphi)^2,\\
R_2^\varphi(\varepsilon) &:= 2\int\limits_{C(\varepsilon,R)} \nabla z_0^\varphi.\nabla y_\varepsilon^\varphi + z_0^\varphi.y_\varepsilon^\varphi,\\
R_3^\varphi(\varepsilon) &:= -\int\limits_{B_\varepsilon} |\nabla z_0^\varphi|^2 + (z_0^\varphi)^2
\end{aligned}$$

where z_0^φ, y_ε^φ are defined by (4.22) and (4.23), respectively.

Let us evaluate each of the terms, $R_1^\varphi(\varepsilon)$, $R_2^\varphi(\varepsilon)$ and $R_3^\varphi(\varepsilon)$. Recall first that for any function

$$|\nabla f|^2 = (\partial_r f)^2 + \frac{1}{r^2}(\partial_\theta f)^2$$

therefore,

$$R_1^\varphi(\varepsilon) := \int\limits_{C(\varepsilon,R)} |\nabla y_\varepsilon^\varphi|^2 + (y_\varepsilon^\varphi)^2 = \int\limits_{C(\varepsilon,R)} \left[(\partial_r y_\varepsilon^\varphi)^2 + \frac{1}{r^2}(\partial_\theta y_\varepsilon^\varphi)^2 + (y_\varepsilon^\varphi)^2\right]$$

with

$$\partial_r y_\varepsilon^\varphi(r,\theta) = -\frac{1}{2}a_0 \frac{\dfrac{I_0'(\varepsilon)}{K_0'(\varepsilon)}}{\dfrac{I_0(R)}{K_0(R)} - \dfrac{I_0'(\varepsilon)}{K_0'(\varepsilon)}} \left(\frac{K_0'(r)}{K_0(R)} - \frac{I_0'(r)}{I_0(R)}\right)$$

$$-\sum_{n=1}^{\infty} \frac{\dfrac{I_n'(\varepsilon)}{K_n'(\varepsilon)}}{\dfrac{I_n(R)}{K_n(R)} - \dfrac{I_n'(\varepsilon)}{K_n'(\varepsilon)}} \left(\frac{K_n'(r)}{K_n(R)} - \frac{I_n'(r)}{I_n(R)}\right)(a_n\cos(n\theta) + b_n\sin(n\theta)),$$

and

$$\frac{1}{r}\partial_\theta y_\varepsilon^\varphi(r,\theta) =$$
$$-\sum_{n=1}^{\infty} \frac{\dfrac{I_n'(\varepsilon)}{K_n'(\varepsilon)}}{\dfrac{I_n(R)}{K_n(R)} - \dfrac{I_n'(\varepsilon)}{K_n'(\varepsilon)}} \frac{n}{r}\left(\frac{K_n(r)}{K_n(R)} - \frac{I_n(r)}{I_n(R)}\right)(b_n\cos(n\theta) - a_n\sin(n\theta)).$$

Using the orhogonality of trigonometric functions in $[0,2\pi]$ and integrating with respect to θ, we find

$$\int\limits_{C(\varepsilon,R)} (\partial_r y_\varepsilon^\varphi)^2 = \pi(a_1^2+b_1^2)\left(\frac{\frac{I_1'(\varepsilon)}{K_1'(\varepsilon)}}{\frac{I_1(R)}{K_1(R)} - \frac{I_1'(\varepsilon)}{K_1'(\varepsilon)}}\right)^2 \int\limits_\varepsilon^R \left(\frac{K_1'(r)}{K_1(R)} - \frac{I_1'(r)}{I_1(R)}\right)^2 r dr + o\left(\varepsilon^2\right)$$

