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Modulational instability of obliquely interacting capillary-gravity waves over infinite depth

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Two coupled time-dependent two dimensional nonlinear Schrödinger equations have been derived using multiscale expansion for two nonlinearly interacting capillary-gravity waves over an infinite depth of water. These equations are then utilised to discuss the modulational (Benjamin-Feir) instability of two Stokes wavetrains due to unidirectional and bidirectional perturbations. It is found from the graphs and the three-dimensional contour plots that the rate of growth of instability for two wave packets interacting obliquely is higher than the instance of modulation of one wave packet. We have likewise examined the influence of capillarity on modulational instability.

Key words: nonlinear Schrödinger equation, crossing sea states, capillary-gravity waves, modulational instability.

1. Introduction

In the analysis of nonlinear evolution of slowly modulated water waves, nonlinear Schrödinger equations are frequently used as they can properly reflect the Benjamin-Feir instability (modulational instability). For deep water waves, Benjamin and Feir [1–3] have described the experimental results of modulational instability. They have showed that when deep water wavetrains are formed at one end of a long tank the waves began to develop irregularities in wavenumber and amplitude. At last, at a large distance from the wavemaker, the wavetrains have disintegrated fully and have appeared to become random in nature.

A physical elucidation for the onset of modulational instability can be comprehended by studying the behavior of a weakly nonlinear deep water wavetrain which contains waves of uniform wavelengths initially. If the wave envelope has a sinusoidal nature, then the waves at the crests of the envelope will propagate forward more rapidly than the waves at the troughs of the envelope and this occurs owing to the nonlinearity of the wave motion. Accordingly, wave numbers will enhance before the envelope crests and diminish after the envelope crests.

Now, dispersive effects cause energy to come towards the crests of the envelope, causing the amplitude at the crests to enhance. This in turn expedites the instability.

The evolution problem of two surface wave packets for counter-propagating or co-propagating or obliquely propagating waves has been studied by different authors [4–7] in several contexts. Stability analysis for surface-gravity wave in an infinite depth of water in the presence of a second wave has been discussed by Roskes [8] using a system of cubic nonlinear Schrödinger equations. In particular, cases of instability and multiphase solitary envelop waves are taken into account. Later on, Dhar and Das [9] have made the same analysis as performed by Roskes [8] based on fourth order nonlinear evolution equation and they have derived two coupled fourth order nonlinear evolution equations for two Stokes wavetrains in an infinite depth of water. Onorato *et al.* [10] have described the stability analysis of two wave packets propagating in the same direction in shallow water. Starting from Zakharov's integral equation, two coupled fourth order nonlinear evolution equations have also been derived by Debsarma and Das [11] for two co-propagating capillary-gravity wave packets and they have observed that the presence of a uniform capillary-gravity wavetrain makes an enhancement in the growth rate of instability of a surface gravity wavetrain.

Recently, there has been great enthusiasm for studying an important subject related to the dynamics of a pair of obliquely interacting wave systems. Onorato *et al.* [4] have argued first about the main cause that the growth rate of instability for two obliquely interacting wave systems is higher than that for a single wavetrain.

Laine-Pearson [5] has argued that instability due to modulation can be considered as a possible mechanism for the generation of large amplitude freak waves in a situation of crossing sea states. After investigating weakly nonlinear interaction of two wave systems spreading along with two separate directions in deep water, he also inferred that the rate of growth of long-wave instability of two waves interacting obliquely is more significant than those due to resonant interaction of short-crested waves. Now, such freak waves may be formed due to both nonlinear effect and statistical, linear effect in which geometrical and spatio-temporal focusing is considered and may exist both in shallow and deep water [6]. Kharif and Pelinovsky [12] have presented a simple statistical analysis of the freak wave probability based on the assumption of a Gaussian wave field with random perturbations. It has been observed that the random perturbations can grow to produce inherently nonlinear water wave systems, generally known as freak waves, through the nonlinear interaction between two coupled water waves.

Starting from Zakharov's integral equation, Onorato *et al.* [4] have developed two coupled nonlinear Schrödinger equations for two nonlinearly interacting waves in deep water with two separate channels of spread. From these equations, they have found the instability growth rate for one dimensional two wave systems and observed that the growth rates rely on not just the wavelength of perturbation and steepness of the primary waves but also on the angle between the two wave systems. This outcome has then been extended by SHUKLA *et al.* [6] for investigating the rate of growth of instability for bidirectional perturbations employing two coupled nonlinear Schrödinger equations as derived by ONORATO *et al.* [4].

Again, ONORATO *et al.* [13] have performed experiments in two-wave basins of different dimensions and stated that the possibility of the generation of freak waves strongly relies upon the directional properties of the waves. Further, SENAPATI *et al.* [14] have determined two coupled nonlinear evolution equations in the case of crossing sea states in the presence of a uniform wind flow in deep water.

All these analyses made by the aforesaid authors are for gravity waves. In the present paper, we have made a stability analysis of capillary-gravity waves in a situation of crossing sea states over an infinite depth of water, starting from two coupled nonlinear Schrödinger equations. Therefore this paper is an extension of the work made by SHUKLA *et al.* [6] to incorporate capillarity, which is effective for small waves. Using a multiple scale method, we have obtained nonlinear Schrödinger equations to study slowly modulated waves, whereas ONORATO *et al.* [4] and SHUKLA *et al.* [6] have derived the nonlinear Schrödinger equations from Zakharov's integral equation. Here two capillary-gravity wavetrains are propagating obliquely and making equal angles with the direction which is taken as the x-axis. The present paper is sorted out as follows: Section 2 comprises of basic equations and supposition. The two coupled nonlinear Schrödinger

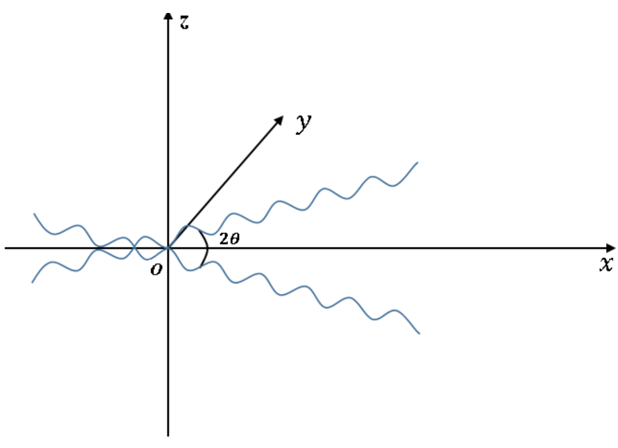

FIG. 1. Schematic diagram of the obliquely interacting two wavetrains on the xy plane.

equations are then derived in Section 3. Section 4 consists of stability of a uniform wavetrain. Finally, a discussion along with the conclusion is exhibited in Section 5.

