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DIMENSION OF THE INTERSECTION OF CERTAIN CANTOR SETS IN THE PLANE

Steen Pedersen and Vincent T. Shaw

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Abstract. In this paper we consider a retained digits Cantor set T based on digit expansions with Gaussian integer base. Let F be the set all x such that the intersection of T with its translate by x is non-empty and let F_β be the subset of F consisting of all x such that the dimension of the intersection of T with its translate by x is β times the dimension of T. We find conditions on the retained digits sets under which F_β is dense in F for all $0 \leq \beta \leq 1$. The main novelty in this paper is that multiplication the Gaussian integer base corresponds to an irrational (in fact transcendental) rotation in the complex plane.

Keywords: Cantor set, fractal, self-similar, translation, intersection, dimension, Minkowski dimension.

Mathematics Subject Classification: 28A80, 51F99.

1. INTRODUCTION

Cantor sets occur in mathematical models of many naturally occurring object, play a role in number theory, signal processing, ergodic theory, limit-theorems in probability, and in interior design. For their role in designing relaxing floors, see [32]. We study the "size" of the intersection of two Cantor sets. The significance of this problem was noted by Furstenberg [9] and Palis [27]. Some papers dealing with aspects of our problem are: $[2, 3, 6, 7, 17–19, 21, 22, 25, 31, 33–35]$, and [23]. The literature in this subject and its applications is vast. The list above is represents a small sample of the literature closely related to our problem.

For a set T of complex number and a complex number x, the translate of T by x is $x + T := \{x + t \mid t \in T\}$. Let F be the set of all x such that $T \cap (x + T)$ is non-empty. Palis [27] conjectured that for dynamically defined Cantor sets in the real line typically the corresponding set F either has Lebesgue measure zero or contains an interval. The papers [5] and [24] investigate this conjecture for random retained digits sets and solve it in the affirmative in the deterministic case.

For $0 \leq \beta \leq 1$, let F_β be the set of $x \in F$, such that $T \cap (x + T)$ has dimension $\beta \dim (T)$. In this paper, we consider certain Cantor subsets T of the complex plane and present conditions under which F_β is dense in F for all $0 \leq \beta \leq 1$. For certain Cantor subsets of the real line this was investigated in [4, 12, 26], and [28].

2. PRIOR WORK ON THIS PROBLEM

2.1. PRIOR WORK IN THE REAL LINE

Most prior work closely related to our problem has been done for subsets of the real line, we will first summarize some of what is known in that case.

Let $n \geq 2$ be an integer. Any real number $x \in [0, 1]$ has at least one representation in base n

$$x = 0._n x_1 x_2 \ldots := \sum_{k=1}^{\infty} \frac{x_k}{n^k}$$

where each x_k is in the digit set

$$D = D_n := \{0, 1, \ldots n - 1\}.$$

Deleting some element from the full digit set D we get a retained set of *digits* $D^* := \{d_k \mid k = 1, 2, \ldots, m\}$ with $m < n$ digits $d_k < d_{k+1}$ and a corresponding *retained digits Cantor set*

$$T = T_{n,D^*} := \{0._n x_1 x_2 \ldots \mid x_k \in D^* \text{ for all } k \in \mathbb{N}\}. \tag{2.1}$$

For T to contain more than one point we need $m \geq 2$. We are interested in the dimension of the sets $T \cap (x + T)$. Since the problem is invariant under translation we will assume $d_1 = 0$.

Formulas for the dimension of $T \cap (x + T)$ can be found in [3, 4, 16, 26], and [29] under various conditions on D^*. Examples in [28] shows there are retained digits Cantor sets T_{n,D^*} for which F_β is not dense in F for some $0 \leq \beta \leq 1$. Hence, to show F_β is dense in F, some conditions must be imposed on D^*. See [4, 12, 26, 28] for condition under which F_β is dense in F for all $0 \leq \beta \leq 1$. It is known that $T \cap (x + T)$ only is self-similar for a small set of x, see [22] and [30].

Some of the cited papers only consider the middle thirds Cantor set, some only consider rational t, and in some of the papers dimension means Minkowski dimension and in some it means Hausdorff dimension.

2.2. PRIOR WORK IN THE PLANE

For subsets of the plane previous work closely related to our problem has been restricted to Sierpinski carpets type fractals. More specifically, the dimension of $T \cap (x + T)$ is investigated in [23] for certain Sierpinski carpets T a certain class of translation vectors x. In [2] the authors consider the intersection of a Sierpinski carpet with its rational translates. They use methods and obtain results similar to the ones in [26]. Both in [23] and [2] the authors study cases where the "base" is a real number and the "digits" are vectors parallel to the coordinate axes. In particular, the scaling matrix is a constant multiple of the identity matrix. In this paper we consider cases where the "base" is a complex number whose phase is an irrational multiple of π and the "digits" are real numbers.

3. PLANAR GAUSSIAN FRACTALS

As described in Section 2 numbers in the real line can be represented using an integer base $n > 1$ and a digit set $D = D_n = \{0, 1, \ldots, n-1\}$. Using these representations Cantor sets can be obtained by only using digits from some proper subset D^* of D. Similarly, in the complex plane will use bases $b := -n + i$ and digit set $D = D_b := \{0, 1, 2, \ldots, n^2\}$, for an integer $n \geq 1$. Our choice of base and digit set is motivated by [10, Theorem 40], where it is shown that if $m \neq 0$ is an integer, then $\{0, 1, 2, \ldots, m^2\}$ is a complete set of representatives for the Gaussian integers modulo $a := m + i$ and by [14] where it is shown that every Gaussian integer has a unique representation of the form $\sum_{k=0}^{\ell} d_{-k} a^k, d_{-k} \in D = \{0, 1, 2, \ldots, m^2\}$ if and only if $m < 0$.

