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AN OPERATIONAL CALCULUS MODEL FOR THE CENTRAL DIFFERENCE AND EXPONENTIAL-TRIGONOMETRIC AND HYPERBOLIC FIBONACCI SEQUENCES

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ABSTRACT

In the paper, there has been constructed such a non-classical Bittner operational calculus model, in which the derivative is understood as a central difference $D_n\{x(k)\}:=\{x(k+n)-x(k-n)\}$. The discussed model has been generalized by considering the operation $D_{n,b}\{x(k)\}:=\{x(k+n)-bx(k-n)\}$, where $b\in\mathbb{C}\setminus\{0\}$. In the D_1-difference model exponential-trigonometric and hyperbolic Fibonacci sequences have been introduced.

Key words:
operational calculus, derivative, integrals, limit conditions, central difference, Fibonacci sequences.

Research article

INTRODUCTION

From the integral calculus fundamental theorems [1] it follows that for all functions $\{f(t)\} \in C^0([a,b],\mathbb{R}), \{x(t)\} \in C^1([a,b],\mathbb{R})$ and for each $t, t_0 \in [a,b]$ holds the below:

$$\frac{d}{dt}\int_{t_0}^{t} f(\tau)\,d\tau = f(t), \quad \int_{t_0}^{t} x'(\tau)\,d\tau = x(t) - x(t_0). \tag{1}$$

Hence, if

$$S\{x(t)\} := \{x'(t)\}, \quad T_{t_0}\{f(t)\} := \Big\{\int_{t_0}^{t} f(\tau)\,d\tau\Big\}, \quad s_{t_0}\{x(t)\} := \{x(t)\}|_{t=t_0} \equiv \{x(t_0)\}, \tag{2}$$

then on the basis of (1) we get

$$S\,T_{t_0} f = f, \quad T_{t_0} S\,x = x - s_{t_0} x, \tag{3}$$

where $f = \{f(t)\}, x = \{x(t)\}$[1].

In view of the properties (3), the operations (2) are a classic example of the so-called *Bittner operational calculus* [2–5]. Namely, we say that the system

$$\Big(C^0([a,b],\mathbb{R}), C^1([a,b],\mathbb{R}), \frac{d}{dt}, \int_{t_0}^{t}, |_{t=t_0}, t_0 \in [a,b]\Big)$$

is a *continuous model* (*representation*) of this calculus with the *ordinary derivative* $S := d/dt$.

In the difference calculus [7, 10, 12], to the derivative

$$x'(t) \equiv \frac{dx(t)}{dt} := \lim_{h\to 0} \frac{d_h x(t)}{d_h t},$$

where

$$d_h x(t) := x(t+h) - x(t),$$

[1] $\{f(t)\}$ denotes the symbol of a function f, i.e. $f = \{f(t)\}$, whereas $f(t)$ means the value of the function $\{f(t)\}$ for an argument t. This denotation is derived from J. Mikusiński [15].

there corresponds a *forward difference*

$$\Delta x(k) := x(k+1) - x(k).$$

In [6] there have been considered operations S, T_{k_0}, s_{k_0} where $k_0 \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$[2], which are determined on a linear space $C(\mathbb{N}_0, \mathbb{R})$ of real sequences $x = \{x(k)\}_{k \in \mathbb{N}_0}$ with the following formulas

$$S\,x \equiv \Delta x, \quad T_{k_0} x := \Big\{\sum_{i=0}^{k-1} x(i) - \sum_{i=0}^{k_0-1} x(i)\Big\}, \quad s_{k_0} x := \{x(k_0)\}.$$

They form a *discrete model* of the Bittner operational calculus, because they possess properties analogical to (3), that is

$$S\,T_{k_0} x = x, \quad T_{k_0} S\,x = x - s_{k_0} x.$$

A generalization of the above representation is a model with S as an n^{th}-*order* ($n \in \mathbb{N}$) *forward difference*

$$S\,x \equiv \Delta_n x := \{x(k+n) - x(k)\},$$

which was introduced in [22].

A generalization of the ordinary derivative $x'(t)$ is the *Schwarz derivative* $x'_s(t)$, also known as a *symmetric derivative* [14]. It is defined by the formula

$$x'_s(t) := \lim_{h \to 0} \frac{\delta_h x(t)}{\delta_h t},$$

where

$$\delta_h x(t) := x(t+h) - x(t-h).$$

If there exists $x'(t)$, then there also exists $x'_s(t)$ and $x'(t) = x'_s(t)$. Moreover, e.g. for $x(t) := |t|$ we have $x'_s(0) = 0$, while $x'(0)$ does not exist [20].

In numerical methods, the *symmetric difference* δ_h is applied to the approximation of the ordinary derivative $x'(t)$ [13]. Namely, if $\{x(t)\} \in C^3([a,b], \mathbb{R})$, then

$$x'(t) \approx \frac{\delta_h x(t)}{2h}.$$

[2] $\mathbb{N}$ denotes a set of natural numbers.

Likewise, for $\{x(t)\} \in C^5([a,b],\mathbb{R})$ we have

$$x'(t) \approx \frac{8\delta_h x(t) - \delta_{2h} x(t)}{12h}.$$

In the calculus of differences, similarly as before, to the symmetric difference δ_h there corresponds a *central difference*

$$Dx(k) \equiv D_1 x(k) := x(k+1) - x(k-1).$$

In this paper, we will construct such a *discrete* model of the Bittner operational calculus, in which the derivative S will be understood as an operation

$$D_n\{x(k)\} := \{x(k+n) - x(k-n)\}, \quad n \in \mathbb{N} \tag{4}$$

determined on the space of two-sided sequences, i.e. if $k \in \mathbb{Z}$[3]. We will determine the operations T_{k_0} and s_{k_0}, where $k_0 \in \mathbb{Z}$ in such a way that the relations (3) are fulfilled.

FOUNDATIONS OF THE NON-CLASSICAL BITTNER OPERATIONAL CALCULUS

The *Bittner operational calculus* is a system

$$CO(L^0, L^1, S, T_q, s_q, Q)^4, \tag{5}$$

where L^0 and L^1 are linear spaces (over the same scalar field Γ) such that $L^1 \subset L^0$. A linear operation $S : L^1 \longrightarrow L^0$ (denoted as $S \in \mathscr{L}(L^1, L^0)$), called a *derivative*, is a surjection. Moreover, Q is a set of indices q for the operations $T_q \in \mathscr{L}(L^0, L^1)$ and $s_q \in \mathscr{L}(L^1, L^1)$ such that $ST_q f = f, f \in L^0$ and $s_q x = x - T_q S x, x \in L^1$. T_q and s_q are called *integrals* and *limit conditions*, respectively. The kernel of S, i.e. $\operatorname{Ker} S$ is called a set of *constants* for the derivative S. The limit conditions s_q are projections of L^1 onto the subspace $\operatorname{Ker} S$.

By means of induction, we determine a sequence of spaces $L^n, n \in \mathbb{N}$ in such a way that

$$L^n := \{x \in L^{n-1} : Sx \in L^{n-1}\}.$$

[3] $\mathbb{Z}$ denotes a set of integer numbers.

[4] The abbreviation CO is derived from the French *calcul opératoire* (operational calculus).