2. Basic equations and assumption

We consider a Cartesian system of coordinates $oxyz$, where oxy represents the uninterrupted free surface of the water and z axis is taken vertically upwards. We assume that two wavetrains progress on the xy plane having basic wave numbers $\mathbf{k}_1 = (k, l)$ and $\mathbf{k}_2 = (k, -l)$, respectively. We take, in the perturbed state, $z = \eta(x, y, t)$ as the equation of the undulating free surface at time t. The perturbed velocity potential ϕ for irrotational and inviscid motion satisfies the following three dimensional Laplace equations,

$$\nabla^2\phi = 0, \qquad -\infty < z < \eta. \tag{2.1}$$

The kinematic condition is given by

$$\frac{\partial\phi}{\partial z} - \frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y} \quad \text{on } z = \eta. \tag{2.2}$$

At the free surface, the dynamic boundary condition can be expressed as

$$\begin{aligned} &\frac{\partial\phi}{\partial t} + g\eta \\ &\quad = -\frac{1}{2}(\nabla\phi)^2 + s\left[\frac{\eta_{xx}(1+\eta_y^2) + \eta_{yy}(1+\eta_x^2) - 2\eta_{xy}\eta_x\eta_y}{(1+\eta_x^2+\eta_y^2)^{\frac{3}{2}}}\right] \quad \text{on } z = \eta. \end{aligned} \tag{2.3}$$

Also ϕ should satisfy the following condition

$$\phi \to 0 \quad \text{as } z \to -\infty, \tag{2.4}$$

where the parameter s is the surface tension coefficient T divided by the density ρ of the bulk fluid and g is the gravitational acceleration.

It is assumed that the disturbance being a progressive wave, we consider the solutions of the above equations as follows

$$\begin{aligned} P = P_{00} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[&P_{mn}\exp\{i(m\psi_1 + n\psi_2)\} \\ &+ P_{mn}^*\exp\{-i(m\psi_1 + n\psi_2)\}], \quad (m, n) \neq (0, 0), \end{aligned} \tag{2.5}$$

where $\psi_1 = kx + ly - \omega t$ and $\psi_2 = kx - ly - \omega t$ represent the phase functions of the first and second wavetrains, respectively and P symbolizes for ϕ and η;

ϕ_{00}, ϕ_{mn}, ϕ^*_{mn} are functions of the slow modulation variables $x_1 = \delta x$, $y_1 = \delta y$, $t_1 = \delta t$ and z; η_{00}, η_{mn}, η^*_{mn} are functions of x_1, y_1, t_1. Here '$*$' indicates complex conjugate and δ is a slowness parameter which measures the weakness of nonlinearity.

In the present paper, we are dealing with two Stokes wavetrains whose amplitudes are finite and small, so we have introduced the slow modulation variables x_1, y_1, t_1 and we have obtained here analytical results for small wavenumbers of perturbations and small wave steepness.

The nonlinear spatio-temporal evolution of slowly modulated water-surface waves can be described by the nonlinear Schrödinger equation when the wave steepness is small ($\ll 1$) and the bandwidth is sufficiently narrow ($\ll 1$). Typically, one assumes that the bandwidth and the wave steepness are of the same order of magnitude $O(\delta)$, for which the leading dispersive and nonlinear effects balance at the third order $O(\delta^3)$. However, when dispersive and nonlinear effects are in balance, the solitary envelope waves, also called envelope solitons, may exist and multiphase solitary envelope wave solutions have been found under two conditions [8]. Now, the instabilities and solitary wave solutions for these conditions may be ascribed as self-modulation and self-focusing effects. Thus, δ represents both the slow modulations and the wave steepness (wave amplitude). Again, the derivation of cubic Schrödinger equations (3.13) and (3.14) requires that δ be a small parameter and describes a balance between nonlinearity and wave dispersion.

We now discuss the physical justification and mathematical explanation regarding the fundamental assumption that the modes $(2,0)$, $(0,2)$, $(1,1)$, $(1,-1)$ have weaker amplitudes than the modes $(1,0)$, $(0,1)$. In the expression for η, given by (2.5), the first term η_{00} is a real function, slowly varying in space and time and it represents the surface elevation brought about by the radiation stress of the waves. Again, η_{10} and η_{01} represent the leading first order amplitudes of order $O(\delta)$, where $\delta \ll 1$. Accordingly, $\eta_{10}\exp(i\psi_1)$ and $\eta_{01}\exp(i\psi_2)$ are known as primary or carrier wavetrains. Further, the amplitudes η_{mn} $[(m,n) = (2,0),(0,2),(1,1),(1,-1)]$ appear either due to the self-interaction of any one of the two wavetrains or due to the interaction of any one of the two wavetrains with the other one and are of order $O(\delta^2)$, as η_{mn} are the product of two leading first order amplitudes. Proceeding in this way we obtain third order amplitudes η_{mn} $[(m,n) = (3,0),(0,3),\ldots]$ and thus we get a convergent infinite series containing successively the weaker order of amplitudes. In other words, we may state that the primary waves having finite but small amplitudes produce higher harmonics either through nonlinear self-interaction or through nonlinear wave-wave interaction. As a result, the generated higher harmonics will be of amplitudes smaller than that of the primary harmonics.