Gaussian retained digits Cantor sets are sets

$$T = T_{b,D^*} := \left\{ \sum_{k=1}^{\infty} d_k b^{-k} \mid d_k \in D^* \right\}. \tag{3.1}$$

obtained by only using digits from a proper subset D^* of $D = D_b$. For T to contain more than one point we need D^* to contain more than one digit. Hence we will assume D^* contains at least two digits, in particular $n \geq 2$. Note that $T = \bigcup_{d \in D^*} b^{-1}(d + T)$. So T is a self-similar set. Multiplication by $b^{-1} = |b|^{-1} e^{i\theta}$ can be thought of as $|b|^{-1} \times M_\theta$, where M_θ is a rotation matrix. Since $\arctan(x)$ is an irrational (actually transcendental) multiple of π, when x is a rational number other than $0, \pm 1$, the angle of rotation θ is an irrational multiple of π.

4. REPRESENTATIONS IN BASE $b = -n + i$

Every complex number is within $\frac{1}{\sqrt{2}}$ of some Gaussian integer $\mathbb{Z}[i]$. Hence, every complex number z is within $\frac{1}{\sqrt{2}}$ of some complex number of the form $\sum_{k=0}^{m} d_{-k} b^k$, where $d_{-k} \in D$. Now, pick a Gaussian integer $\sum_{k=0}^{m} d_{-k} b^k, d_{-k} \in D$ within $\frac{1}{\sqrt{2}}$ of bz. Then $\frac{1}{b} \sum_{k=0}^{m} d_{-k} b^k = \sum_{j=-1}^{m-1} d_{-j+1} b^j$ is within $\frac{1}{|b|\sqrt{2}}$ of z. Similarly, given any complex number z, we can find $\sum_{k=-l}^{m} d_{-k} b^k$, where $d_{-k} \in D$, within $\frac{1}{|b|^l \sqrt{2}}$ of z. It follows that every complex number has at least one representation of the form

$$z = \sum_{k=-\infty}^{m} d_{-k} b^k = \sum_{k=0}^{m} d_{-k} b^k + \sum_{k=1}^{\infty} d_k b^{-k}, \quad d_k \in D.$$

We say that z has *radix representation*

$$z = d_{-m} \dots d_{-1} d_0 ._b d_1 d_2 \dots := \sum_{k=-\infty}^{m} d_{-k} b^k. \tag{4.1}$$

Radix representations with base b do not have all the properties of the real number representations with base $2, 3, 4, \dots$ For example, it is well known that $0._{10}999\dots = 1$. But, for example, if $b = -2 + i$ with digits set $D = \{0, 1, 2, 3, 4\}$, then $0._{-2+i}444\dots = \frac{2}{5}(-3 - i)$ and $3._{-2+i}0 + 142._{-2+i}0 = 0$. Nevertheless, we find it convenient to use radix representations with Gaussian integer base b. See [11] for an expository article on the geometry and algebra of radix representations of complex numbers in a Gaussian integer bases.

When studying Cantor subsets of the real line using digit representations to some base, the basic set arising from using all the digits is the unit interval $[0, 1]$. The corresponding set in our situation is the set

$$T_0 := \{0.d_1 d_2 \dots \mid d_k \in D\}. \tag{4.2}$$

Figure 1 of illustrates T_0 when $n = 2$.

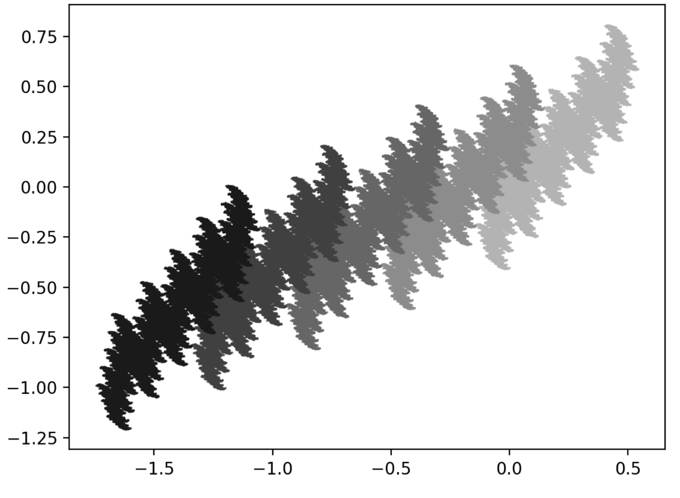

Fig. 1. T_0, when $b = -2 + i$, darker grays correspond to larger d_1

Sets of this type are studied extensively in the literature, among many other facts, it is know that: (i) $\bigcup_{x \in \mathbb{Z}[i]} (x + T_0) = \mathbb{C}$, (ii) if $x \neq y$ in $\mathbb{Z}[i]$ then $(x + T_0) \cap (y + T_0)$ has Lebesgue measure zero, (iii) T_0 is the closure of its interior, and (iv) the boundary of T_0 has Lebesgue measure zero and dimension greater than one, see for example [1, 13, 15, 20], and the references therein. The facts (i) and (ii) mean that T_0 tiles the complex plane by translation by the Gaussian integers.

Using radix representations we can write the Gaussian retained digits Cantor set T_{b,D^*} as

$$T = \{0._b d_1 d_2 \ldots \mid d_k \in D^*\} .$$

Write $D^* := \{d_1, d_2, \ldots, d_m\}$ where $d_k < d_{k+1}$. Since our problem is invariant under translation, we will assume $d_1 = 0$.