Then

$$\ldots \subset L^n \subset L^{n-1} \subset \ldots \subset L^1 \subset L^0$$

and

$$S^n(L^{m+n}) = L^m,$$

where

$$\mathscr{L}(L^n, L^0) \ni S^n := \underbrace{S \circ S \circ \ldots \circ S}_{n-\text{times}}, \quad m \in \mathbb{N}_0,\ n \in \mathbb{N}.$$

A MODEL WITH THE 2nTH-ORDER CENTRAL DIFFERENCE

Let $\mathbb{C}$ be a set of complex numbers. What is more, let $C(\mathbb{Z}, \mathbb{C})$ be a linear space of two-sided complex sequences $x = \{x(k)\}_{k\in\mathbb{Z}}$ with usual sequences addition and sequences multiplication by complexes.

The operation (4), determined on $C(\mathbb{Z}, \mathbb{C})$, will be called the $2n^{th}$*-order central difference*, where n is a given natural number.

Let us notice that any element $c \in \operatorname{Ker} D_n$ is a $2n$*-periodic* sequence, since for each $k \in \mathbb{Z}$ we have

$$c(k+n) - c(k-n) = 0 \Longleftrightarrow c(k+2n) = c(k).$$

Then, for any sequence $c \in \operatorname{Ker} D_n$ there exist numbers $a_0, a_1, \ldots, a_{2n-1} \in \mathbb{C}$ such that

$$c = \{a_0\varepsilon_0^k + a_1\varepsilon_1^k + \cdots + a_{2n-1}\varepsilon_{2n-1}^k\},$$

where

$$\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{2n-1} \tag{6}$$

are $2n^{\text{th}}$ roots of unity, i.e.

$$\varepsilon_j = \cos\frac{j\pi}{n} + \mathrm{i}\,\sin\frac{j\pi}{n}, \quad j \in \overline{0, 2n-1}\,^5,$$

while 'i' denotes the imaginary unit.

[5] $\overline{0, 2n-1} := \{0, 1, \ldots, 2n-1\}$.

In what follows, we will use the below properties of the sequence (6):

$$\varepsilon_j^{k+2\ell n} = \varepsilon_j^k, \quad j \in \overline{0, 2n-1}, k, \ell \in \mathbb{Z},$$
$$\varepsilon_0^m + \varepsilon_1^m + \ldots + \varepsilon_{2n-1}^m = 0, \quad m \neq 2\ell n, \ell, m \in \mathbb{Z}, n \in \mathbb{N}.$$

We will prove the following

Theorem. *The system* (5)*, where* $x = \{x(k)\} \in L^0 = L^1 := C(\mathbb{Z}, \mathbb{C}), k_0 \equiv q \in Q := \mathbb{Z}$ *and*

$$S\,x := \{x(k+n) - x(k-n)\}, \tag{7}$$

$$T_{k_0}x := \begin{cases} -\dfrac{1}{2n}\displaystyle\sum_{j=0}^{2n-1}\sum_{i=k+n}^{k_0+n-1} \varepsilon_j^{k+n-i}\, x(i) & for \quad k < k_0 \\ 0 & for \quad k = k_0 \\ \dfrac{1}{2n}\displaystyle\sum_{j=0}^{2n-1}\sum_{i=k_0+n}^{k+n-1} \varepsilon_j^{k+n-i}\, x(i) & for \quad k > k_0 \end{cases}, \quad k \in \mathbb{Z}, \tag{8}$$

$$s_{k_0}x := \left\{\frac{1}{2n}\sum_{j=0}^{2n-1}\sum_{i=k_0}^{k_0+2n-1} \varepsilon_j^{k-i}\, x(i)\right\} \tag{9}$$

forms a discrete model of the Bittner operational calculus[6].

Proof. It is obvious that the operations (7)–(9) are linear. Let $\{y(k)\} := T_{k_0}\{x(k)\}$. Then, for $k = k_0$ we obtain

$$S\{y(k)\}|_{k=k_0} = \{y(k_0+n) - y(k_0-n)\}$$
$$= \left\{\frac{1}{2n}\sum_{j=0}^{2n-1}\sum_{i=k_0+n}^{k_0+2n-1} \varepsilon_j^{k_0-i}\, x(i) + \frac{1}{2n}\sum_{j=0}^{2n-1}\sum_{i=k_0}^{k_0+n-1} \varepsilon_j^{k_0-i}\, x(i)\right\}$$
$$= \left\{\frac{1}{2n}\sum_{j=0}^{2n-1}\sum_{i=k_0}^{k_0+2n-1} \varepsilon_j^{k_0-i}\, x(i)\right\}$$
$$= \left\{x(k_0) + \frac{1}{2n}\sum_{i=k_0+1}^{k_0+2n-1} (\,\varepsilon_0^{k_0-i} + \varepsilon_1^{k_0-i} + \cdots + \varepsilon_{2n-1}^{k_0-i})\, x(i)\right\} = \{x(k)\}|_{k=k_0}. \tag{10}$$

[6] Given the definition of integrals T_{k_0}, we assume that $\sum_{i=k_0+n}^{k_0+n-1} f(i) := 0$.

For $k < k_0$ and $k + n = k_0$, i.e. $k = k_0 - n$, we get in turn

$$S\{y(k)\} = \{y(k+n) - y(k-n)\} = \{y(k_0) - y(k_0 - 2n)\}$$

$$= \left\{0 + \frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{i=k_0-n}^{k_0+n-1} \varepsilon_j^{k_0-n-i}\, x(i)\right\}$$

$$= \left\{x(k_0 - n) + \frac{1}{2n} \sum_{i=k_0-n+1}^{k_0+n-1} (\, \varepsilon_0^{k_0-n-i} + \varepsilon_1^{k_0-n-i} + \cdots + \varepsilon_{2n-1}^{k_0-n-i})\, x(i)\right\} = \{x(k_0 - n)\}.$$

Therefore,

$$S\{y(k)\} = \{x(k_0 - n)\}, \quad \text{that is} \quad S\{y(k)\} = \{x(k)\}.$$

For $k < k_0$ and $k + n < k_0$, i.e. $k < k_0 - n$, we have

$$S\{y(k)\} = \{y(k+n) - y(k-n)\}$$

$$= \left\{\frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{i=k}^{k_0+n-1} \varepsilon_j^{k-i}\, x(i) - \frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{i=k+2n}^{k_0+n-1} \varepsilon_j^{k-i}\, x(i)\right\}$$

$$= \left\{\frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{i=k}^{k+2n-1} \varepsilon_j^{k-i}\, x(i)\right\}. \tag{11}$$

Hence, similarly as in (10), we conclude that $S\{y(k)\} = \{x(k)\}$.

If $k < k_0$ and $k + n > k_0$, i.e. $k_0 - n < k < k_0$, we get in turn

$$S\{y(k)\} = \{y(k+n) - y(k-n)\}$$

$$= \left\{\frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{i=k_0+n}^{k+2n-1} \varepsilon_j^{k-i}\, x(i) + \frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{i=k}^{k_0+n-1} \varepsilon_j^{k-i}\, x(i)\right\}$$

$$= \left\{\frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{i=k}^{k+2n-1} \varepsilon_j^{k-i}\, x(i)\right\}.$$

So, by analogy to (11), we have $S\{y(k)\} = \{x(k)\}$.

If $k > k_0$ and $k - n = k_0$, i.e. $k = k_0 + n$, then

$$S\{y(k)\} = \{y(k+n) - y(k-n)\} = \{y(k_0+2n) - y(k_0)\}$$

$$= \left\{\frac{1}{2n}\sum_{j=0}^{2n-1}\sum_{i=k_0+n}^{k_0+3n-1}\varepsilon_j^{k_0+n-i}\,x(i) - 0\right\}$$

$$= \left\{x(k_0+n) + \frac{1}{2n}\sum_{i=k_0+n+1}^{k_0+3n-1}(\,\varepsilon_0^{k_0+n-i} + \varepsilon_1^{k_0+n-i} + \cdots + \varepsilon_{2n-1}^{k_0+n-i})\,x(i)\right\} = \{x(k_0+n)\},$$

that is $S\{y(k)\} = \{x(k)\}$.