As a_{mn}, b_{mn} for $(m,n) = (2,0),(0,2),(1,1),(1,-1)$ are of order $O(\delta^2)$, it follows from Eqs. (3.6) (or (3.7)) that η_{mn} are of order $O(\delta^2)$ for the said values of (m,n). Solving η_{mn}, for $(m,n) = (2,0),(0,2),(1,1),(1,-1)$ in terms of η_{10} and η_{01}, which are given in the Appendix A, we have found that $\eta_{11} = C\eta_{10}\eta_{01}$, where C depends on some known parameters. Now η_{10} and η_{01} are the leading first order of magnitude $O(\delta)$, where $\delta \ll 1$, whereas η_{mn}, for $(m,n) = (2,0),(0,2),(1,1),(1,-1)$ are the second order of magnitude $O(\delta^2)$. Thus we may conclude that the amplitudes η_{mn} for said values of (m,n) are weaker than η_{10} (or η_{01}).

For either wavetrain, the dispersion relation is given by

$$\omega^2 - gk_0 - s{k_0}^3 = 0 \tag{2.6}$$

and the group velocity is

$$c_g = \frac{d\omega}{dk_0} = \frac{g + 3sk_0^2}{2\omega}, \tag{2.7}$$

where $k = k_0 \cos\theta$ and $l = k_0 \sin\theta$, 2θ being the angle between two wavetrains.

3. Derivation of nonlinear evolution equations

Substituting Eq. (2.5) in Eq. (2.1) and then equating the coefficients of $\exp\{i(m\psi_1 + n\psi_2)\}$ on both sides of the aforesaid equations for $(m,n) = (1,0)$, $(0,1),(1,1),(1,-1),(2,0),(0,2)$ we obtain

$$\frac{d^2\phi_{mn}}{dz^2} = \Delta_{mn}^2 \phi_{mn}, \tag{3.1}$$

where

$$\Delta_{mn} = \left[\left\{(m+n)k - i\delta\frac{\partial}{\partial x_1}\right\}^2 + \left\{(m-n)l - i\delta\frac{\partial}{\partial y_1}\right\}^2\right]^{1/2}.$$

Therefore the solution of Eq. (3.1) satisfying condition (2.4) is

$$\phi_{mn} = e^{z\Delta_{mn}} A_{mn}, \tag{3.2}$$

where A_{mn} is a function of x_1, y_1, t_1.

For $(m,n) = (0,0)$, employing the Fourier transform w.r.t. x_1 and y_1 defined by

$$\overline{f}(K_x, K_y) = \frac{1}{2\pi} \iint\limits_{-\infty}^{\infty} f(x_1, y_1) e^{-i(K_x x_1 + K_y y_1)}\, \mathrm{d}x_1\, \mathrm{d}y_1, \tag{3.3}$$

we obtain the following equation

$$\frac{d^2\overline{\phi}_{00}}{dz^2} = \delta^2 K^2 \overline{\phi}_{00}, \tag{3.4}$$

where $K^2 = K_x^2 + K_y^2$ and K_x, K_y are Fourier transform parameters. On solving Eq. (3.4) along with the condition (2.4) we get

$$\overline{\phi}_{00} = e^{\delta k z}\overline{A}_{00}, \tag{3.5}$$

where $\overline{A}_{00}$ is a function of K_x, K_y, t_1.

We have selected only six pairs of m and n as the two third-order nonlinear terms in the cubic Schrödinger equations (3.13) and (3.14) appear due to the interactions of second harmonics η_{20}, η_{02}, η_{11}, η_{1-1} with the primary harmonics η_{10} and η_{01} together with the cubic interactions of η_{10} and η_{01}. The nonlinear terms due to the interactions of modes $(1,0)$ and $(0,1)$ with the other modes are of fourth and higher orders and therefore those modes are not required in deriving our third order nonlinear Schrödinger equations.

Inserting Eq. (2.5) into Eqs. (2.2) and (2.3) expanded in the Taylor series about $z = 0$ up to the second order and after equating the coefficients of $\exp\{i(m\psi_1 + n\psi_2)\}$ on both sides of aforesaid equations for $(m,n) = (1,0), (0,1), (2,0), (0,2), (1,1), (1,-1)$, we arrive at the following equations

$$\Delta_{mn}A_{mn} + iW_{mn}\eta_{mn} = a_{mn}, \tag{3.6}$$

$$-iW_{mn}A_{mn} + g\eta_{mn} + s\Delta_{mn}^2\eta_{mn} = b_{mn}, \tag{3.7}$$

where $W_{mn} = (m+n)\omega + i\delta\frac{\partial}{\partial t_1}$ and a_{mn}, b_{mn} are due to nonlinear terms. Applying Fourier transform for $(m,n) = (0,0)$ in (2.2) and (2.3) we have

$$\delta k\overline{A}_{00} - \delta\frac{\partial\overline{\eta}_{00}}{\partial t_1} = \overline{a}_{00}, \tag{3.8}$$

$$\delta\frac{\partial\overline{A}_{00}}{\partial t_1} + g\overline{\eta}_{00} + s\delta^2K^2\overline{\eta}_{00} = \overline{b}_{00}. \tag{3.9}$$

If we now eliminate A_{mn} between (3.6) and (3.7) on account of $(m,n) = (1,0)$, $(0,1)$, we arrive at

$$[W_{10}^2 - g\Delta_{10} - s\Delta_{10}^3]\eta_{10} = -iW_{10}a_{10} - \Delta_{10}b_{10}, \tag{3.10}$$

$$[W_{01}^2 - g\Delta_{01} - s\Delta_{01}^3]\eta_{01} = -iW_{01}a_{01} - \Delta_{01}b_{01}. \tag{3.11}$$

In order to solve the aforesaid two sets of equations given by (3.6)–(3.9) we introduce the perturbation expansions (following DHAR and DAS [9] and

SENAPATI *et al.* [14]) of the quantities A_{mn} and η_{mn} for $(m, n) = (1, 0), (0, 1), (2, 0), (0, 2), (1, 1), (1, -1), (0, 0)$ as follows:

$$(A_{mn}, \eta_{mn}) = \sum_j \delta^j (A_{mn}^{(j)}, \eta_{mn}^{(j)}). \tag{3.12}$$