The Gaussian retained digits Cantor set T can be obtained from the tile T_0 using a refinement process similar to the way the usual middle thirds Cantor set C is constructed from the closed interval $C_0 := [0, 1]$ by removing the open middle thirds of intervals giving the sets

$$C_k := \{0._3 d_1 d_2 \ldots \mid d_j \in \{0, 2\}, j \leq k, d_j \in \{0, 1, 2\}, j > k\} .$$

Then $C_{k+1} \subseteq C_k$ and $C = \bigcap_{k=0}^{\infty} C_k$. Similarly, for an integer $k \geq 0$, the set

$$T_k = \{0._b d_1 d_2 \ldots \mid d_i \in D^*, i \leq k, d_i \in D, i \geq k\}$$

is called a *refinement of* T_0 *at the kth stage.*

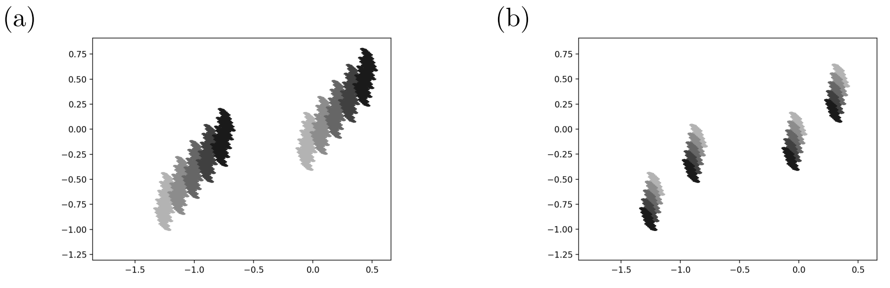

Fig. 2. Refinements of Figure 1: (a) T_1, when $b = -2 + i$, $D^* = \{0, 3\}$, darker grays correspond to larger d_2; (b) T_2, when $b = -2 + i$, $D^* = \{0, 3\}$, darker grays correspond to larger d_3

See Figure 2 for the refinements T_1 and T_2 when $n = 2$ and $D^* = \{0, 3\}$. Clearly, $T_{k+1} \subseteq T_k$ for all k and $T = \bigcap_{k=0}^{\infty} T_k$. Moreover,

$$T_{k+1} = \bigcup_{d \in D^*} b^{-1} (d + T_k) ,$$

where for a complex number z and a set S of complex numbers,

$$zS := \{zs \mid s \in S\} .$$

Similarly

$$T_k = \bigcup_{d_i \in D^*} \left(0._b d_1 d_2 \ldots d_k + b^{-k} T_0\right) .$$

We call sets of the form $T_k + z$ *subtiles* and sets of the form $0._b d_1 d_2 \ldots d_k + b^{-k} T_0$, $d_j \in D^*$ *subtiles of* T_k.

Clearly subtiles are similar to T_0, there are $|D^*|^k$ subtiles of T_k, T_k is the union of its subtiles, and each of these subtiles has Lebesgue measure $|b|^{-2k} = \left(n^2+1\right)^{-k}$. When $n = 2$ and $D^* = \{0, 3\}$, Figure 4 illustrates the two subtiles of T_1 and Figure 4 the four subtiles of T_2.

We say the set T_{k+1} is obtained by *refining* the set T_k; that is, by removing from T_k the complex numbers $0._b d_1 d_2 \ldots$ with digit d_{k+1} not in D^*.

5. DIMENSION

When we say *dimension* we mean box-counting dimension (also know as entropy dimension, Kolmogorov dimension, or Minkowski dimension among other terminology). In this paper we calculate the dimension of $T \cap (x + T)$. In order to do so we adapt the covering sets used in the definition of dimension to our situation.

Definition 5.1. Let $E \subset \mathbb{R}^n$ such that E is nonempty. Then the *dimension of* E is defined as

$$\dim E = \lim_{\delta \to 0} \frac{\log\left(N_\delta(E)\right)}{-\log\left(\delta\right)}$$

where $N_\delta(E)$ denotes the smallest number of sets each of diameter at most δ needed to cover E. The lower box-counting dimension is obtained by using the limit inferior in place of the limit.

Various definitions of $N_\delta(E)$ appear in the literature:

- the smallest amount of closed balls of radius δ that cover E;
- the least amount of cubes of side length δ that cover E;
- the number of δ-mesh cubes that intersect E;
- the smallest number of sets of diameter at most δ that cover E;
- the largest number of disjoint balls of radius δ with centers in E.

In [8] it is shown that all of these give equivalent definitions of dimension. We extend the list to include the smallest number of sets of the form $0._b d_1 d_2 \ldots d_k + b^{-k} T_0$, $d_j \in D$, for some integer k, that cover the set E.

Lemma 5.2. *Let E be a non-empty subset of T_0. For a fixed integer $k \geq 1$, let $N_k(E)$ denote the smallest number of sets of the form $0._b d_1 d_2 \ldots d_k + b^{-k} T_0$, $d_j \in D$. Then the dimension of E exists if and only if $\lim_{k\to\infty} \frac{\log(N_k(E))}{k \log(|b|)}$ exists, and in the affirmative case this limit is the dimension of E.*

Proof. For a fixed k, let $N_k(E)$ denote the smallest number of sets of the form $0._b d_1 d_2 \ldots d_k + b^{-k} T_0$, $d_j \in D$, needed to cover E. Each of these sets has diameter $\delta_k := |b|^{-k} \operatorname{diam}(T_0)$, where $\operatorname{diam}(T_0)$ denotes the diameter of T_0. Let $M_\delta(E)$ be the number of δ-mesh squares $[m_1\delta, (m_1+1)\delta] \times [m_2\delta, (m_2+1)\delta]$, $m_1, m_2 \in \mathbb{Z}$, that intersect E. Since a δ-mesh square has diameter $\sqrt{2}\delta$ we have

$$N_k(E) \leq M_{\delta_k/\sqrt{2}}(E)$$

and since any set of diameter at most δ is contained in 9 δ-mesh squares we have

$$\tfrac{1}{9} M_{\delta_k}(E) \leq N_k(E).$$

Hence

$$\lim_{k\to\infty} \frac{\log(N_k(E))}{-\log(\delta_k)} = \lim_{k\to\infty} \frac{\log(M_{\delta_k}(E))}{-\log(\delta_k)}$$

provided one of the limits exists. □

6. DISJOINTNESS OF SUBTILES

In order to use Lemma 5.2 to calculate dimensions, we need to know how to find the minimal number of subtiles needed to cover a set. Hence, in this section we show that the subtiles of T_k are pairwise disjoint, when D^* satisfies a separation condition. We use this to ensure covers by subtiles contain the minimal number of subtiles. Any condition that ensures that the subtiles are disjoint must depend on n, see Remark 6.2. The dependence on n is due to the rotation in the scaling matrix and was therefore not present in previous studies of the dimension of the intersection of a fractal with its translates.