For $k > k_0$ and $k - n > k_0$, i.e. $k > k_0 + n$, we obtain

$$S\{y(k)\} = \{y(k+n) - y(k-n)\}$$

$$= \left\{\frac{1}{2n}\sum_{j=0}^{2n-1}\sum_{i=k_0+n}^{k+2n-1}\varepsilon_j^{k-i}\,x(i) - \frac{1}{2n}\sum_{j=0}^{2n-1}\sum_{i=k_0+n}^{k-1}\varepsilon_j^{k-i}\,x(i)\right\}$$

$$= \left\{\frac{1}{2n}\sum_{j=0}^{2n-1}\sum_{i=k}^{k+2n-1}\varepsilon_j^{k-i}\,x(i)\right\},$$

which, similarly as in (11), also means that $S\{y(k)\} = \{x(k)\}$.

Lastly, if $k > k_0$ and $k - n < k_0$, i.e. $k_0 < k < k_0 + n$, then

$$S\{y(k)\} = \{y(k+n) - y(k-n)\}$$

$$= \left\{\frac{1}{2n}\sum_{j=0}^{2n-1}\sum_{i=k_0+n}^{k+2n-1}\varepsilon_j^{k-i}\,x(i) + \frac{1}{2n}\sum_{j=0}^{2n-1}\sum_{i=k}^{k_0+n-1}\varepsilon_j^{k-i}\,x(i)\right\}$$

$$= \left\{\frac{1}{2n}\sum_{j=0}^{2n-1}\sum_{i=k}^{k+2n-1}\varepsilon_j^{k-i}\,x(i)\right\} = \{x(k)\}.$$

Finally, we can conclude that the property $S\,T_{k_0}x = x$ is fulfilled.

Let $\{f(k)\} := S\{x(k)\} = \{x(k+n) - x(k-n)\}$. Therefore, for $k < k_0$ we get

$$T_{k_0}S\{x(k)\} = T_{k_0}\{f(k)\} = \left\{-\frac{1}{2n}\sum_{j=0}^{2n-1}\sum_{i=k+n}^{k_0+n-1}\varepsilon_j^{k+n-i}\,f(i)\right\}$$

$$= \left\{\frac{1}{2n}\sum_{j=0}^{2n-1}[\,\sum_{i=k+n}^{k_0+n-1}\varepsilon_j^{k+n-i}\,x(i-n) - \sum_{i=k+n}^{k_0+n-1}\varepsilon_j^{k+n-i}\,x(i+n)\,]\right\}$$

$$= \left\{ \frac{1}{2n} \sum_{j=0}^{2n-1} [\sum_{i=k}^{k_0-1} \varepsilon_j^{k-i}\, x(i) + \sum_{i=k_0}^{k+2n-1} \varepsilon_j^{k-i}\, x(i) \right.$$

$$\left. -(\sum_{i=k_0}^{k+2n-1} \varepsilon_j^{k-i}\, x(i) + \sum_{i=k+2n}^{k_0+2n-1} \varepsilon_j^{k-i}\, x(i))] \right\}$$

$$= \left\{ \frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{i=k}^{k+2n-1} \varepsilon_j^{k-i}\, x(i) \right\} - \left\{ \frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{i=k_0}^{k_0+2n-1} \varepsilon_j^{k-i}\, x(i) \right\}.$$

Hence, on the basis of (11), we eventually obtain

$$T_{k_0} S\{x(k)\} = \{x(k)\} - \left\{ \frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{i=k_0}^{k_0+2n-1} \varepsilon_j^{k-i}\, x(i) \right\} = \{x(k)\} - s_{k_0}\{x(k)\}.$$

Likewise, if $k > k_0$, then

$$T_{k_0} S\{x(k)\} = T_{k_0}\{f(k)\} = \left\{ \frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{i=k_0+n}^{k+n-1} \varepsilon_j^{k+n-i}\, f(i) \right\}$$

$$= \left\{ \frac{1}{2n} \sum_{j=0}^{2n-1} [\sum_{i=k_0+n}^{k+n-1} \varepsilon_j^{k+n-i}\, x(i+n) - \sum_{i=k_0+n}^{k+n-1} \varepsilon_j^{k+n-i}\, x(i-n)] \right\}$$

$$= \left\{ \frac{1}{2n} \sum_{j=0}^{2n-1} [\sum_{i=k}^{k_0+2n-1} \varepsilon_j^{k-i}\, x(i) + \sum_{i=k_0+2n}^{k+2n-1} \varepsilon_j^{k-i}\, x(i) \right.$$

$$\left. -(\sum_{i=k_0}^{k-1} \varepsilon_j^{k-i}\, x(i) + \sum_{i=k}^{k_0+2n-1} \varepsilon_j^{k-i}\, x(i))] \right\}$$

$$= \left\{ \frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{i=k}^{k+2n-1} \varepsilon_j^{k-i}\, x(i) \right\} - \left\{ \frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{i=k_0}^{k_0+2n-1} \varepsilon_j^{k-i}\, x(i) \right\} = \{x(k)\} - s_{k_0}\{x(k)\}.$$

Therefore, the property $T_{k_0} S\, x = x - s_{k_0} x$ is also fulfilled. □

Since for $k \in \overline{k_0 + 1, k_0 + 2n - 1}$ we have

$$\sum_{i=k_0+n}^{k+n-1} (\varepsilon_0^{k+n-i} + \varepsilon_1^{k+n-i} + \cdots + \varepsilon_{2n-1}^{k+n-i}) = 0,$$

then, from the definition (8) of integrals T_{k_0}, we get the following

Corollary 1. *If* $\{y(k)\} := T_{k_0}\{x(k)\}$, *then*

$$y(k) = 0 \quad \textit{for} \quad k \in \overline{k_0, k_0 + 2n - 1}.$$

Corollary 2. *The numbers* $x(k_0), x(k_0 + 1), \ldots, x(k_0 + 2n - 1)$ *form a cycle of the* $2n$*-periodic sequence* $\{c(k)\} = s_{k_0}\{x(k)\}$, *i.e.*

$$c(k) = c(k + 2\ell n) = x(k), \quad k \in \overline{k_0, k_0 + 2n - 1}, \ell \in \mathbb{Z}.$$

Moreover, $s_{k_0} T_{k_0}\{x(k)\} = \{0\}$ *and* $s_{k_0}\{c(k)\} = \{c(k)\}$.