Here the index j begins with $j = 1$ for $(m, n) = (1, 0), (0, 1)$ and $j = 2$ for $(m, n) = (2, 0), (0, 2), (1, 1), (1, -1), (0, 0)$. After calculating the nonlinear terms a_{mn} and b_{mn} for $(m, n) = (2, 0), (0, 2), (1, 1), (1, -1)$, correct up to $O(\delta^2)$ and then substituting the expansion (3.12) in (3.6) and (3.7), we solve for A_{mn} (in terms of η_{10} and η_{01}) for $(m, n) = (1, 0), (0, 1)$ correct up to $O(\delta^2)$. Again, using expansion (3.12) in (3.6) and (3.7), we obtain solutions for A_{mn} and η_{mn} (in terms of η_{10} and η_{01}) on account of $(m, n) = (2, 0), (0, 2), (1, 1), (1, -1)$, correct up to $O(\delta^2)$. These solutions are available in Appendix A. In a similar manner, from Eqs. (3.8) and (3.9) in the case of $(m, n) = (0, 0)$ we find a_{00} and b_{00}, which are at least of $O(\delta^3)$. Now, it follows from Eq. (3.9) that $\overline{\eta}_{00}$ is at least of $O(\delta^3)$ (as $\overline{\eta}_{00}^{(2)} = 0$) and from Eq. (3.8), it follows that, $\overline{A}_{00}$ is of order $O(\delta^2)$. Using these solutions, we then find a_{10} and b_{10} (in terms of η_{10} and η_{01}), correct up to $O(\delta^3)$. Substituting these solutions on right side of Eq. (3.10) and simplifying the left side of the same equation, we finally obtain a nonlinear Schrödinger equation of the first wave packet, whose basic wave number is (k, l), of the following form to describe the motion.

$$i\frac{\partial \eta_{10}}{\partial t_1} + i\gamma_1 \frac{\partial \eta_{10}}{\partial x_1} + i\gamma_2 \frac{\partial \eta_{10}}{\partial y_1} + \gamma_3 \frac{\partial^2 \eta_{10}}{\partial x_1^2} + \gamma_4 \frac{\partial^2 \eta_{10}}{\partial y_1^2} + \gamma_5 \frac{\partial^2 \eta_{10}}{\partial x_1 \partial y_1} = \Lambda_1 \eta_{10}^2 \eta_{10}^* + \Lambda_2 \eta_{10} \eta_{01} \eta_{01}^*. \tag{3.13}$$

The coefficients of Eq. (3.13) are available in Appendix B. It is important to note that the first nonlinear term on right side of Eq. (3.13) is due to self interaction of the first wave packet whereas, the second term arises due to nonlinear interaction of the first wave packet with the second one.

In a similar manner, for the other wave packet with basic wave number $(k, -l)$, we have obtained the Schrödinger equation from (3.11) as follows

$$i\frac{\partial \eta_{01}}{\partial t_1} + i\gamma_1 \frac{\partial \eta_{01}}{\partial x_1} - i\gamma_2 \frac{\partial \eta_{01}}{\partial y_1} + \gamma_3 \frac{\partial^2 \eta_{01}}{\partial x_1^2} + \gamma_4 \frac{\partial^2 \eta_{01}}{\partial y_1^2} - \gamma_5 \frac{\partial^2 \eta_{01}}{\partial x_1 \partial y_1} = \Lambda_1 \eta_{01}^2 \eta_{01}^* + \Lambda_2 \eta_{01} \eta_{10} \eta_{10}^*. \tag{3.14}$$

It is to be noted that, for deriving the third order coupled nonlinear Schrödinger equations, we have not used any linearity assumption. Again, the importance of introducing the general problem formulation is that in the studies of nonlinear evolution of water waves, nonlinear Schrödinger equations are often used as they can properly reflect the modulational instability or the Benjamin-Feir instability. That is why we have derived the said nonlinear equation using a well-known multiple scale method to describe slowly modulated waves. Further, for small wave numbers of perturbations and small amplitudes, the most successful and elegant procedure is through the use of nonlinear Schrödinger equation.

Equations (3.13) and (3.14) have been made dimensionless by considering the transformations with their tildes dropped:

$$(3.15)\qquad \begin{aligned} &(\tilde{x},\tilde{y},\tilde{t}) = (k_0x_1, k_0y_1, \sqrt{gk_0}t_1), \qquad (\tilde{k},\tilde{l}) = \left(\frac{k}{k_0},\frac{l}{k_0}\right),\\ &\tilde{\omega} = \frac{\omega}{\sqrt{gk_0}}, \qquad \tilde{s} = \frac{sk_0^2}{g}, \qquad (\tilde{\eta}_{10},\tilde{\eta}_{01}) = (k_0\eta_{10}, k_0\eta_{01}). \end{aligned}$$

For $s = 0$, the above coefficients of Eqs. (3.13) and (3.14) are in agreement with the similar coefficients of the equations obtained by Onorato *et al.* [4]. Again, for $s = 0$, the coefficients of equations (3.13) and (3.14) become the same as those of the corresponding coefficients of Senapati *et al.* [14] in case of $U = 0$, $r = 0$.

4. Stability analysis

We choose the solutions of the Schrödinger equations (3.13) and (3.14) in the form

$$(4.1)\qquad \eta_{10} = \alpha_0 e^{-it\Delta\omega_1} \equiv \eta_{10}^{(0)}, \qquad \eta_{01} = \beta_0 e^{-it\Delta\omega_2} \equiv \eta_{01}^{(0)},$$

where α_0 and β_0 are real constants indicating the wave steepness of the two wavetrains and the nonlinear frequency shifts $\Delta\omega_1$ and $\Delta\omega_2$ satisfy the following relations:

$$(4.2)\qquad \Delta\omega_1 = \Lambda_1{\alpha_0}^2 + \Lambda_2{\beta_0}^2, \qquad \Delta\omega_2 = \Lambda_2{\alpha_0}^2 + \Lambda_1{\beta_0}^2.$$

