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Modeling and control of an unstable system using probabilistic fuzzy inference system

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A new type Fuzzy Inference System is proposed, a Probabilistic Fuzzy Inference system which model and minimizes the effects of statistical uncertainties. The blend of two different concepts, degree of truth and probability of truth in a unique framework leads to this new concept. This combination is carried out both in Fuzzy sets and Fuzzy rules, which gives rise to Probabilistic Fuzzy Sets and Probabilistic Fuzzy Rules. Introducing these probabilistic elements, a distinctive probabilistic fuzzy inference system is developed and this involves fuzzification, inference and output processing. This integrated approach accounts for all of the uncertainty like rule uncertainties and measurement uncertainties present in the systems and has led to the design which performs optimally after training. In this paper a Probabilistic Fuzzy Inference System is applied for modeling and control of a highly nonlinear, unstable system and also proved its effectiveness.

Key words: inverted pendulum and cart system, probabilistic fuzzy set, probabilistic fuzzy relation, probabilistic fuzzy inference system, probabilistic fuzzy logic controller.

1. Introduction

Basically the methods used to design controllers for complex and highly nonlinear systems fall into two categories. A nonlinear system is linearized and then applied classical linear control laws in the first category, but the performance of the control system might be deteriorated because of the assumptions made in the approximation of the nonlinear system. The methods of second category implement directly nonlinear controllers based on nonlinear systems, which preserve the nonlinear characteristics of the system. But to design such a controller accurate mathematical models of the process are required, which are very difficult to obtain because the real world systems are highly complex with unknown dynamics and also the classical control methods could not perform well. Hence for modeling nonlinear functions Fuzzy Systems can be considered as

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a good tool. In fuzzy theory fuzzy modeling is a key issue because the linguistic If -Then Fuzzy rules carry much information about the system behavior [25].

The Fuzzy Logic Control (FLC), most useful method to collect human knowledge and proficiency, has been the accent for the study of various plants which are poorly modeled or the model uncertainty in the dynamics is not known. Fuzzy controls have achieved worldwide success in large commercial products and applications due to two advantages: little dependence on the process model and adequate control performance developed by using human control knowledge and experience [11]. In a complex and uncertain described system a fuzzy model has excellent capability and specifically suitable for modeling the nonlinear systems using a set of local linear models corresponding to different operating points which are combined by a fuzzy inference mechanism which can preserve the nonlinear characteristics [15, 16]. However, the nonlinearities existing in the dynamic system are not known in advance, then the performance is degraded quickly to unacceptable levels. When the number of the fuzzy local models is greater, then adaptive on line control provides an alternative approach that is global stability analysis with a minimum modeling effort and has promising potential for the task of tackling the presence of unknown parameters with better performance [18].

To control completely an inverted pendulum and cart system (IPC), two different control processes of swinging up the pendulum and positioning the pendulum and cart are considered. A hybrid fuzzy controller that consists of a fuzzy swing-up controller and a fuzzy controller using the parallel distributed pole assignment scheme is presented in [17]. Three dimensional FLC based on 3-D fuzzy set and the third dimension of the spatial information had been developed for spatially distributed dynamic systems, which controls the overall behavior of the space domain instead of accurately the variable of interest at each sensing location like conventional FLC [21]. Different modeling methods based on fuzzy clustering techniques for IPC are presented and compared in [14], but they do not consider the modeling error due to measurement noises. A robust adaptive control architecture in which two control rules are applied first to swing up the pendulum from pendant position to an upward position by driving the cart back and forth and then stabilized the inverted pendulum by applying the adaptive control law. This control architecture uses the fuzzy system to adaptively model the plant nonlinearities [4]. An integrative optimization approach for the design of robust quadratic-optimal parallel distributed compensation controllers for Takagi-Sugeno (TS) fuzzy model based control system involves both parametric uncertainties and approximation error by minimizing directly the quadratic integral performance index. This integrative method complementarily fuses the orthogonal function approach, the hybrid Taguchi-genetic algorithm and the linear matrix inequalities technique for ensuring that the closed loop TS-fuzzy model based control systems with both elemental parametric uncertainties and norm bounded approximation error can be stabilized [19].