Example. Using the *Mathematica*® program, we will list the terms of the sequence $\{y(k)\} := T_{k_0}\{x(k)\}$ for $k \in \overline{-22, 19}$ in the case when $n = 2$ and $k_0 = -2$. By means of (8) we obtain:

k	$y(k)$	k	$y(k)$
-22	$-x(-20) - x(-16) - x(-12) - x(-8) - x(-4)$	-1	0
-21	$-x(-19) - x(-15) - x(-11) - x(-7) - x(-3)$	0	0
-20	$-x(-18) - x(-14) - x(-10) - x(-6) - x(-2)$	1	0
-19	$-x(-17) - x(-13) - x(-9) - x(-5) - x(-1)$	2	$x(0)$
-18	$-x(-16) - x(-12) - x(-8) - x(-4)$	3	$x(1)$
-17	$-x(-15) - x(-11) - x(-7) - x(-3)$	4	$x(2)$
-16	$-x(-14) - x(-10) - x(-6) - x(-2)$	5	$x(3)$
-15	$-x(-13) - x(-9) - x(-5) - x(-1)$	6	$x(0) + x(4)$
-14	$-x(-12) - x(-8) - x(-4)$	7	$x(1) + x(5)$
-13	$-x(-11) - x(-7) - x(-3)$	8	$x(2) + x(6)$
-12	$-x(-10) - x(-6) - x(-2)$	9	$x(3) + x(7)$
-11	$-x(-9) - x(-5) - x(-1)$	10	$x(0) + x(4) + x(8)$
-10	$-x(-8) - x(-4)$	11	$x(1) + x(5) + x(9)$
-9	$-x(-7) - x(-3)$	12	$x(2) + x(6) + x(10)$
-8	$-x(-6) - x(-2)$	13	$x(3) + x(7) + x(11)$
-7	$-x(-5) - x(-1)$	14	$x(0) + x(4) + x(8) + x(12)$
-6	$-x(-4)$	15	$x(1) + x(5) + x(9) + x(13)$
-5	$-x(-3)$	16	$x(2) + x(6) + x(10) + x(14)$
-4	$-x(-2)$	17	$x(3) + x(7) + x(11) + x(15)$
-3	$-x(-1)$	18	$x(0) + x(4) + x(8) + x(12) + x(16)$
-2	0	19	$x(1) + x(5) + x(9) + x(13) + x(17)$

A CERTAIN GENERALIZATION

The operation

$$S_b\{x(k)\} := \{x(k + n) - b\, x(k - n)\}, \tag{12}$$

where $\{x(k)\} \in L^0 = L^1 := C(\mathbb{Z}, \mathbb{C}), b \in \mathbb{C} \setminus \{0\}$, is a generalization of the central difference (7).

(12) will be called a *$2n^{th}$-order central difference with the base b*.

While constructing an operational calculus model corresponding to the derivative (12), we will use the method of solving the equation $x(k+1) - b(k)x(k) = f(k)$ described in [12], as well as the following auxiliary theorems:

Lemma 1 (Th. 3 [5]). *An abstract differential equation*

$$Sx = f, \quad f \in L^0, x \in L^1$$

with the limit condition

$$s_q x = x_{0,q}, \quad x_{0,q} \in \operatorname{Ker} S$$

has exactly one solution

$$x = x_{0,q} + T_q f.$$

Lemma 2 (Th. 4 [5]). *With a given derivative $S \in \mathscr{L}(L^1, L^0)$, the projection $s_q \in \mathscr{L}(L^1, \operatorname{Ker} S)$ determines the integral $T_q \in \mathscr{L}(L^0, L^1)$ from the condition*

$$x = T_q f \quad \textit{if and only if} \quad Sx = f, s_q x = 0.$$

Moreover, the projection s_q is a limit condition corresponding to the integral T_q.

One of the elements of $\operatorname{Ker} S_b$ is the sequence

$$e(k) := b^{\frac{k}{2n}}, \quad k \in \mathbb{Z}.$$

Then

$$e(k+n) = b\, e(k-n), \quad k \in \mathbb{Z}.$$

Let us consider the following difference equation

$$S_b\{x(k)\} = \{f(k)\},$$

i.e.

$$x(k+n) - b\, x(k-n) = f(k), \quad k \in \mathbb{Z}. \tag{13}$$

Hence, we have

$$\frac{x(k+n)}{e(k+n)} - \frac{x(k-n)}{e(k-n)} = \frac{f(k)}{e(k+n)}, \quad k \in \mathbb{Z},$$

so

$$y(k+n) - y(k-n) = g(k), \quad k \in \mathbb{Z}, \tag{14}$$

where

$$y(k) := \frac{x(k)}{e(k)}, \quad g(k) := \frac{f(k)}{e(k+n)}, \quad k \in \mathbb{Z}. \tag{15}$$

The equation (14) can be presented in the form of

$$S\{y(k)\} = \{g(k)\}, \tag{16}$$

where $S \equiv D_n$ is the operation (7).

From Lemma 1 it follows that the solution of the equation (16) is a sequence

$$\{y(k)\} = s_{k_0}\{y(k)\} + T_{k_0}\{g(k)\},$$

where T_{k_0} and s_{k_0} are operations (8) and (9).

From (15) we get $x(k) = e(k)\,y(k), k \in \mathbb{Z}$. Finally,

$$\{x(k)\} = \{e(k)\}s_{k_0}\left\{\frac{x(k)}{e(k)}\right\} + \{e(k)\}T_{k_0}\left\{\frac{f(k)}{e(k+n)}\right\} \tag{17}$$

is a solution of (13).

If

$$\{\widetilde{c}(k)\} := s_{k_0}\left\{\frac{x(k)}{e(k)}\right\},$$

then $\{\widetilde{c}(k)\} \in \operatorname{Ker} S$, thus

$$\widetilde{c}(k+n) - \widetilde{c}(k-n) = 0, \quad k \in \mathbb{Z}.$$

Let

$$s_{b,k_0}\{x(k)\} := \{e(k)\}s_{k_0}\left\{\frac{x(k)}{e(k)}\right\}, \quad k_0 \in Q := \mathbb{Z}, \{x(k)\} \in L^1. \tag{18}$$

Therefore, for each $k \in \mathbb{Z}$ we obtain

$$\begin{aligned} S_b s_{b,k_0} x(k) &= e(k+n)\,\widetilde{c}(k+n) - b\,e(k-n)\,\widetilde{c}(k-n) \\ &= e(k+n)(\widetilde{c}(k+n) - \widetilde{c}(k-n)) = e(k+n)\cdot 0 = 0, \end{aligned}$$

that is $s_{b,k_0} \in \mathscr{L}(L^1, \operatorname{Ker} S_b)$. Moreover, since $s_{k_0}\{\widetilde{c}(k)\} = \{\widetilde{c}(k)\}$, then for each $k \in \mathbb{Z}$ we have

$$\begin{aligned} s^2_{b,k_0} x(k) &= s_{b,k_0}[e(k)\widetilde{c}(k)] = e(k)s_{k_0}\left[\frac{e(k)\widetilde{c}(k)}{e(k)}\right] \\ &= e(k)s_{k_0}\widetilde{c}(k) = e(k)\widetilde{c}(k) = s_{b,k_0}x(k). \end{aligned}$$

Finally, s_{b,k_0} is a projection of L^1 onto $\operatorname{Ker} S_b$ for each $k_0 \in \mathbb{Z}$. From Lemma 2 it follows that the projection s_{b,k_0} determines an *integral* T_{b,k_0} from the formula (17). Namely,

$$T_{b,k_0}\{f(k)\} := \{e(k)\}T_{k_0}\left\{\frac{f(k)}{e(k+n)}\right\}, \quad k_0 \in Q, \{f(k)\} \in L^0. \tag{19}$$

What is more, s_{b,k_0} is a *limit condition* corresponding to the integral (19). Hence, we arrive at the

Corollary 3. *The system* (12), (18), (19) *forms a discrete model of the Bittner operational calculus*

$$CO\Big(C(\mathbb{Z},\mathbb{C}), C(\mathbb{Z},\mathbb{C}), S_b, T_{b,k_0}, s_{b,k_0}, \mathbb{Z}\Big). \tag{20}$$