Next, we employ the harmonic perturbations of the aforesaid uniform solutions as follows:

$$(4.3)\qquad \eta_{10} = \eta_{10}^{(0)}(1+\alpha'), \qquad \eta_{01} = \eta_{01}^{(0)}(1+\beta'),$$

where the infinitesimal perturbations α', β' being complex quantities given by $\alpha' = \alpha'_r + i\alpha'_i$ and $\beta' = \beta'_r + i\beta'_i$, where α'_r, α'_i, β'_r and β'_i being real. Inserting these perturbed solutions (4.3) into (3.13), (3.14), linearising and separating those equations with respect to α', β' into two parts, real and imaginary, we obtain four equations. Employing Fourier transform on the aforesaid equations in α_r', α_i', β_r' and β_i' and assuming the t dependence in the form $e^{-i\Omega t}$, we obtain the equations as follows:

$$(4.4)\quad i(\Omega-\lambda\gamma_1-\mu\gamma_2)\tilde{\alpha}_i'-(\lambda^2\gamma_3+\mu^2\gamma_4+\lambda\mu\gamma_5+2\Lambda_1\alpha_0^2)\tilde{\alpha}_r'-2\Lambda_2\beta_0^2\tilde{\beta}_r'=0,$$

$$(4.5)\quad -(\lambda^2\gamma_3+\mu^2\gamma_4+\lambda\mu\gamma_5)\tilde{\alpha}_i'-i(\Omega-\lambda\gamma_1-\mu\gamma_2)\tilde{\alpha}_r'=0,$$

$$(4.6)\quad i(\Omega-\lambda\gamma_1+\mu\gamma_2)\tilde{\beta}_i'-(\lambda^2\gamma_3+\mu^2\gamma_4-\lambda\mu\gamma_5+2\Lambda_1\alpha_0^2)\tilde{\beta}_r'-2\Lambda_2\alpha_0^2\tilde{\alpha}_r'=0,$$

$$(4.7)\quad -(\lambda^2\gamma_3+\mu^2\gamma_4-\lambda\mu\gamma_5)\tilde{\beta}_i'-i(\Omega-\lambda\gamma_1+\mu\gamma_2)\tilde{\beta}_r'=0,$$

where

$$(4.8)\quad (\tilde{\alpha}_r',\tilde{\alpha}_i',\tilde{\beta}_r',\tilde{\beta}_i')=\frac{1}{2\pi}\iint_{-\infty}^{\infty}(\alpha_r',\alpha_i',\beta_r',\beta_i')e^{-i(\lambda x+\mu y)}\,\mathrm{d}x\,\mathrm{d}y$$

and (λ,μ) represents the perturbed wave number vector. From the condition of nontrivial solution of Eqs. (4.4)–(4.7), we obtain the following nonlinear

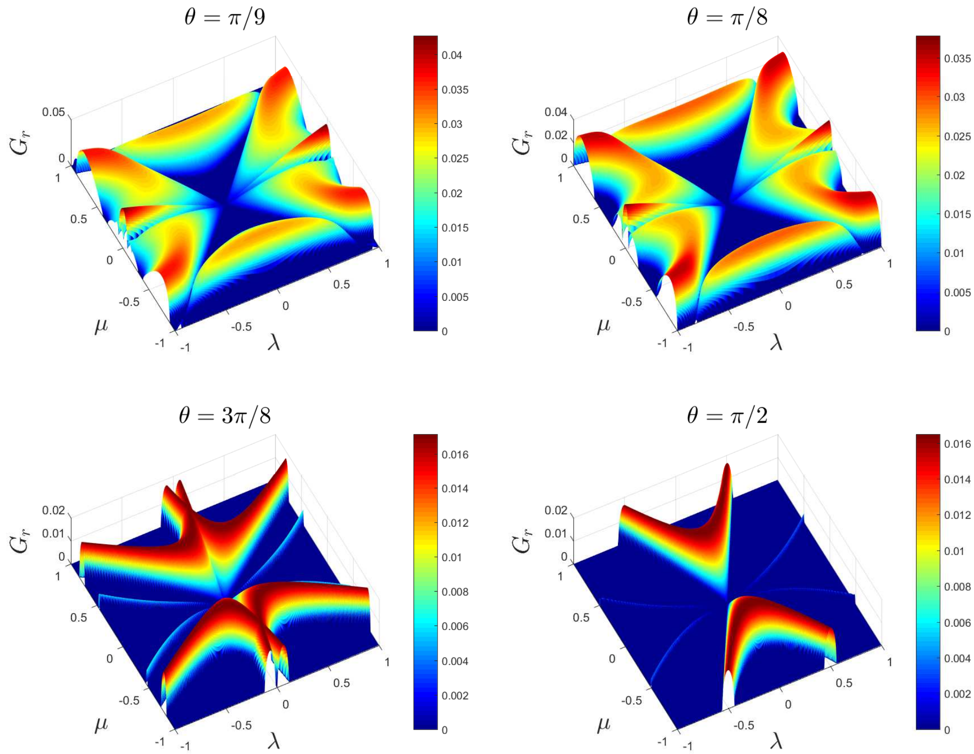

Fig. 2. 3-dimensional Contour plot $G_r=\mathrm{Im}(\Omega)$ in the perturbed wave numbers plane for $\alpha_0=0.1$, $\beta_0=0.1$, s=0.035 and $\theta=\frac{\pi}{9},\frac{\pi}{8},\frac{3\pi}{8},\frac{\pi}{2}$.

dispersion relation

$$(4.9)\qquad [(\Omega - Q_+)^2 - P_+(P_+ - 2\Lambda_1\alpha_0^2)][(\Omega - Q_-)^2 - P_-(P_- - 2\Lambda_1\beta_0^2)] = 4P_+P_-\Lambda_2^2\alpha_0^2\beta_0^2,$$

where

$$P_\pm = -(\lambda^2\gamma_3 + \mu^2\gamma_4 \pm \lambda\mu\gamma_5), \qquad Q_\pm = \lambda\gamma_1 \pm \mu\gamma_2.$$

The unidirectional perturbation can be found in the x-direction, by putting $\mu = 0$ in the following simplified from

$$(4.10)\quad \Omega = \gamma_1\lambda \pm \sqrt{(\gamma_3\lambda^2)^2 + \gamma_3\lambda^2\Lambda_1(\alpha_0^2+\beta_0^2) \mp \gamma_3\lambda^2\sqrt{\Lambda_1^2(\alpha_0^2-\beta_0^2)^2 + 4\Lambda_2^2\alpha_0^2\beta_0^2}}.$$

This equation is again in agreement with the Eq. (11) of [4].