Our proof that the subtiles of T_k are pairwise disjoint is based on the following lemma.

Lemma 6.1. *Let $n \geq 2$ be an integer, $b = -n + i$, $D = \{0, 1, \ldots, n^2\}$ and $T_0 = \{0.d_1 d_2 \ldots \mid d_k \in D\}$. If $d, d' \in \mathbb{C}$ have the same imaginary part and $|d - d'| \geq n + 1$, then $(d + T_0) \cap (d' + T_0) = \emptyset$.*

Proof. By translation it is enough to show $T_0 \cap (d + T_0) = \emptyset$ when d is a real number and $d \geq n+1$. Let $z := \frac{n^2}{2}$. Denote $c_s := \frac{s}{b} + \sum_{k=2}^{\infty} z b^{-k}$, where $s \in D$ is fixed. Abusing the radix representation notation we will write $c_s = 0._b szzz\ldots$, also when z is not an integer. We think of c_s as representing the "center" of the subtile $b^{-1}(s + T_0) \subseteq T_0$. Let

$$r = r(n) := \frac{n^2}{2\left(n^2 + 1 - \sqrt{n^2+1}\right)}. \tag{6.1}$$

Clearly, $r(2) \approx 0.72$, $r(n)$ is decreasing on the interval $(0, \infty)$, and $r(n) \to \frac{1}{2}$ as $n \to \infty$. Let $s \in D$. For $x \in b^{-1}(s + T_0)$, write $x = 0._b s x_2 x_3 \ldots$ where $x_j \in D$. Hence

$$|x - c_s| = \left|\sum_{j=2}^{\infty} (x_j - z) b^{-j}\right| \leq \sum_{j=2}^{\infty} z |b|^{-j} = \frac{n^2 |b|^{-2}}{2\left(1 - |b|^{-1}\right)} = r,$$

since $|b| = \sqrt{n^2+1}$. Hence, $b^{-1}(s + T_0)$ is a subset of $B_r(c_s)$, where $B_\alpha(\beta)$ is the open ball with center $\beta \in \mathbb{C}$ and radius $\alpha > 0$. As $b^{-1}(s + T_0) \subseteq B_r(c_s)$, $s \in D$, we get $T_0 \subseteq \bigcup_{s\in D} B_r(c_s)$. Similarly, $d + T_0 \subseteq \bigcup_{t\in D} B_r(c_t + d)$.

Therefore, to show $T_0 \cap (T_0 + d) = \emptyset$, it is sufficient to show that $\bigcup_{s\in D} B_r(c_s)$ and $\bigcup_{t\in D} B_r(c_t + d)$ are disjoint. Hence we must show $B_r(c_s) \cap B_r(d + c_t) = \emptyset$ for all $s, t \in D$. Figure 3 illustrates this when $b = -2 + i$ and $D^* = \{0, 3\}$.

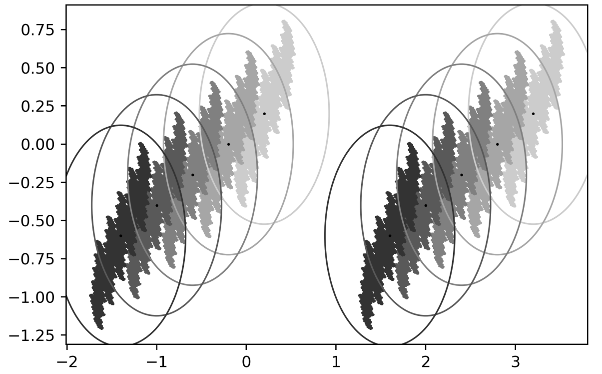

Fig. 3. When $b = -2 + i$ and $d = 3$ the "Minkowski sausages" $\bigcup_{s=0}^{n^2} B_r(c_s)$ and $\bigcup_{t=0}^{n^2} B_r(c_t + d)$ do not intersect. The circles look like ellipses, since there are different scales on the two axes. The dots are the centers c_s and $c_t + d$

For any $s, t \in D$,

$$|c_s - (d + c_t)| = |0._b szzz\ldots - d._b tzzz\ldots| = |d._b(t - s)|$$

is the distance between the centers of the balls $B_r(c_s)$ and $B_r(d + c_t)$ and these balls have radii r. Hence it is sufficient to show that $|d._b(t - s)| > 2r$, for all $s, t \in D$. Algebraically, we get

$$|d._b(t - s)|^2 = d^2 - \frac{2dn}{n^2 + 1}(t - s) + \frac{1}{n^2 + 1}(t - s)^2.$$

Since r determined by Eqn. (6.1) is independent of $s, t \in D = \left\{0, 1, \ldots, n^2\right\}$, we minimize

$$\xi(\tau) := d^2 - \frac{2dn}{n^2 + 1}\tau + \frac{1}{n^2 + 1}\tau^2$$

over $\tau \in \left[-n^2, n^2\right]$. Now

$$\xi'(\tau) = -\frac{2dn}{n^2 + 1} + \frac{2}{n^2 + 1}\tau.$$

Hence $\xi'(dn) = 0$ and the minimal value of $\xi(\tau)$ over $\tau \in \mathbb{R}$ is $\xi(dn) = \frac{d^2}{n^2+1}$ and when $n < d$ the minimal value of $\xi(\tau)$ over $\left[-n^2, n^2\right]$ is $\xi\left(n^2\right) > \xi(dn)$.