In particular, to the derivative $\overline{S} := \frac{1}{2}S_{-1}$ understood as a $2n^{th}$*-order central mean*

$$\overline{S}\{x(k)\} \equiv M\{x(k)\} = \left\{\frac{x(k-n) + x(k+n)}{2}\right\},$$

there correspond the below integrals and limit conditions

$$\overline{T}_{k_0} = 2T_{-1,k_0}, \quad \overline{s}_{k_0} = s_{-1,k_0}.$$

THE λ-FIBONACCI SEQUENCES

A special case of the $2n^{\text{th}}$-order central difference is a derivative

$$S\,x \equiv D_1 x = \{x(k+1) - x(k-1)\}, \tag{21}$$

to which, in line with Theorem 1 (cf. [21]), there correspond the following integrals

$$T_{k_0}x := \begin{cases} -\dfrac{1}{2}\displaystyle\sum_{i=k+1}^{k_0}[1-(-1)^{k-i}]x(i) & \text{for} \quad k < k_0 \\ 0 & \text{for} \quad k = k_0 \\ \dfrac{1}{2}\displaystyle\sum_{i=k_0+1}^{k}[1-(-1)^{k-i}]x(i) & \text{for} \quad k > k_0 \end{cases}, \quad k \in \mathbb{Z} \tag{22}$$

and limit conditions

$$s_{k_0}x := \{\frac{1}{2}[x(k_0) + x(k_0+1)] + \frac{1}{2}(-1)^{k-k_0}[x(k_0) - x(k_0+1)]\}. \tag{23}$$

Let $C(\mathbb{Z}, \mathbb{R})$ be a space of two-sided real sequences. For $x, y \in C(\mathbb{Z}, \mathbb{R})$ and $z := x + y\,\mathrm{i}$, we define $S z := S x + \mathrm{i}\, S y$. Similarly, we determine $T_{k_0} z$ and $s_{k_0} z$.

A sequence $z = \{z(k)\}$, which is a solution of the problem

$$\begin{cases} S z = \lambda z, & \lambda \in \mathbb{C} \\ s_{k_0} z = c, & c \in \operatorname{Ker} S \end{cases}, \tag{24}$$

will be called an *exponential element* (with λ *exponent*) corresponding to the derivative (21)[7] (cf. [5]).

This element is determined uniquely, because the only solution of the problem (24) for $c = 0$ is $z = 0$.

Let us consider such a case of (24), in which the limit condition for $k_0 = 0$ is induced by the initial conditions

$$z(0) = 0, \quad z(1) = 1.$$

Then, the problem (24) will take the form of

$$\begin{cases} z(k+1) - z(k-1) = \lambda z(k) \\ s_0 z(k) = \frac{1}{2}[\, 1 - (-1)^k \,] \end{cases}, \quad k \in \mathbb{Z}$$

and for $\lambda \neq \pm 2\mathrm{i}$ has a solution given by the λ*-Binet formula*

$$z(k) = \frac{\Phi_\lambda^k - \varphi_\lambda^k}{\Phi_\lambda - \varphi_\lambda}, \quad k \in \mathbb{Z}, \tag{25}$$

where

$$\Phi_\lambda := \frac{\lambda + \sqrt{\lambda^2 + 4}}{2}, \quad \varphi_\lambda := \frac{\lambda - \sqrt{\lambda^2 + 4}}{2}.$$

The solution (25) is called a λ*-Fibonacci sequence*.

Real λ-Fibonacci sequences for $\lambda \in \mathbb{R}_{>0}$ and $k \in \mathbb{Z}$ were already considered by Gazalé [9] and Stakhov [18], while for $\lambda \in \mathbb{N}$ and $k \in \mathbb{N}_0$ — by Falcón and Plaza [8].

The number sequence $\{\Phi_\lambda\}_{\lambda \in \mathbb{N}}$ was called a *family of metallic means* (*proportions*) by de Spinadel in [16, 17]. For $\lambda = 1, 2, 3$ the proportions are called *golden*, *silver* and *bronze*, respectively.

---

[7] By analogy to the exponential function $\{e^{\lambda(t-t_0)}c\}$ corresponding to the ordinary derivative $S := d/dt$.

Let $\lambda := \alpha + \beta\,\mathrm{i}$, where $\alpha^2 + \beta^2 > 0$. By separating real and imaginary parts in the problem (24), we obtain a system of equations

$$\begin{cases} S\,x = \alpha\,x - \beta\,y \\ S\,y = \beta\,x + \alpha\,y \end{cases} \tag{26}$$

with limit conditions

$$s_0 x = c, \quad s_0 y = 0, \tag{27}$$

where $c = \{\frac{1}{2}[\,1 - (-1)^k\,]\}$.

From (26), (27) it follows that the sequences x, y are the solutions of the equation

$$[(S - \alpha I)^2 + \beta^2 I]u = 0^{8} \tag{28}$$

with the corresponding limit conditions

$$s_0 x = c, s_0 S\,x = \alpha\,c \tag{29}$$

$$s_0 y = 0, s_0 S\,y = \beta\,c. \tag{30}$$

Let us denote these solutions as

$$x := \{\mathrm{Exp}\,(\alpha)\mathrm{Cos}\,(\beta)F(k)\}, \quad y := \{\mathrm{Exp}\,(\alpha)\mathrm{Sin}\,(\beta)F(k)\}$$

and let us call them *exponential-trigonometric Fibonacci sequences*.

Likewise, let

$$z = x + y\,\mathrm{i} \;=: \{\mathrm{Exp}\,(\alpha + \beta\,\mathrm{i}\,)F(k)\}$$

denote an $(\alpha + \beta\,\mathrm{i}\,)$-Fibonacci sequence. This sequence is an exponential element with the exponent $\lambda = \alpha + \beta\,\mathrm{i}$ and it corresponds to the central difference (21).

Hence, we get the *Euler-Fibonacci formula* (cf. [5])

$$\mathrm{Exp}\,(\alpha + \beta\,\mathrm{i}\,)F(k) = \mathrm{Exp}\,(\alpha)\mathrm{Cos}\,(\beta)F(k) + \;\mathrm{i}\;\mathrm{Exp}\,(\alpha)\mathrm{Sin}\,(\beta)F(k), \quad k \in \mathbb{Z}, \tag{31}$$

which is an analogon of the classic Euler formula.

We will use the equivalent notations

[8] I is the identity operation defined on $L^0 = C(\mathbb{Z}, \mathbb{R})$.

$$\begin{gathered}
\operatorname{Exp} F(k) \equiv \operatorname{Exp}(1) F(k), \\
\operatorname{Exp} \operatorname{Sin} F(k) \equiv \operatorname{Exp}(1) \operatorname{Sin}(1) F(k), \\
\operatorname{Exp} \operatorname{Cos} F(k) \equiv \operatorname{Exp}(1) \operatorname{Cos}(1) F(k), \\
\operatorname{Sin} F(k) \equiv \operatorname{Exp}(0) \operatorname{Sin}(1) F(k), \\
\operatorname{Cos} F(k) \equiv \operatorname{Exp}(0) \operatorname{Cos}(1) F(k),
\end{gathered}$$

whereby the sequences

$$\{\operatorname{Cos} F(k)\}, \quad \{\operatorname{Sin} F(k)\}$$

will be called a *Fibonacci cosine* and *sine*, respectively.