In Figs. 2 and 3 we have plotted 3-dimensional contour maps of instability growth rate for bidirectional perturbations on account of several values of θ $\left(\theta = \frac{\pi}{9}, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{\pi}{2}\right)$ and s $(s = 0.035, 0)$ by considering $\alpha_0 = 0.1$, $\beta_0 = 0.1$. From

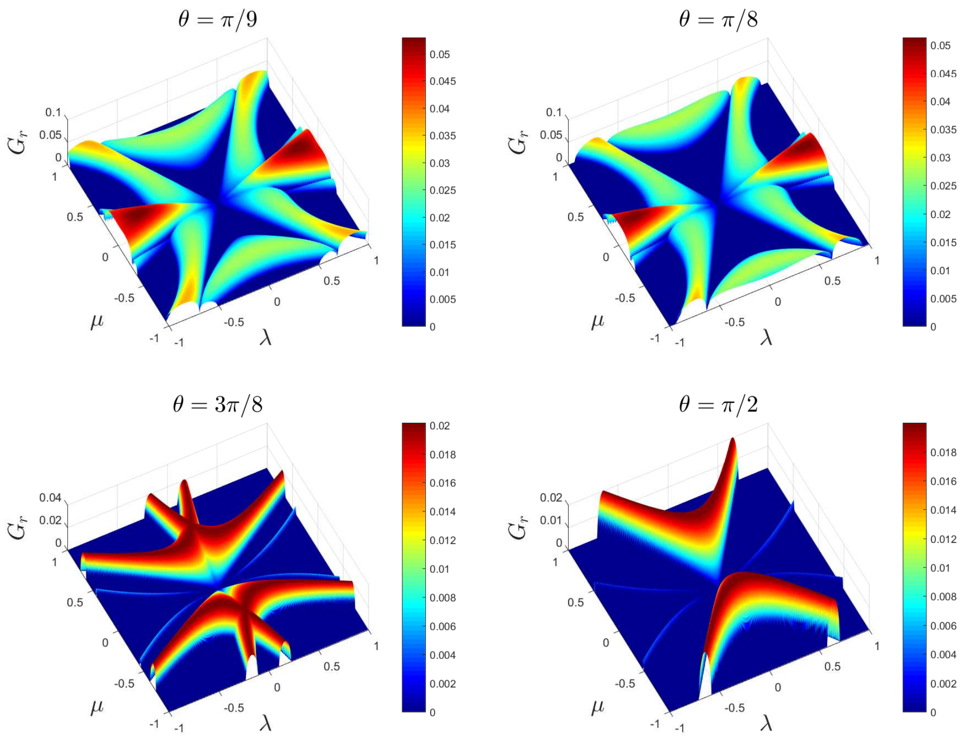

FIG. 3. 3-dimensional Contour plot of $G_r = \mathrm{Im}(\Omega)$ in the perturbed wave numbers plane for $\alpha_0 = 0.1$, $\beta_0 = 0.1$, $s = 0$ and $\theta = \frac{\pi}{9}, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{\pi}{2}$.

these figures we have observed that the effect of capillarity results in a decrease in the growth rate producing a stabilizing influence. Further comparing Fig. 2 with the Fig. 5 of [14] for $\theta = \frac{3\pi}{8}$ we have inferred that due to capillarity the instability growth rate decreases.

In Figs. 4 and 5 we have portrayed the same contour plots of the growth rate of instability on account of several values of θ and s in which $\alpha_0 \neq \beta_0$. Examining Figs. 4 and 5 for $\alpha_0 \neq \beta_0$ with the corresponding Figs. 2 and 3 for $\alpha_0 = \beta_0$ we have observed that the instability regions for $\alpha_0 = \beta_0$ are symmetric about the lines $\lambda = 0$ and $\mu = 0$ whereas for $\alpha_0 \neq \beta_0$ it is not so. As before in this case also, it is found that the capillary effect produces a stabilizing influence.

In Fig. 6 we have drawn the instability growth rate G_r for unidirectional perturbation for different values of θ ($\theta = 16°, 19°$), β_0 ($\beta_0 = 0.09, 0.15$) and s ($s = 0.035, 0$). From these figures it is observed that the capillary effect produces a decrease in the instability growth rate G_r up to a certain value of wave number λ giving a stabilizing influence. After that the effect of capillarity gives rise to an increase in the instability growth rate, producing a destabilizing influence. Further, these figures portray that G_r increases as θ decreases.