Hence, $|d._b(t - s)| > 2r$ is true for all $s, t \in D$ if and only if one of the following two conditions is true:

$$(2r)^2 < \xi(dn) = \frac{d^2}{n^2 + 1}, \tag{6.2}$$

$$(2r)^2 < \xi\left(n^2\right) = \frac{d^2\left(n^2 + 1\right) + n^4 - 2dn^3}{n^2 + 1} \text{ and } n < d. \tag{6.3}$$

Using Eqn. (6.1) we see Eqn. (6.2) is equivalent to

$$\left(\frac{n^2}{n^2+1-\left(n^2+1\right)^{1/2}}\right)^2 < \frac{d^2}{n^2+1}.$$

Solving for d it follows $(2r)^2 < \xi\,(dn)$ is equivalent to

$$\frac{n^2\left(n^2+1\right)^{1/2}}{n^2+1-\left(n^2+1\right)^{1/2}} < d. \tag{6.4}$$

Clearly, the expression on the left hand side is close to n, when n is large. Hence, we will look for bounds on

$$\eta\,(n) := \frac{n^2\left(n^2+1\right)^{1/2}}{n^2+1-\left(n^2+1\right)^{1/2}} - n = \frac{n^2\left(n^2+1\right)^{1/2} - n^3 - n + n\left(n^2+1\right)^{1/2}}{n^2+1-\left(n^2+1\right)^{1/2}}$$

for $n \geq 2$. Using $\left(n^2+1\right)^{1/2} < n+1$ we get

$$\eta\,(n) < \frac{n^2\,(n+1) - n^3 - n + n\,(n+1)}{n^2+n-(n+1)} = \frac{2n^2}{n^2-1}.$$

Hence, $\eta(n) < 2$ and therefore Eqn. (6.4) holds for all $n \geq 2$, when $d \geq n+2$. Similarly, using $n < \left(n^2+1\right)^{1/2}$ we get

$$\eta\,(n) > \frac{n^2\cdot n - n^3 - n + n\cdot n}{n^2+n-(n+1)} = \frac{n^2-n}{n^2-1}.$$

Hence, $\frac{2}{3} \leq \eta\,(n)$ when $n \geq 2$, and therefore therefore Eqn. (6.4) fails for all $n \geq 2$, when $d \leq n + \frac{2}{3}$.

Hence using Eqn. (6.2) and Eqn. (6.3) we see the only case remaining case where $|d._b(t-s)| > 2r$ could be true for all $s, t \in D$ is when $d = n+1$ and $(2r)^2 < \xi\left(n^2\right)$. When $d = n+1$,

$$\xi\left(n^2\right) = \frac{2n^2+2n+1}{n^2+1}.$$

So using Eqn. (6.1) and $d = n+1$, the condition $(2r)^2 < \xi\left(n^2\right)$ can be written as

$$\left(\frac{n^2}{n^2+1-\left(n^2+1\right)^{1/2}}\right)^2 < \frac{2n^2+2n+1}{n^2+1}. \tag{6.5}$$

Plugging in $n = 2$, shows this is true when $n = 2$. Using $\left(n^2+1\right)^{1/2} < n+1$ we see Eqn. (6.5) follows from

$$\left(\frac{n^2}{n^2+1-(n+1)}\right)^2 \leq \frac{2n^2+2n+1}{n^2+1},$$

a bit of algebra simplifies this to

$$0 \leq n^4 - 2n^3 - 2n^2 + 1. \tag{6.6}$$

This inequality fails when $n = 2$ and is true when $n = 3$. The derivative is $4n^3 - 6n^2 - 4n$, which has roots $n = -\frac{1}{2}, 0, 2$. Hence $n^4 - 2n^3 - 2n^2 + 1$ is increasing on the interval $(2, \infty)$. Consequently, Eqn. (6.6) is true for all $n \geq 3$. Thus Eqn. (6.5) is true for all $n \geq 2$. This completes the proof. □

Remark 6.2.

(i) Consider Figure 3. The angle between the line containing the centers $c_s = \frac{s}{b} + 0._b 0zzz\ldots, s \in D$ and the positive $x-$axis is $\frac{\pi}{2} + \arctan(n)$ which increases to π as $n \to \infty$. We need $B_r(c_0)$ to the "left" of $B_r(d + c_{n^2})$. For large n we have $\frac{1}{b} = \frac{-n-i}{n^2+1} \approx -\frac{1}{\sqrt{n^2+1}}$. Hence $c_{n^2} \approx -\frac{n^2}{\sqrt{n^2+1}} + 0._b 0zzz\ldots$ and $c_0 = 0._b 0zzz\ldots$ Since $r \gtrsim \frac{1}{2}$ we need we need $d - \frac{n^2}{\sqrt{n^2+1}} > 1$ for large n. Using $\frac{n^2}{\sqrt{n^2+1}} \approx n$, we see the bound $d \geq n+1$ is either best possible or close to best possible.

(ii) A numerical calculation shows the condition $d \geq n+1$ is best possible for all $n \leq 100$. When $d \leq n$, for some s, t, the balls $B_r(c_s)$ and $B_r(d + c_t)$ will overlap. The Python code used is in Section 8.

Now, we establish that the subtiles of T_k are disjoint provided D^* satisfies the separation condition.