and

$$\int\limits_{C(\varepsilon,R)} \frac{1}{r^2}(\partial_\theta y_\varepsilon^\varphi)^2 = \pi(a_1^2+b_1^2)\left(\frac{\frac{I_1'(\varepsilon)}{K_1'(\varepsilon)}}{\frac{I_1(R)}{K_1(R)} - \frac{I_1'(\varepsilon)}{K_1'(\varepsilon)}}\right)^2 \int\limits_\varepsilon^R \frac{1}{r}\left(\frac{K_1(r)}{K_1(R)} - \frac{I_1(r)}{I_1(R)}\right)^2 dr + o\left(\varepsilon^2\right)$$

by taking into account that $\dfrac{I_n'(\varepsilon)}{K_n'(\varepsilon)} = O\left(\varepsilon^{2n}\right)$ $(n \geq 1)$ and $\dfrac{I_0'(\varepsilon)}{K_0'(\varepsilon)} = O\left(\varepsilon^{2}\right)$.

Clearly, $\int\limits_{C(\varepsilon,R)} (y_\varepsilon^\varphi)^2 = o\left(\varepsilon^2\right)$. Thus,

$$\mathcal{R}_1^\varphi(\varepsilon) = \pi(a_1^2+b_1^2)\left[\frac{\frac{I_1'(\varepsilon)}{K_1'(\varepsilon)}}{\frac{I_1(R)}{K_1(R)} - \frac{I_1'(\varepsilon)}{K_1'(\varepsilon)}}\right]^2 \mathcal{R}_{1,1}(\varepsilon) + o\left(\varepsilon^2\right)$$

with

$$\mathcal{R}_{1,1}(\varepsilon) := \int\limits_\varepsilon^R \left[\left(\frac{K_1'(r)}{K_1(R)} - \frac{I_1'(r)}{I_1(R)}\right)^2 + \frac{1}{r^2}\left(\frac{K_1(r)}{K_1(R)} - \frac{I_1(r)}{I_1(R)}\right)^2\right] rdr.$$

Now, by the properties (4.24-4.25), given from Lemma 2, this yields

$$\mathcal{R}_1^\varphi(\varepsilon) = \frac{\pi(a_1^2+b_1^2)}{4I_1^2(R)}\varepsilon^2 + o\left(\varepsilon^2\right). \tag{4.28}$$

And for $R_2^\varphi(\varepsilon)$, we have

$$\begin{aligned} R_2^\varphi(\varepsilon) &:= 2\int\limits_{C(\varepsilon,R)} \left[\nabla z_0^\varphi . \nabla y_\varepsilon^\varphi + z_0^\varphi y_\varepsilon^\varphi\right] \\ &= 2(\int\limits_{C(\varepsilon,R)} y_\varepsilon^\varphi\left(-\Delta z_0^\varphi + z_0^\varphi\right) + \int\limits_{\Gamma_R} y_\varepsilon^\varphi \partial_n z_0^\varphi + \int\limits_{\Gamma_\varepsilon} y_\varepsilon^\varphi \partial_n z_0^\varphi) \\ &= -2\int\limits_{\Gamma_\varepsilon} y_\varepsilon^\varphi \partial_n y_\varepsilon^\varphi \end{aligned}$$

because $y_\varepsilon^\varphi = 0$ on Γ_R and $\partial_n z_0^\varphi = -\partial_n y_\varepsilon^\varphi$ on Γ_ε. Observe that $\partial_n y_\varepsilon^\varphi = -\partial_r y_\varepsilon^\varphi$, since the normal vector n is outer to the ring $C(\varepsilon, R)$. So, in view of Lemma 2, we obtain

$$\begin{aligned} R_2^\varphi(\varepsilon) &= 2\pi\left(a_1^2+b_1^2\right)\left(\frac{\frac{I_1'(\varepsilon)}{K_1'(\varepsilon)}}{\frac{I_1(R)}{K_1(R)} - \frac{I_1'(\varepsilon)}{K_1'(\varepsilon)}}\right)^2 \int\limits_\varepsilon^R \frac{K_1'(r)\,K_1(r)}{K_1^2(R)} rdr + o\left(\varepsilon^2\right) \\ &= -\frac{\pi\left(a_1^2+b_1^2\right)}{2I_1(R)^2}\varepsilon^2 + o\left(\varepsilon^2\right). \end{aligned} \tag{4.29}$$