By means of $\{\operatorname{Exp}(\lambda) F(k)\}$, we can also determine the below *hyperbolic Fibonacci sequences* (*cosine* and *sine*)

$$\left.\begin{aligned}
\operatorname{Cosh} F(k) &:= \frac{\operatorname{Exp}(1) F(k) + \operatorname{Exp}(-1) F(k)}{2} \\
\operatorname{Sinh} F(k) &:= \frac{\operatorname{Exp}(1) F(k) - \operatorname{Exp}(-1) F(k)}{2}
\end{aligned}\right\}, \quad k \in \mathbb{Z}.$$

Then, using the Binet formula (25), we obtain

$$\left.\begin{aligned}
\operatorname{Cosh} F(k) &= \begin{cases} 0 & \text{for} \quad k = 2\ell \\ \operatorname{ch}F\mathrm{s}(k) & \text{for} \quad k = 2\ell + 1 \end{cases} \\
\operatorname{Sinh} F(k) &= \begin{cases} \operatorname{sh}F\mathrm{s}(k) & \text{for} \quad k = 2\ell \\ 0 & \text{for} \quad k = 2\ell + 1 \end{cases}
\end{aligned}\right\}, \quad \ell \in \mathbb{Z},$$

where

$$\operatorname{ch}F\mathrm{s}(k) := \frac{\Phi_1^k + \Phi_1^{-k}}{\sqrt{5}}, \quad \operatorname{sh}F\mathrm{s}(k) := \frac{\Phi_1^k - \Phi_1^{-k}}{\sqrt{5}}, \quad k \in \mathbb{Z}$$

are so-called *symmetric hyperbolic Fibonacci* (*cosine* and *sine*) *sequences*, introduced by Stakhov and Rozin in [19] (see also [18]).

Then, we have

$$F(k) = \operatorname{Sinh} F(k) + \operatorname{Cosh} F(k), \quad k \in \mathbb{Z},$$

where $\{F(k)\}$ is a classic two-sided 1-Fibonacci sequence,
whereby $F(-k) = (-1)^{k+1} F(k), k \in \mathbb{N}$:

k	…	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	…
$F(k)$	…	13	−8	5	−3	2	−1	1	0	1	1	2	3	5	8	13	…

We will now discuss a special case of λ-Fibonacci sequences, when $\lambda := 1 + \mathrm{i}$. From (28)–(30) it follows that $x = \{\mathrm{Exp\,Cos}\, F(k)\}$ and $y = \{\mathrm{Exp\,Sin}\, F(k)\}$ are solutions of the homogeneous 4th-order difference equation

$$u(k+2) - 2u(k+1) + 2u(k-1) + u(k-2) = 0, \quad k \in \mathbb{Z}$$

with the respective initial conditions

$$x(-1) = 1, x(0) = 0, x(1) = 1, x(2) = 1$$
$$y(-1) = 0, y(0) = 0, y(1) = 0, y(2) = 1.$$

We can solve the above problems using *Mathematica*®. By means of the RSolve command, after arranging the obtained results, we eventually get

$$\mathrm{Exp\,Cos}\, F(k) = \frac{\sqrt{2-\mathrm{i}}\,(\Psi_1^k - \psi_1^k) + \sqrt{2+\mathrm{i}}\,(\Psi_2^k - \psi_2^k)}{2\sqrt{10}} \tag{32}$$

$$\mathrm{Exp\,Sin}\, F(k) = \frac{\sqrt{-2+\mathrm{i}}\,(\psi_1^k - \Psi_1^k) + \sqrt{-2-\mathrm{i}}\,(\psi_2^k - \Psi_2^k)}{2\sqrt{10}}, \tag{33}$$

where $k \in \mathbb{Z}$ and

$$\Psi_{1,2} := \left(\frac{1}{2} \pm \frac{1}{2}\mathrm{i}\right) + \sqrt{1 \pm \frac{1}{2}\mathrm{i}}\ , \quad \psi_{1,2} := \left(\frac{1}{2} \pm \frac{1}{2}\mathrm{i}\right) - \sqrt{1 \pm \frac{1}{2}\mathrm{i}}\ .$$

By using *Mathematica*®, we can also determine any terms of the sequences (32), (33). The below tables pertain to the range $k \in \overline{-10, 12}$:

k	…	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
Exp Cos $F(k)$	…	39	−43	32	−19	9	−3	0	1	−1	1	0

k	1	2	3	4	5	6	7	8	9	10	11	12	…
Exp Cos $F(k)$	1	1	1	0	−3	−9	−19	−32	−43	−39	5	128	…

k	…	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
Exp Sin $F(k)$	…	87	−36	8	4	−7	6	−4	2	−1	0	0

k	1	2	3	4	5	6	7	8	9	10	11	12	…
Exp Sin $F(k)$	0	1	2	4	6	7	4	−8	−36	−87	−162	−244	…

It is also worth noticing that $\operatorname{Exp}\operatorname{Cos} F(k)$ and $\operatorname{Exp}\operatorname{Sin} F(k)$ for $k \in \mathbb{N}_0$ are presented in the *OEIS*®[9] database as sequences A143056 and A272665.

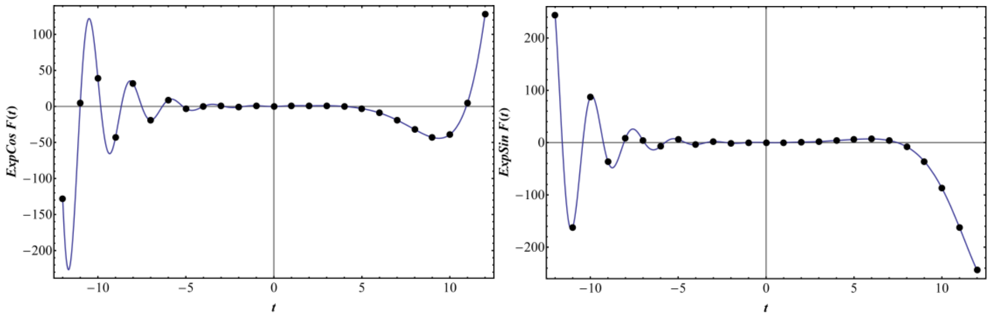

Fig. 1. Graphs of Fibonacci exponential-trigonometric functions and sequences

If we apply the Re, ComplexExpand and FullSimplify *Mathematica*® commands to the formulas (32), (33), then, after arranging the obtained results, we get the mentioned sequences in real forms. Namely,

$$\operatorname{Exp}\operatorname{Cos} F(k) = \frac{1}{\sqrt{2\sqrt{5}}}\left[\sqrt{\left(\Phi_1 + \sqrt{\Phi_1}\right)^k}\cos\left(\frac{1}{2}\operatorname{arc\,cot}(2) - k\operatorname{arc\,tan}\left(\sqrt{\Phi_1^{-3}}\right)\right)\right.$$
$$\left. - \sqrt{\left(\Phi_1 - \sqrt{\Phi_1}\right)^k}\cos\left(\frac{1}{2}\operatorname{arc\,cot}(2) - k\left(\pi - \operatorname{arc\,tan}\left(\sqrt{\Phi_1^{-3}}\right)\right)\right)\right]$$

$$\operatorname{Exp}\operatorname{Sin} F(k) = \frac{1}{\sqrt{2\sqrt{5}}}\left[\sqrt{\left(\Phi_1 - \sqrt{\Phi_1}\right)^k}\sin\left(\frac{1}{2}\operatorname{arc\,cot}(2) - k\left(\pi - \operatorname{arc\,tan}\left(\sqrt{\Phi_1^{-3}}\right)\right)\right)\right.$$
$$\left. - \sqrt{\left(\Phi_1 + \sqrt{\Phi_1}\right)^k}\sin\left(\frac{1}{2}\operatorname{arc\,cot}(2) - k\operatorname{arc\,tan}\left(\sqrt{\Phi_1^{-3}}\right)\right)\right],$$

where $k \in \mathbb{Z}$.