Corollary 6.3. *Let $n \geq 2$ be an integer, $b = -n + i$, $D = \{0, 1, \ldots, n^2\}$, and $T_0 = \{0._b d_1 d_2 \ldots \mid d_k \in D\}$. Suppose $D^* \subseteq D$ satisfies the separation condition $|d - d'| \geq n + 1$, for $d \neq d'$ in D^*. If $d_j, d'_j \in D^*$ for all j and $d_j \neq d'_j$ for at least one j, then the subtiles $0._b d_1 d_2 \ldots d_k + b^{-k} T_0$ and $0._b d'_1 d'_2 \ldots d'_k + b^{-k} T_0$ of $T_k = \{0._b d_1 d_2 \ldots \mid d_j \in D^* \text{ when } j \leq k, d_j \in D \text{ when } j > k\}$ are disjoint.*

Proof. Let $d_j, d'_j \in D^*$ such that $d_j \neq d'_j$ for at least one j. Suppose $d_k = d'_k$ for all $k < i$ and $d_i \neq d'_i$. After a translation and rescaling, if necessary, we may and will assume $i = 1$. Let

$$e_j := \begin{cases} d_j - d'_j & \text{when } d_j > d'_j, \\ 0 & \text{when } d_j \leq d'_j \end{cases} \quad \text{and} \quad e'_j := \begin{cases} 0 & \text{when } d_j > d'_j, \\ d'_j - d_j & \text{when } d_j \leq d'_j. \end{cases}$$

Then e_j, e'_j are in D, hence

$$A := 0._b e_2 e_3 \ldots e_k + b^{1-k} T_0 \quad \text{and} \quad B := 0._b e'_2 e'_2 \ldots e'_k + b^{1-k} T_0$$

are subsets of T_0. Since, $|e_1 - e'_1| = |d_1 - d'_1| \geq n + 1$, it follows from Lemma 6.1 that $T_0 + e_1$ and $T_0 + e'_1$ are disjoint. So $e_1 + A \subseteq e_1 + T_0$ and $e'_1 + B \subseteq e'_1 + T_0$ are disjoint. Consequently,

$$0._b e_1 e_2 e_3 \ldots e_k + b^{-k} T_0 = b^{-1}(e_1 + A) \quad \text{and} \quad 0._b e'_1 e'_2 e'_2 \ldots e'_k + b^{-k} T_0 = b^{-1}(e'_1 + B)$$

are disjoint. Let

$$f_j = \begin{cases} d'_j & \text{when } d_j > d'_j, \\ d_j & \text{when } d_j \le d'_j \end{cases}$$

and $x := 0._b f_1 f_2 \dots$ Then

$$x + 0._b e_1 e_2 e_3 \dots e_k + b^{-k} T_0 = 0._b d_1 d_2 \dots d_k + b^{-k} T_0,$$
$$x + 0._b e'_1 e'_2 e'_2 \dots e'_k + b^{-k} T_0 = 0._b d'_1 d'_2 \dots d'_k + b^{-k} T_0.$$

Since the left hand sides are disjoint, so are the right hand sides. This completes the proof. □

Using Lemma 5.2, we can now calculate the dimension of T using the subtiles of T_k. This can of course also be done using self-similarity.

Corollary 6.4. *Let $n \ge 2$ be an integer, $b = -n + i$, and $D = \{0, 1, \dots, n^2\}$. Suppose $D^* \subseteq D$ satisfies the separation condition $|d - d'| \ge n + 1$, for $d \ne d'$ in D^*. Then $T = \{0._b d_1 d_2 \dots \mid d_k \in D^*\}$ has dimension $\frac{\log(|D^*|)}{\log(|b|)}$, where $|D^*|$ denotes the cardinality of D^*.*

Proof. The number of subtiles in $T_k = \{z \in T_0 \mid d_i \in D^*, \forall i \le k\}$ is $|D^*|^k$, hence $N_k(T) \le |D^*|^k$. Since the subtiles of T_k are disjoint by Corollary 6.3 and each subtile contains elements of T, we have $N_k(T) \ge |D^*|^k$. Hence, $N_k(T) = |D^*|^k$ and therefore

$$\frac{\log(N_k(T))}{k \log(|b|)} = \frac{\log(|D^*|)}{\log(|b|)}.$$

Consequently, it follows form Lemma 5.2 that the dimension of T is equal to $\frac{\log(|D^*|)}{\log(|b|)}$. □

7. DIMENSION AND DENSITY OF F_β IN F

We use the self-similarity construction of T to study $T \cap (x + T)$ and use these results to find conditions under which we can prove F_β is dense in F.

Note that $T \cap (x + T) \ne \emptyset$ if and only if $z = w + x$ for some $z, w \in T$. Therefore, $x = z - w$ and $F = T - T := \{z_1 - z_2 \mid z_1, z_2 \in T\}$. Because the digits of the elements of T are restricted to D^*, we get $F = \left\{\sum_{k=1}^{\infty} e_k b^{-k} \mid e_k \in \Delta\right\}$, where $\Delta := D^* - D^*$.

Since $T_{k+1} \subseteq T_k$ and $T = \bigcap_{k=0}^{\infty} T_k$ we have

$$T \cap (x + T) = \bigcap_{k=0}^{\infty} (T_k \cap (x + T_k)).$$

Unfortunately, this is not what we need to be able to use Lemma 5.2. More relevant for Lemma 5.2 would be

$$T \cap (x + T) = \bigcap_{k=0}^{\infty} (T_k \cap (\lfloor x \rfloor_k + T_k)), \tag{7.1}$$

where for $x = 0._b d_1 d_2 d_3 \dots$ we let $\lfloor x \rfloor_k$ denote the truncation to the first k places, i.e.,

$$\lfloor x \rfloor_k := 0._b d_1 d_2 d_3 \dots d_k.$$

Since that allow us to use Lemma 5.2. We will establish Eqn. (7.1) under some assumptions on D^*. To do so we begin by establishing some related facts. The first of these facts is a version of Corollary 6.3.