There remains $R_3^\varphi(\varepsilon)$. As

$$R_3^\varphi(\varepsilon) := -\int\limits_{B_\varepsilon}\left[|\nabla z_0^\varphi|^2 + (z_0^\varphi)^2\right] = -\int_{\Gamma_\varepsilon} z_0^\varphi \partial_n z_0^\varphi$$
$$= -\int_{\Gamma_\varepsilon} z_0^\varphi \partial_r z_0^\varphi$$

with z_0^φ given by (4.22) and

$$\partial_r z_0^\varphi = \frac{1}{2}a_0\frac{I_0'(r)}{I_0(R)} + \sum_{n=1}^{\infty}\frac{I_n'(r)}{I_n(R)}(a_n\cos(n\theta) + b_n\sin(n\theta)),$$

we find, by the same manner, that

$$R_3^\varphi(\varepsilon) = -\left(\frac{\pi a_0^2}{4I_0(R)^2} + \frac{\pi\left(a_1^2+b_1^2\right)}{4I_1(R)^2}\right)\varepsilon^2 + o\left(\varepsilon^2\right). \tag{4.30}$$

Finally, upon collecting the formulas (4.28), (4.29) and (4.30), the result follows. ∎

5. Topological derivative of shape functional

5.1. Conical derivative of z^ε in Ω

As seen previously, the singular perturbation of the geomerical domain is replaced by a regular perturbation of the bilineair form, for which we use the standard sensitivity analysis of variational inequalities over polyhedric sets. Then, the first order expansion of the solution with respect to small parameter is obtained. To this end, we recall an abstract result, which is a generalization of the implicit function theorem for variational inequalities (see Sokolowski and Zolesio, 1992, Theorem 4.14, p. 177).

In a closed convex set K of a Hilbert space V, consider the following family of variational inequalites, depending on the parameter $t \in [0, \delta)$ with $\delta > 0$,

$$y_t \in K,\ a_t(y_t, \varphi - y_t) \geq \langle f_t, \varphi - y_t\rangle_{V',V} \quad \forall\varphi \in K \tag{5.31}$$

with $a_t(\cdot,\cdot) : V \times V \to \mathbb{R}$ being a bilinear form, $f_t \in V'$ and $y_t := \mathcal{P}_t(f_t)$ being the solution of (5.31).

Theorem 1 *Assume that the bilinear form $a_t(\cdot,\cdot) : V \times V \to \mathbb{R}$ is coercive and continuous uniformly with respect to $t \in [0, \delta)$. Let $\mathcal{A}_t \in \mathcal{L}(V; V')$ be the linear operator defined by: $a_t(\phi, \varphi) = \langle \mathcal{A}_t\phi, \varphi\rangle_{V',V}$ for all $\phi, \varphi \in V$. We suppose that:*

1. *there exists $\mathcal{A}' \in \mathcal{L}(V; V')$ such that*

$$\mathcal{A}_t = \mathcal{A}_0 + t\mathcal{A}' + o(t) \quad \text{in } \mathcal{L}(V; V')$$

2. *for $t > 0$ small enough,*

$$f_t = f_0 + tf' + o(t) \qquad \text{in } V'$$

where $f_0, f' \in V'$.

3. *for the solutions to the variational inequality*

$$\Pi f := \mathcal{P}_0(f) \in K, \quad a_0(\Pi f, \varphi - \Pi f) \geq \langle f, \varphi - \Pi f \rangle_{V',V} \quad \forall \varphi \in K$$

the following differential stability result holds

$$\forall h \in V', \quad \Pi(f_0 + \varepsilon h) = \Pi f_0 + \varepsilon \Pi' h + o(\varepsilon) \quad \text{in } V$$

for $\varepsilon > 0$ small enough, where the mapping $\Pi' : V' \to V$ is continuous and positively homogeneous and $o(\varepsilon)$ is uniform with respect to $h \in V'$ on compact subsets of V'.