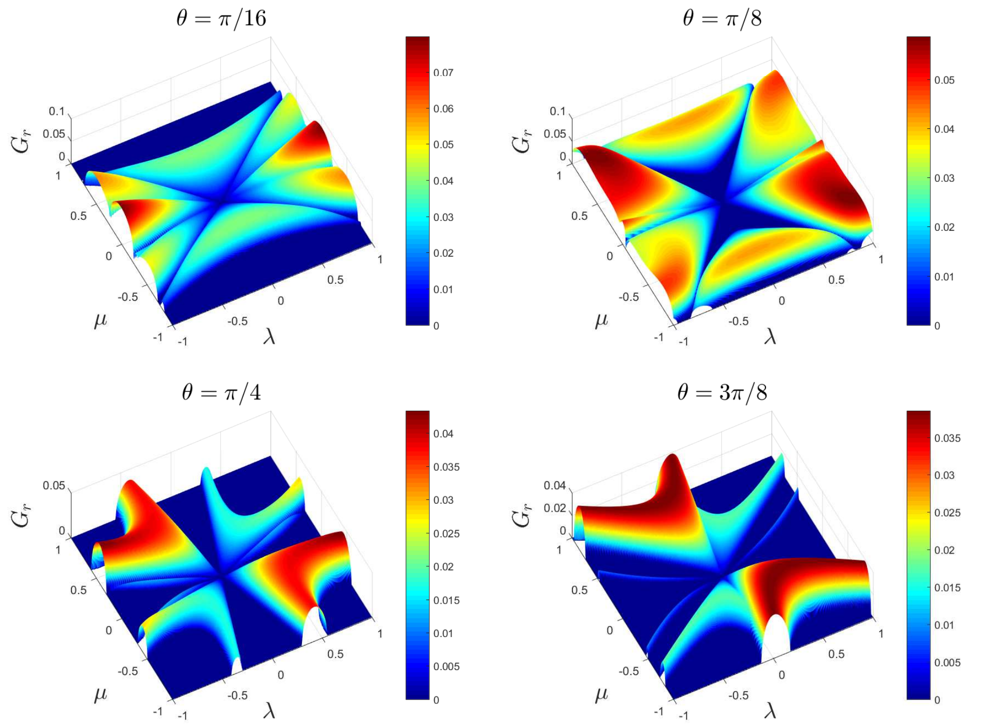

Fig. 4. 3-dimensional Contour plot of $G_r = \text{Im}(\Omega)$ in the perturbed wave numbers plane for $\alpha_0 = 0.1$, $\beta_0 = 0.15$, $s = 0.035$ and $\theta = \frac{\pi}{24}, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}$.

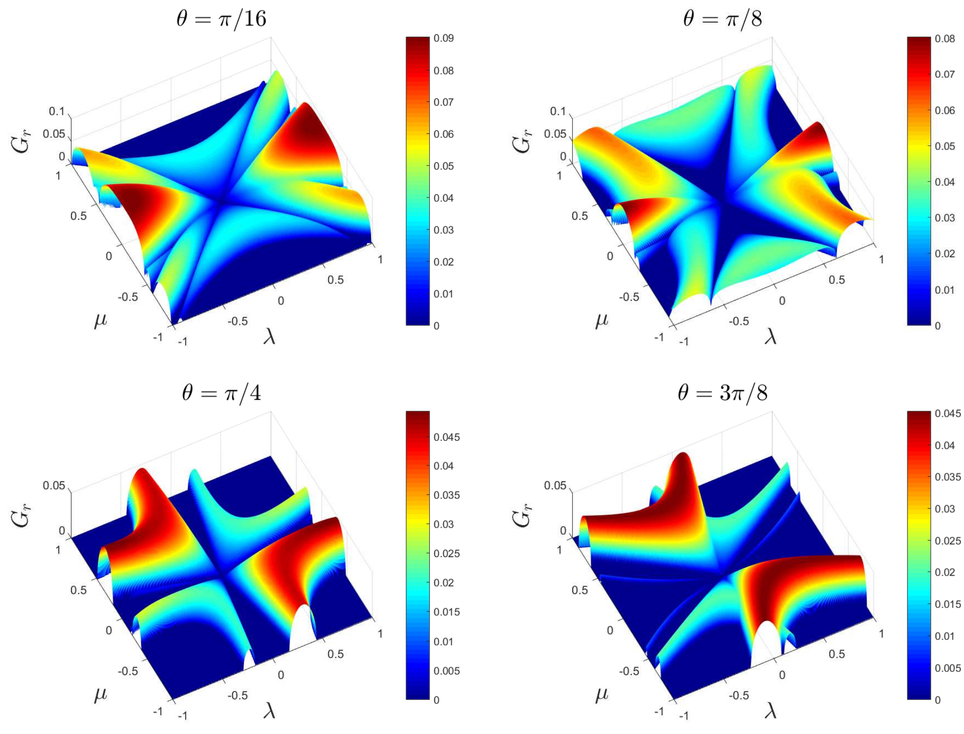

FIG. 5. 3-dimensional Contour plot of $G_r = \mathrm{Im}(\Omega)$ in the perturbed wave numbers plane for $\alpha_0 = 0.1$, $\beta_0 = 0.15$, $s = 0$ and $\theta = \frac{\pi}{24}, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}$.

Examining Fig. 6 with Fig. 1 of [15] for $V = 0$, it is found that the instability growth rate in the situation of crossing seas is higher than the case for a single wave.

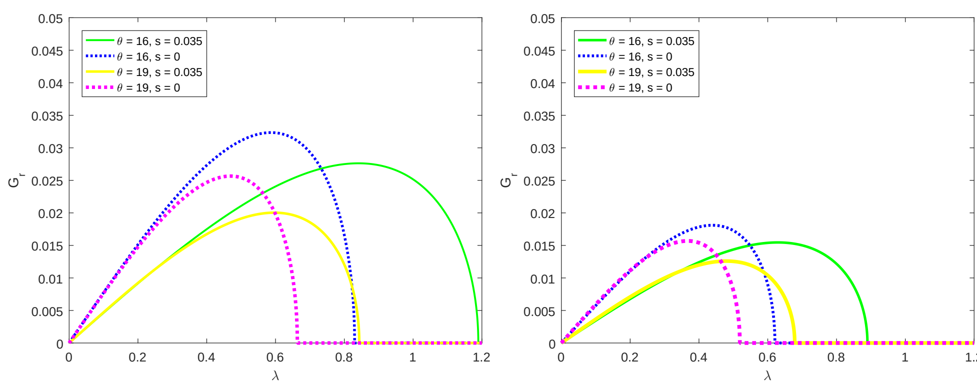

FIG. 6. $G_r = \mathrm{Im}(\Omega)$ as a function of wave number λ. $\alpha_0 = 0.1$, $\beta_0 = 0.15$ (left), $\beta_0 = 0.09$ (right), $\theta = 16^\circ,\ 19^\circ$. For continuous line $s = 0.035$, for dotted line $s = 0$.