Lemma 7.1. *Let $n \geq 2$ be an integer, $b = -n + i$, $D = \{0, 1, \dots, n^2\}$, and $T_0 = \{0._b d_1 d_2 \dots \mid d_k \in D\}$. Suppose $D^* \subseteq D$, $d \leq \frac{n^2}{2}$ for all $d \in D^*$ and $\Delta = D^* - D^*$ satisfies the separation condition $|d - d'| \geq n + 1$ for all $d \neq d'$ in Δ. If $d_j, d_j' \in \Delta$ for all j and $d_j \neq d_j'$ for at least one j, then the subtiles $0.d_1 d_2 \dots d_k + b^{-k} T_0$ and $0._b d_1' d_2' \dots d_k' + b^{-k} T_0$ of $T_k = \{0._b d_1 d_2 \dots \mid d_j \in D^* \text{ when } j \leq k, d_j \in D \text{ when } j > k\}$ are disjoint.*

Proof. Let $d_j, d_j' \in \Delta$ for all j and $d_j \neq d_j'$ for at least one j, and let $d_{\max}$ be the largest element of D^*. Let $E^* := \Delta + d_{\max}$, $e_j = d_j + d_{\max}$, and $e_j' := d_j'$. Then E^*, e_j, e_j' satisfies the assumption on D^*, d_j, d_j' in Corollary 6.3, hence the sets $0._b e_1 e_2 \dots e_k + b^{-k} T_0$ and $0._b e_1' e_2' \dots e_k' + b^{-k} T_0$ are disjoint. Translating these sets by $0._b a_1 a_2 \dots a_k$, where $a_j := -d_{\max}$ for all j, gives the desired conclusion. □

The following result is a characterization of $T_k \cap (\lfloor x \rfloor_k + T_k)$ in terms of the digits of x and D^*.

Lemma 7.2. *Assume the hypotheses of Lemma 7.1. For $x_j \in \Delta$,*

$$T_k \cap (0._b x_1 x_2 \dots x_k + T_k) = \{0._b u_1 u_2 \dots u_k \mid u_j \in D^* \cap (x_j + D^*)\} + b^{-k} T_0, \quad (7.2)$$

where for sets A, B of complex numbers $A + B := \{a + b \mid a \in A, b \in B\}$.

Proof. Let $t \in T_k \cap (0._b x_1 x_2 \dots x_k + T_k)$. Since

$$t \in T_k = b^{-k}\left(b^{k-1} D^* + \dots + b D^* + D^* + T_0\right),$$

it follows that $t = b^{-k}\left(b^{k-1} u_1 + \dots + b u_{k-1} + u_k + y\right)$ for some $y \in T_0$ and $u_j \in D^*$. Using

$$\begin{aligned} t &\in 0._b x_1 x_2 \dots x_k + T_k \\ &= b^{-k}\left(b^{k-1} x_1 + \dots + b x_{k-1} + x_k + b^{k-1} D^* + \dots + b D^* + D^* + T_0\right) \end{aligned}$$

we have $t = b^{-k}\left(b^{k-1} x_1 + \dots + b x_{k-1} + x_k + b^{k-1} v_1 + \dots + b v_{k-1} + v_k + z\right)$ for some $z \in T_0$ and $v_j \in D^*$. Hence,

$$b^{k-1}(u_1 - v_1) + \dots + b(u_{k-1} - v_{k-1}) + (u_k - v_k) + y = b^{k-1} x_1 + \dots + b x_{k-1} + x_k + z.$$

Using $u_j - v_j, x_j \in \Delta$ when $j \leq k$ and Lemma 7.1 it follows that $y = z$ and $u_j - v_j = x_j$, when $j \leq k$. Hence, $u_j - x_j = v_j \in D^*$, when $j \leq k$. This establishes $\subseteq$ in Eqn. (7.2).

Conversely, suppose

$$z \in \{0.u_1 \dots u_k \mid u_j \in D^* \cap (x_j + D^*)\} + b^{-k} T_0.$$

Then $z = 0.u_1 \dots u_k d_{k+1} d_{k+2} \dots$ where $u_j \in D^* \cap (x_j + D^*)$ for some $0 \le j \le k$ and $d_{k+i} \in D$ for $i \ge 1$. Since $u_j \in D^*$, when $j \le k$, then by definition of T_k, $z \in T_k$. Since $u_j \in D^* + x_j$ when $j \le k$, the first k digits of z can be expressed as $u_j = x_j + d_j^{'}$ for $d_j^{'} \in D^*$. This shows that $z \in 0.x_1 \dots x_k + T_k$. Therefore, $z \in T_k \cap (0.x_1 \dots x_k + T_k)$. This shows $\supseteq$ in Eqn. (7.2).

Thus we have established equality in Eqn. (7.2). □

We now establish the version of the obvious

$$T_{k+1} \cap (x + T_{k+1}) \subseteq T_k \cap (x + T_k)$$

that is relevant for our application of Lemma 5.2.

Lemma 7.3. *Assume the hypotheses of Lemma 7.1. For $x \in F$ we have*

$$T_{k+1} \cap (\lfloor x \rfloor_{k+1} + T_{k+1}) \subseteq T_k \cap (\lfloor x \rfloor_k + T_k),$$

for all $k \ge 0$, where $x_j \in \Delta$.

Proof. For any $k \ge 0$, and any $x_j \in \Delta$ we have

$$\begin{aligned}
&T_{k+1} \cap (0._b x_1 x_2 \dots x_k x_{k+1} + T_{k+1}) \\
&= \{0._b u_1 u_2 \dots u_k u_{k+1} \mid u_j \in D^* \cap (x_j + D^*)\} + b^{-k-1} T_0 && (7.3) \\
&\subseteq \{0._b u_1 u_2 \dots u_k u_{k+1} \mid u_j \in D^* \cap (x_j + D^*), j \le k, u_{k+1} \in D\} + b^{-k-1} T_0 && (7.4) \\
&= \{0._b u_1 u_2 \dots u_k \mid u_j \in D^* \cap (x_j + D^*)\} + b^{-k} T_0 && (7.5) \\
&= T_k \cap (0._b x_1 x_2 \dots x_k + T_k) && (7.6)
\end{aligned}$$

The first and last equalities (7.3) and (7.6) are by Lemma 7.2. The subset inclusion (7.4) is obvious, since $D^* \subseteq D$. The middle equality (7.5) follows from

$$T_0 = b^{-1}(D + T_0) = \{0._b u \mid u \in D\} + b^{-1} T_0. \qquad \square$$

Using the lemmas we can now establish a formula for the dimension of $T \cap (x + T)$. In this formula we use the limit inferior, since the limit need not exist.