Then the solutions to the variational inequality (5.31) are right differentiable with respect to t at $t = 0$, and for t small enough,

$$\mathrm{y}_t = \mathrm{y}_0 + t\mathrm{y}' + o(t) \quad \text{in } V$$

where

$$\mathrm{y}' = \Pi'(f' - \mathcal{A}'\mathrm{y}_0).$$

The following theorem ensures the existence of the conical differential of the mapping

$$(0, \varepsilon_0] \ni \varepsilon \to z_\varepsilon^R \in H^1(\Omega_R)$$

where z_ε^R is the restriction of the solution z_ε to Ω_R. For this aim, consider, for $z^R \in K := K(\Omega_R)$, the restriction of the solution z to Ω_R for $\varepsilon = 0$, the convex cone

$$S_K\left(z^R\right) := \left\{ \begin{array}{c} v \in H^1_{\Gamma_0}(\Omega_R) : v \geq 0 \text{ q.e. on } \Xi\left(z^R\right) \\ a\left(\Omega_R; z^R, v\right) + \left\langle \mathcal{A}_0\left(z^R\right), z^R \right\rangle_{\Gamma_R} = \mathcal{L}(\Omega_R, v) \end{array} \right\} \tag{5.32}$$

where $\Xi\left(z^R\right) := \left\{x \in \Gamma_s : z^R(x) = 0\right\}$ is the coincidence set.

Theorem 2 *For ε sufficiently small, we have the following expansion of z_ε^R with respect to the parameter ε, at 0^+*

$$z_\varepsilon^R = z^R + \varepsilon^2 q^R + o\left(\varepsilon^2\right)$$

where q^R is the unique solution of the following variational inequality

$$q^R \in S_K\left(z^R\right), \quad a\left(\Omega_R, q^R, v - q^R\right) - \left\langle \mathcal{B}\left(z^R\right), v - q^R \right\rangle_{\Gamma_R} \geq 0, \quad \forall v \in S_K\left(z^R\right) \tag{5.33}$$

Proof. We shall apply the abstract result with

$$a_t(\cdot,\cdot) := a(\Omega_R;\cdot,\cdot) + \langle \mathcal{A}_0(\cdot),\cdot\rangle_{\Gamma_R} - t\langle \mathcal{B}(\cdot),\cdot\rangle_{\Gamma_R},$$

$f_t \equiv f$ and $t = \varepsilon^2$, to get an asymptotic expansion of the solution z_ε^R with respect to ε. Observe that the first assumption is satisfied with

$$\langle \mathcal{A}'(w), v\rangle_{\Gamma_R} = -\langle \mathcal{B}(w), v\rangle_{\Gamma_R} \text{ for all } w, v \in H^{\frac{1}{2}}(\Gamma_R)$$

and the second assumption is obviously satisfied. Let us consider now $z^t = \mathcal{P}_t(f) \in K$, the solution of

$$a_t\left(z^t, v - z^t\right) \geq \mathcal{L}\left(\Omega_R; v - z^t\right) \quad \forall v \in K$$

and for $t = 0$, let $z^R := \mathcal{P}_0(f) \in K$, the solution of the variational inequality for $t = 0$, that is,

$$a\left(\Omega_R; z^R, v - z^R\right) + \left\langle \mathcal{A}_0\left(z^R\right), z^R\right\rangle_{\Gamma_R} \geq \mathcal{L}\left(\Omega_R; v - z^R\right) \quad \forall v \in K.$$