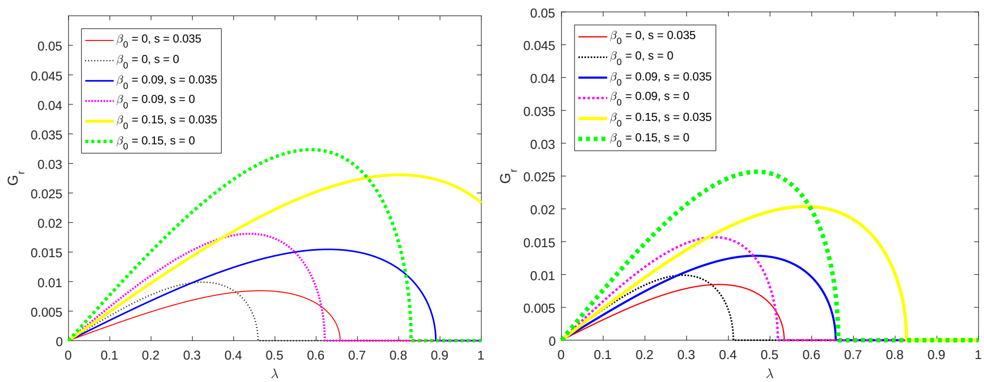

Fig. 7. $G_r = \text{Im}(\Omega)$ as a function of wave number λ. Here $\theta = 16°$ (left), $\theta = 19°$ (right) $\alpha_0 = 0.1$, $\beta_0 = 0.09$. For continuous line $s = 0.035$, for dotted line $s = 0$.

Figure 7 exhibits that the instability growth rate G_r increases with the increase of the amplitude β_0 of the second wavetrain for fixed value of θ.

5. Discussion along with conclusion

Starting from third order two space dimensional coupled nonlinear Schrödinger equations namely (3.13) and (3.14) under the position of crossing sea states, we have discussed analytically the modulational instability of two obliquely interacting capillary-gravity wave packets for infinitely deep water due to unidirectional as well as bidirectional perturbations. Using a multiscale expansion, we have derived nonlinear Schrödinger equations to study slowly modulated waves. It is well known that the effects due to capillarity are significant for short waves. Comparing Fig. 2, (on account of capillarity, $s = 0.035$) with Fig. 3 (in the absence of capillarity, $s = 0$), for $\alpha_0 = \beta_0 = 0.1$, we conclude that the effect of capillarity produces a decrease in the growth rate of instability G_r. Similarly, considering Fig. 4 (on account of $s = 0.035$) and Fig. 5 (on account of $s = 0$) for $\alpha_0 \neq \beta_0$, we observe a stabilizing influence due to capillarity in the growth rate as in the previous case. For unidirectional perturbation it is found from Figs. 6 and 7 that growth rate of instability decreases due to the effect of capillarity up to certain value of the wave number λ and then instability growth rate increases. Further, it is observed from Figs. 6 and 7 for unidirectional perturbation and from Figs. 2–5 for 3-dimensional Contour maps that the growth rate of instability for two obliquely interacting wave packets is higher than the case of modulation of a single wave packet [15]. It has been additionally found that the instability growth rate increases as the angle 2θ between two wave systems decreases.

Appendix A

$$A_{10} = -\frac{i\omega}{k_0}\left[1 + i\delta\left(\frac{k}{k_0^2}\frac{\partial}{\partial x_1} + \frac{l}{k_0^2}\frac{\partial}{\partial y_1} + \frac{1}{\omega}\frac{\partial}{\partial t_1}\right)\right]\eta_{10},$$

$$A_{01} = -\frac{i\omega}{k_0}\left[1 + i\delta\left(\frac{k}{k_0^2}\frac{\partial}{\partial x_1} - \frac{l}{k_0^2}\frac{\partial}{\partial y_1} + \frac{1}{\omega}\frac{\partial}{\partial t_1}\right)\right]\eta_{01},$$

$$\eta_{20} = \frac{\omega^2 k_0 \eta_{10}^2}{2\omega^2 - gk_0 - 4sk_0^3}, \qquad A_{20} = \frac{i\omega(\omega^2 - gk_0 - 4sk_0^3)\eta_{10}^2}{2\omega^2 - gk_0 - 4sk_0^3},$$

$$\eta_{02} = \frac{\omega^2 k_0 \eta_{01}^2}{2\omega^2 - gk_0 - 4sk_0^3}, \qquad A_{02} = \frac{i\omega(\omega^2 - gk_0 - 4sk_0^3)\eta_{01}^2}{2\omega^2 - gk_0 - 4sk_0^3},$$

$$\eta_{11} = \frac{2\omega^2 k(k^2 + 2l^2 - 2kk_0)\eta_{10}\eta_{01}}{k_0^2(gk + 4sk^3 - 2\omega^2)},$$

$$A_{11} = \frac{2i\omega}{k_0^2}\left[kk_0 - \frac{\omega^2(k^2 + 2l^2 - 2kk_0)}{gk + 4sk^3 - 2\omega^2}\right]\eta_{10}\eta_{01},$$

$$\eta_{1-1} = \frac{2l^2\omega^2\eta_{10}\eta_{01}^*}{k_0^2(g + 4l^2 s)}, \qquad A_{1-1} = 0.$$

Appendix B

$$\gamma_1 = \frac{k(3s+1)}{2\omega}, \quad \gamma_2 = \frac{l(3s+1)}{2\omega}, \qquad \gamma_3 = -\frac{k^2}{8\omega^3} + \frac{l^2 + 3s(2k^2 + l^2)}{4\omega},$$

$$\gamma_4 = -\frac{l^2}{8\omega^3} + \frac{k^2 + 3s(2k^2 + l^2)}{4\omega}, \quad \gamma_5 = -\frac{kl}{4\omega^3} - \frac{kl(3s+1)}{2\omega},$$

$$\Lambda_1 = \frac{\omega(3\omega^2 - 3 - 12s)}{2\omega^2 - 1 - 4s} + 2\omega - \frac{s}{2\omega}\left\{3l^2k^2 - \frac{3}{2}(k^4 + l^4)\right\},$$

$$\Lambda_2 = \frac{s}{2\omega}\{2l^2k^2 + 3(k^4 + l^4)\} + \omega(k^2 - l^2) + 2\omega k^2 - \frac{\omega^3 l^2(k^2 - l^2 - 2)}{2(1 + 4l^2 s)} - \frac{\omega k(k^2 + l^2 - 2k)^2}{k + 4sk^2 - 2\omega^2}.$$

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