A classic logarithmic spiral in a complex plane is a graph of the function

$$z(t) = e^{(\alpha+\beta\,\mathrm{i})t}.$$

On the basis of the Euler-Fibonacci formula (31), the $(1 + \mathrm{i})$-Fibonacci numbers can be also interpreted as complex plane points

$$z_k := (\operatorname{Exp}\operatorname{Cos} F(k), \operatorname{Exp}\operatorname{Sin} F(k)), \quad k \in \mathbb{Z},$$

[9] *OEIS*® — The On-Line Encyclopedia of Integer Sequences®, https://oeis.org/.

lying on the $(1 + \mathrm{i})$-*Fibonacci spiral* (fig. 2) with a parametric description

$$x(t) := \operatorname{Exp}\operatorname{Cos} F(t), \quad y(t) := \operatorname{Exp}\operatorname{Sin} F(t), \qquad t \in \mathbb{R}.$$

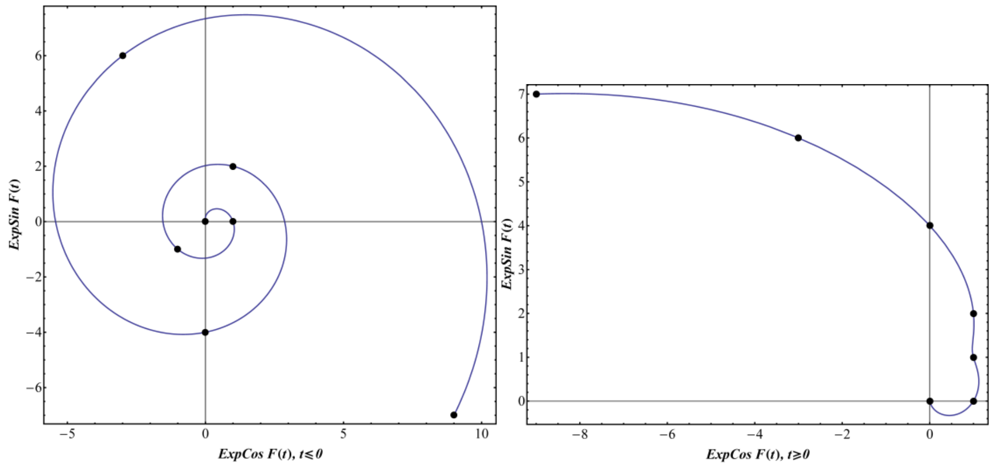

Fig. 2. Fibonacci spiral and $(1 + \mathrm{i})$-Fibonacci numbers

The trigonometric Fibonacci sequences $x = \{\operatorname{Cos} F(k)\}$ and $y = \{\operatorname{Sin} F(k)\}$ are obtained from (28)–(30) for $\alpha = 0$ and $\beta = 1$, i.e. for $\lambda := \mathrm{i}$. Thus, x and y fulfil the equation

$$u(k+2) - u(k) + u(k-2) = 0, \qquad k \in \mathbb{Z}$$

and initial conditions

$$\begin{array}{l} x(-1) = 1, x(0) = 0, x(1) = 1, x(2) = 0 \\ y(-1) = 0, y(0) = 0, y(1) = 0, y(2) = 1 \end{array},$$

taking the forms of

$$\operatorname{Cos} F(k) = \frac{1}{\sqrt{3}}\Big[\cos\Big(\frac{k\pi}{6}\Big) - \cos\Big(\frac{5k\pi}{6}\Big)\Big] = \frac{2}{\sqrt{3}}\sin\Big(\frac{k\pi}{2}\Big)\sin\Big(\frac{k\pi}{3}\Big) \tag{34}$$

$$, \quad k \in \mathbb{Z}.$$

$$\operatorname{Sin} F(k) = \frac{1}{\sqrt{3}}\Big[\sin\Big(\frac{k\pi}{6}\Big) - \sin\Big(\frac{5k\pi}{6}\Big)\Big] = -\frac{2}{\sqrt{3}}\sin\Big(\frac{k\pi}{3}\Big)\cos\Big(\frac{k\pi}{2}\Big) \tag{35}$$

It is easy to verify that

$$\begin{array}{c} \operatorname{Cos} F(-k) = \operatorname{Cos} F(k) = -\operatorname{Cos} F(k+6) \\ \operatorname{Sin} F(-k) = -\operatorname{Sin} F(k) = \operatorname{Sin} F(k+6) \\ \operatorname{Cos} F(k+3) = -\operatorname{Sin} F(k), \ \operatorname{Sin} F(k+3) = \operatorname{Cos} F(k) \end{array}, \qquad k \in \mathbb{Z}.$$

We also have

$$\operatorname{Cos} F(k+12) = \operatorname{Cos} F(k), \quad \operatorname{Sin} F(k+12) = \operatorname{Sin} F(k), \qquad k \in \mathbb{Z}.$$

Thus, the sequences (34), (35) are 12-periodic. The following tables present their cycles, i.e. values for $k \in \overline{0, 11}$:

k	0	1	2	3	4	5	6	7	8	9	10	11
$\operatorname{Cos} F(k)$	0	1	0	0	0	−1	0	−1	0	0	0	1

k	0	1	2	3	4	5	6	7	8	9	10	11
$\operatorname{Sin} F(k)$	0	0	1	0	1	0	0	0	−1	0	−1	0

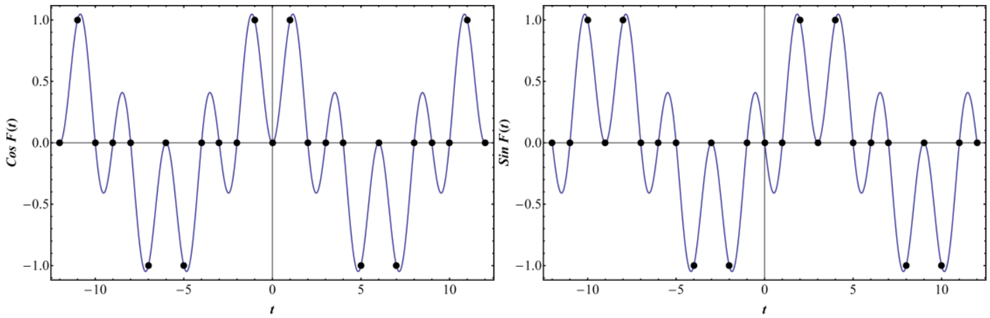

Fig. 3. Graphs of Fibonacci trigonometric functions and sequences

The sequence $\operatorname{Cos} F(k), k \in \mathbb{N}_0$ can be found in *OEIS*® under A110161.