Since the convex set K is polyhedric (see Sokolowski and Zolesio, 1992), the conical derivative of the metric projection onto K at z^R is given by the metric projection onto the cone $S_K\left(z^R\right)$ (Haraux, 1977; Mignot, 1976), that is, one has

$$P_K(z^R + th) = P_K(z^R) + t\mathcal{P}_{S_K(z^R)}(h) + o(t)$$

where the remainder $o(t)$ is uniform on compact subsets of $H^1_{\Gamma_0}(\Omega_R)$. So, the derivative $\mathcal{P}_{S_K(z^R)}(h) \in S_K\left(z^R\right)$ is the unique solution of the variational inequality

$$a(\Omega_R, \mathcal{P}_{S_K(z^R)}(h) - h, v - \mathcal{P}_{S_K(z)}(h)) \geq 0 \quad \forall v \in S_K\left(z^R\right).$$

Therefore, the solution mapping $\Pi : f \to \Pi f \equiv \mathcal{P}_0(f)$, the solution of the variational inequality for $t = 0$, is conically differentiable, with the conical derivative $\Pi' h$ given by the unique solution to the variational inequality

$$\Pi' h \in S_K\left(z^R\right), \quad a\left(\Omega_R, \Pi' h - h, v - \Pi' h\right) \geq 0 \quad \forall v \in S_K\left(z^R\right).$$

Hence, all assumptions of Theorem 1 having been checked, we conclude the conical differentiability of solutions with respect to regular perturbations of bilineair form and

$$z_\varepsilon^R = z^R + \varepsilon^2 q^R + o\left(\varepsilon^2\right)$$

with $q^R := \Pi'(\mathcal{B}z^R)$ being the solution of (5.33). ∎

This Theorem leads to our main results.

5.2. Topological derivative of shape functional

Our first result gives the topological derivative of the tracking type shape functional. Indeed, we have

THEOREM 3 *For the shape functional $\mathcal{J}$, given by*

$$\mathcal{J}(\Omega_\varepsilon) = \frac{1}{2}\int\limits_{\Omega_w} (z_\varepsilon - z_d)^2 = \frac{1}{2}\int\limits_{\Omega_w} (z_\varepsilon^R - z_d)^2 : = j(\varepsilon),$$

the topological derivative is given by

$$\mathcal{T}(\mathcal{O}) = j''(0) = \int\limits_{\Omega_w} (z^R - z_d) q^R,$$

where q^R is the solution of (5.33).

Proof. The shape functional $\mathcal{J}$ is composed of two functions, scalar product in $L^2(\Omega)$ and the state in singular domain $\varepsilon \to z_\varepsilon$, so by the chain rule, we find

$$j'(\varepsilon) = \int_{\Omega_w} \left(z_\varepsilon^R - z_d\right) . \left(z_\varepsilon^R\right)',$$

where the prime denotes the derivative with respect to the small parameter ε, and the fact that there is no first correction in the asymptotic expansion for z_ε^R leads to

$$j'\left(0^+\right) = \lim_{\varepsilon \to 0} j'(\varepsilon) = 0.$$

Now, for the second derivative, we have

$$j''(\varepsilon) = \int_{\Omega_w} \left(z_\varepsilon^R - z_d\right)' . z_\varepsilon^R + \left(z_\varepsilon^R - z_d\right) . \left(z_\varepsilon^R\right)''$$

at $\varepsilon = 0^+$. The first term in integral vanishes, since $\left(z_\varepsilon^R\right)' |_{\varepsilon=0^+} = 0$, and for the second term, we have $\left(z_\varepsilon^R\right)'' |_{\varepsilon=0^+} = q^R$. Finally, we have

$$\mathcal{T}(\mathcal{O}) = j''\left(0^+\right) = \int\limits_{\Omega_w} (z^R - z_d) q^R.$$

■

The second result is devoted to the topological derivative of the energy functional.

THEOREM 4 *The topological derivative of the energy functional is*

$$\mathcal{T}(\mathcal{O}) = \int\limits_{\Omega_R} \nabla z^R \nabla q^R + z^R q^R - \int\limits_{\Omega_w} f q^R - \frac{1}{2}\left\langle \mathcal{B}\left(z^R\right), z^R \right\rangle_{\Gamma_R}$$

with q^R being the solution of (5.33), z^R the solution of the problem in Ω_R and $\mathcal{B}$ defined by the asymptotic expansion (3.13) of the Steklov-Poincaré operator.