Theorem 7.4. *Let $n \geq 2$ be an integer, $b = -n + i$, $D = \left\{0, 1, \ldots, n^2\right\}$, and $T_0 = \{0._b d_1 d_2 \ldots \mid d_k \in D\}$. Suppose $D^* \subseteq D$, $d \leq \frac{n^2}{2}$ for all $d \in D^*$ and $\Delta = D^* - D^*$ satisfies the separation condition $|d - d'| \geq n + 1$ for all $d \neq d'$ in Δ. If $T = \{0._b d_1 d_2 \ldots \mid d_j \in D^*\}$, then*

$$\underline{\dim}\,(T \cap (x + T)) = \liminf_{k \to \infty} \frac{\log (M_k (x))}{k \log (|b|)}$$

where

$$M_k (x) := |D^* \cap (x_1 + D^*)| \cdot |D^* \cap (x_2 + D^*)| \cdot \ldots \cdot |D^* \cap (x_k + D^*)|,$$

when $x = 0._b x_1 x_2 \ldots$ with $x_j \in \Delta$.

Proof. It follows from the lemmas that $T \cap (x + T) \subseteq T_k \cap (\lfloor x \rfloor_k + T_k)$, that the subtiles of T_k contained in $T \cap (\lfloor x \rfloor_k + T_k)$ are disjoint, that every such subtile contains points from $T \cap (x + T)$, and the number of subtiles in $T_k \cap (\lfloor x \rfloor_k + T_k)$ equals $M_k (x)$. Hence the claim follows from Lemma 5.2. □

Our main result is a direct consequence of Theorem 7.4.

Corollary 7.5. *Under the hypotheses of Theorem 7.4, F_β is dense in F, for any $0 \leq \beta \leq 1$.*

Proof. Let $0 \leq \beta \leq 1$ be given and let $x \in F$. Write $x = 0._b x_1 x_2 \ldots$ with $x_j \in \Delta$. Fix $r > 0$ and pick k, such that $y = 0._b x_1 x_2 \ldots x_k y_{k+1} y_{k+2} \ldots$ implies $|x - y| < r$ for any $y_j \in \Delta$.

Let h_j be positive integers such that $h_j \leq j\beta < 1 + h_j$, then $h_j / j \to \beta$ as $j \to \infty$. Since $0 \leq \beta \leq 1$ we have $h_j \leq h_{j+1} \leq 1 + h_j$. Suppose $0 < \beta < 1$. For $j > k$ set

$$y_j = \begin{cases} d_{\max} & \text{if } h_j = h_{j-1}, \\ 0 & \text{if } h_j = 1 + h_{j-1}, \end{cases}$$

where $d_{\max}$ is the largest element of D^*. Notice $D^* \cap (y_j + D^*) = D^*$ if $y_j = 0$ and $D^* \cap (y_j + D^*) = \{d_{\max}\}$ if $y_j = d_{\max}$. Hence, if $b_k = \prod_{j=1}^{k} |D^* \cap (x_j + D^*)|$ and $\ell > k$, then

$$M_\ell (x) = b_k \prod_{j=k+1}^{\ell} |D^*|^{h_j - h_{j-1}} = b_k \, |D^*|^{h_\ell - h_k}.$$

So

$$\frac{\log (M_\ell (x))}{\ell \log (|b|)} = \frac{\log (b_k)}{\ell \log (|b|)} + \frac{h_\ell \log (|D^*|)}{\ell \log (|b|)} - \frac{h_k \log (|D^*|)}{\ell \log (|b|)}.$$

Thus $\frac{\log(M_\ell(x))}{\ell\log(|b|)}$ converges to

$$\beta\frac{\log(|D^*|)}{\log(|b|)} = \beta\dim(T)$$

as $\ell \to \infty$, using $\frac{h_\ell}{\ell} \to \beta$ by the choice of h_ℓ and that $\dim(T) = \frac{\log(|D^*|)}{\log(|b|)}$ by Corollary 6.4. An application of Theorem 7.4 completes the proof. The modifications needed to deal with the cases $\beta = 0$ and $\beta = 1$ are left for the reader. □

8. PYTHON CODE FOR REMARK 6.2

```
import numpy as np

def r(n):
    m = n ** 2
    return m / (2 * (m + 1 - np.sqrt(m + 1)))

def xi(n, d, tau):
    return d ** 2 + (tau ** 2 - 2 * d * n * tau) / (n ** 2 + 1)

def min_xi(n, d):
    a = 0 - n ** 2
    b = xi(n, d, a)
    while a <= n ** 2:
        bb = xi(n, d, a)
        if bb < b:
            b = bb
        a += 1
    return b

for n in range(2, 101):
    d = -1
    small = -1
    while small < 0:
        d += 1
        small = min_xi(n, d) - 4 * (r(n) ** 2)
    print(n, d, small)
```

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Steen Pedersen (corresponding author)
steen.pedersen@wright.edu

Wright State University
Department of Mathematics
3640 Col Glenn Hwy, Dayton, OH 45435, USA

Vincent T. Shaw
shaw.135@wrigth.edu

Wright State University
Department of Mathematics
3640 Col Glenn Hwy, Dayton, OH 45435, USA

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