Fig. 4 presents a graph of the *Fibonacci circle*

$$x(t) := \operatorname{Cos} F(t), \quad y(t) := \operatorname{Sin} F(t), \qquad t \in [0, 12).$$

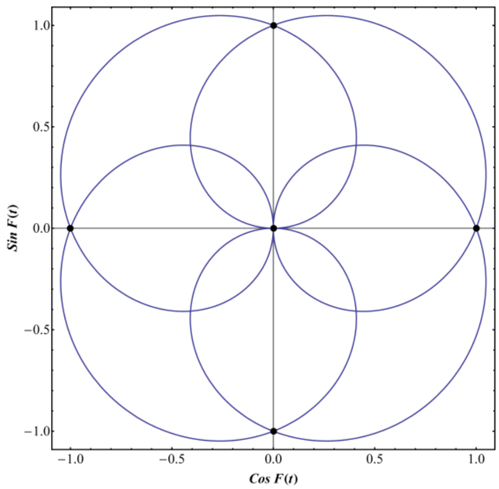

Fig. 4. Fibonacci circle

Using (34) and (35), we obtain the *Fibonacci trigonometric identity*

$$\operatorname{Cos}^2 F(k) + \operatorname{Sin}^2 F(k) = F(k) \pmod 2, \quad k \in \mathbb{Z},$$

where $F(k) \pmod 2 = \frac{4}{3}\sin^2(\frac{k\pi}{3}), k \in \mathbb{Z}$ is a 3-periodic sequence[10]

$$\ldots, 0, 1, 1, 0, 1, 1, 0, 1, 1, \ldots \ .$$

It is not difficult to verify that

$$S\{\operatorname{Cos} F(k)\} = -\{\operatorname{Sin} F(k)\}, \quad S\{\operatorname{Sin} F(k)\} = \{\operatorname{Cos} F(k)\}, \tag{36}$$

where $S \equiv D_1$ is the central difference (21).

From (27) we have

$$s_0\{\operatorname{Cos} F(k)\} = \{\frac{1}{2}(1-(-1)^k\}, \quad s_0\{\operatorname{Sin} F(k)\} = \{0\}, \tag{37}$$

where s_0 is the limit condition (23) for $k_0 = 0$.

From (36) and (37) we get in turn

$$T_0\{\operatorname{Cos} F(k)\} = \{\operatorname{Sin} F(k)\}, \quad T_0\{\operatorname{Sin} F(k)\} = -\{\operatorname{Cos} F(k)\} + \{\frac{1}{2}(1-(-1)^k\},$$

where T_0 is the integral (22) for $k_0 = 0$.

The unilateral $\mathcal{Z}$-transform for one-sided sequences

$$x = \{\operatorname{Cos} F(k)\}_{k\in\mathbb{N}_0}, \quad y = \{\operatorname{Sin} F(k)\}_{k\in\mathbb{N}_0}$$

is equal to

$$X(z) = \frac{z^3 - z}{z^4 - z^2 + 1}, \quad Y(z) = \frac{z^2}{z^4 - z^2 + 1},$$

respectively. Therefore,

$$G_c(z) := X\left(\frac{1}{z}\right) = -X(z) \quad \text{and} \quad G_s(z) := Y\left(\frac{1}{z}\right) = Y(z)$$

are generating functions of these sequences. Hence

$$G_c(z) = \frac{-z^3 + z}{z^4 - z^2 + 1} = \sum_{k=0}^{\infty} \operatorname{Cos} F(k)\, z^k$$

and

$$G_s(z) = \frac{z^2}{z^4 - z^2 + 1} = \sum_{k=0}^{\infty} \operatorname{Sin} F(k)\, z^k.$$

[10] It is a sequence A011655 for $k \in \mathbb{N}_0$ (see also [22]).

In both cases the denominator

$$W(z) = z^4 - z^2 + 1$$

is a *12th-cyclotomic polynomial*, i.e. $W(z) = P_{12}(z)$, where

$$P_n(z) := \prod_{j \in \mathbb{A}_n} (z - \xi_j)$$

and $\xi_j := \cos \frac{2j\pi}{n} + \mathrm{i} \sin \frac{2j\pi}{n}$, $\mathbb{A}_n := \{j \in \overline{1,n} : \quad (j,n) = 1\}$.

$(b, \lambda, 2n)$-BONACCI SEQUENCES

A two-sided sequence $z = \{z(k)\} \in C(\mathbb{Z}, \mathbb{C})$, satisfying the equation

$$z(k+n) - b\, z(k-n) = \lambda\, z(k), \quad b, \lambda \in \mathbb{C} \setminus \{0\}$$

as well as the initial conditions

$$z(k_0) = c_{k_0}, z(k_0+1) = c_{k_0+1}, \ldots, z(k_0+2n-1) = c_{k_0+2n-1}$$

will be called a $(b, \lambda, 2n)$-*Bonacci sequence*.

In the operational calculus model (20) it is an exponential sequence fulfilling the equation

$$S_b z = \lambda\, z$$

and the limit condition

$$s_{b,k_0} z = c = \left\{ \frac{\sqrt[2n]{b^k}}{2n} \sum_{j=0}^{2n-1} \sum_{i=k_0}^{k_0+2n-1} \varepsilon_j^{k-i} \frac{c_i}{\sqrt[2n]{b^i}} \right\}.$$

In [11], Kalman and Mena considered one-sided $(b, \lambda, 2)$-Bonacci sequences, when $k_0 = 0$ and $b, \lambda \in \mathbb{R} \setminus \{0\}$.

A $(1, 1, 2n)$-Bonacci sequence $\mathcal{F} = \{\mathcal{F}(k)\}$, of which $2n$ consecutive terms are Fibonacci numbers

$$\mathcal{F}(k) := F(k) = \frac{\Phi_1^k - \varphi_1^k}{\Phi_1 - \varphi_1} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right], \quad k \in \overline{0, 2n-1}$$

will be called a $(2n)$-*Fibonacci sequence*.

For example, a (4)-Fibonacci sequence is a solution of the equation

$$\mathcal{F}(k+2) - \mathcal{F}(k-2) = \mathcal{F}(k) \tag{38}$$

with conditions

$$\mathcal{F}(0) = F(0) = 0, \mathcal{F}(1) = F(1) = 1, \mathcal{F}(2) = F(2) = 1, \mathcal{F}(3) = F(3) = 2. \tag{39}$$

Using *Mathematica*®, we obtain a solution of the problem (38), (39):

$$\mathcal{F}(k) = \frac{1}{\sqrt{5}}\Big[\Big(\sqrt{\Phi_1^k}\cos\Big(\frac{k\pi}{2}\Big) - \sqrt{\Phi_1^{-k}}\Big)\cos\Big(\frac{k\pi}{2}\Big) + \Big(\sqrt{\Phi_1^{k+3}}\sin\Big(\frac{k\pi}{2}\Big) - \sqrt{\Phi_1^{-(k+3)}}\Big)\sin\Big(\frac{k\pi}{2}\Big)\Big].$$

Hence, we have

$$\mathcal{F}(2k) = F(k), \mathcal{F}(2k-1) = F(k+1), \quad k \in \mathbb{Z}.$$

The sequence $\mathcal{F}(k), k \in \mathbb{N}_0$ can be found in *OEIS*® under A053602:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	…
$\mathcal{F}(k)$	0	1	1	2	1	3	2	5	3	8	5	13	8	21	13	34	21	…

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MODEL RACHUNKU OPERATORÓW DLA RÓŻNICY CENTRALNEJ ORAZ WYKŁADNICZO-TRYGONOMETRYCZNE I HIPERBOLICZNE CIĄGI FIBONACCIEGO

STRESZCZENIE

W artykule skonstruowano model nieklasycznego rachunku operatorów Bittnera, w którym pochodna rozumiana jest jako różnica centralna $D_n\{x(k)\} := \{x(k+n) - x(k-n)\}$. Dokonano uogólnienia opracowanego modelu, rozważając operację $D_{n,b}\{x(k)\} := \{x(k+n) - bx(k-n)\}$, gdzie $b \in \mathbb{C} \setminus \{0\}$. W modelu z różnicą D_1 wprowadzono wykładniczo-trygonometryczne i hiperboliczne ciągi Fibonacciego.

Słowa kluczowe:
rachunek operatorów, pochodna, pierwotne, warunki graniczne, różnica centralna, ciągi Fibonacciego.

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