Since q^R is orthogonal to the solution z^R by the definition of cone (5.32) we get

$$\mathcal{T}(\mathcal{O}) = -\frac{1}{2}\left\langle \mathcal{B}\left(z^R\right), z^R\right\rangle_{\Gamma_R}.$$

Proof. For the energy shape functional we have the following decomposition

$$\begin{aligned}
\mathcal{E}(\Omega_\varepsilon) &:= \frac{1}{2}a(\Omega_\varepsilon, z_\varepsilon, z_\varepsilon) - \mathcal{L}\left(\Omega_w; z_\varepsilon\right) \\
&= \frac{1}{2}a(\Omega_R, z_\varepsilon^R, z_\varepsilon^R) + \frac{1}{2}\left\langle \mathcal{A}_\varepsilon\left(z_\varepsilon\right), z_\varepsilon\right\rangle_{\Gamma_R} - \mathcal{L}\left(\Omega_w; z_\varepsilon^R\right) \\
&= \mathcal{E}(\Omega_R) + \frac{1}{2}\left\langle \mathcal{A}_\varepsilon\left(z_\varepsilon\right), z_\varepsilon\right\rangle_{\Gamma_R} \\
&=: j_1(\varepsilon) + j_2(\varepsilon).
\end{aligned}$$

For j_1, we perform the same analysis as in the previous theorem, and so

$$j_1\left(\varepsilon\right) = \frac{1}{2}\int\limits_{\Omega_R} \left|\nabla z_\varepsilon^R\right|^2 + (z_\varepsilon^R)^2 - \int\limits_{\Omega_w} f z_\varepsilon^R$$

and the first derivative of j_1 is given by

$$j_1'\left(\varepsilon\right) = \int\limits_{\Omega_R} \nabla z_\varepsilon^R \nabla\left(z_\varepsilon^R\right)' + z_\varepsilon^R\left(z_\varepsilon^R\right)' - \int\limits_{\Omega_w} f\left(z_\varepsilon^R\right)'.$$

Since $\left(z_\varepsilon^R\right)'|_{\varepsilon=0^+} = 0$, this yields

$$j'\left(\varepsilon\right)|_{\varepsilon=0^+} = 0$$

and the second derivative at $\varepsilon = 0^+$,

$$j_1''\left(\varepsilon\right)|_{\varepsilon=0^+} = \int\limits_{\Omega_R} \nabla z^R \nabla q^R + z^R q^R - \int\limits_{\Omega_w} f q^R$$

leads to the topological derivative of j_1 being equal zero, taking into account the orthogonality condition in the definition of cone (5.32).

For j_2, we use the asymptotic formula of Steklov-Poincaré operator,

$$\left\langle \mathcal{A}_\varepsilon\left(\varphi\right), \varphi\right\rangle_{\Gamma_R} = \left\langle \mathcal{A}_0\left(\varphi\right), \varphi\right\rangle_{\Gamma_R} - \left\langle \mathcal{B}\left(\varphi\right), \varphi\right\rangle_{\Gamma_R}\varepsilon^2 + o\left(\varepsilon^2\right)$$

to deduce that $j_2''\left(0\right) = -\frac{1}{2}\left\langle \mathcal{B}\left(\varphi\right), \varphi\right\rangle_{\Gamma_R}$ with φ replaced by the trace of z^R.

Hence, the topological derivative of the energy shape functional is given by

$$\mathcal{T}(\mathcal{O}) = j_1''\left(0\right) - \frac{1}{2}\left\langle \mathcal{B}\left(z^R\right), z^R\right\rangle_{\Gamma_R} = -\frac{1}{2}\left\langle \mathcal{B}\left(z^R\right), z^R\right\rangle_{\Gamma_R}.$$

■

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