In all the above cases of conventional Fuzzy Inference Systems (FISs) are employed. When real-world problems are considered, uncertainty cannot be ignored. At the experimental level, uncertainty is a unified companion of any measurement, resulting from a combination of resolution limits of measuring devices and unavoidable measurement errors. In many real world systems the uncertainty is a result of any information deficiency, which means the information may be contradictory, not completely reliable, fragmentary, deficient, incomplete, or vague in some other way. These various deficiencies might result in various types of uncertainty. Out of these, only two types of uncertainties are accepted: linguistic and random. The linguistic uncertainties are associated with words whereas the random uncertainties are associated with unpredictability. Fuzzy logic is capable to handle only linguistic uncertainties. The FIS, as known well, comprises of rules. The knowledge which is utilized to create rules is uncertain. These uncertainty results in rules whose consequents and/or antecedents are uncertain, which transforms into uncertain membership functions (MFs) for consequents and/or antecedents. The conventional FIS, whose MFs are conventional fuzzy sets, is not able to model and minimize the effects of rule uncertainties [24].

Therefore to overcome the shortcomings of conventional FIS, a new concept of Probabilistic Fuzzy Inference System (PFIS) is introduced by integrating fuzzy theory and probability theory, which has been discussed in [1-3,5,6,9,10], but these works present only the relationship of randomness and fuzziness, and are not applied to process control engineering applications. In this PFIS, the MFs of the antecedent and consequent are Probabilistic Fuzzy Sets (PFS), whose membership grades for each element of this set is a fuzzy number in (0,1), hence useful for incorporating uncertainties [13]. A probability density function (PDF) in probability theory, personifies total information about random uncertainties. In most real world applications, it is difficult to define PDF since an infinite number of moments are required to characterize it completely. When the PDF is Gaussian, as known well, its first two moments, i.e., mean and variance are sufficient to completely specify it. But infinite number of moments is required for most PDFs, which is impossible in practice. Instead, one can compute as many moments as necessary to extract as much information from the data for establishing probabilistic modeling of random uncertainty [8]. Using just the first order moment is not useful since random uncertainty requires knowledge of dispersion about the mean, which is indicated by the variance. The output of conventional FISs may be viewed as similar to the mean of a PDF. Hence FIS requires some dispersion measure to capture more information about random uncertainties.

A PFIS provides this dispersion measure and appears to be very essential to design systems that incorporate linguistic and/or numerical uncertainties, which transform into rule uncertainties, like variance is to the mean. A PFS was also introduced by L.A. Zadeh in 1975 as an extended version of the conventional fuzzy set (henceforth called as first order fuzzy set), whose grades of membership themselves fuzzy, so it could be called a "fuzzy-fuzzy set" (second order fuzzy set). It provides measures of dispersion, which can model and minimize the effects of random uncertainties. This is very useful in situations where it is difficult to define the shape of MFs or some of its parameters and hence it helps to incorporate uncertainties [12].

The systematic framework for the proposed Probabilistic Fuzzy Inference System is presented in this paper for process control applications. Like FIS, the PFIS also includes fuzzifier, rule base, fuzzy inference engine and defuzzification. The fuzzifier maps the crisp input into a fuzzy set, which is a Probabilistic Fuzzy Set (PFS). A Probabilistic Fuzzy Relation, which is a fuzzy relation of higher order, has been considered as one way of emerging fuzziness of a relation, which means increased capacity to handle indefinite information in a logical truthful manner [12, 23]. The PFIS is characterized by the same IF–THEN rules but its antecedents and/or consequents are PFSs, hence it is able to capture information with random uncertainties. Defuzzification process of PFIS is difficult since MFs of PFS is obtained by probabilistic method, but with the proposed unique defuzzification method, the PFIS can be viewed as a random version of FIS for the sake of understanding and implementation.

The organization of the paper is as follows. Section 2 discusses the design concepts of the Probabilistic Fuzzy Inference System. In Section 3 the mathematical model of an inverted pendulum and cart system, a highly nonlinear and an unstable system, which is considered for simulation study is presented. Section 4 presents the simulation results of the application of the Probabilistic Fuzzy Inference System for modeling and balancing control of an IPC system. In section 5 the conclusions drawn from the simulation studies are presented.

2. Design of Probabilistic Fuzzy Inference System

A PFIS like conventional FIS has fuzzification, fuzzy rules, inference engine and defuzzification as presented in Fig. 1. But here fuzzification and defuzzification are implemented on PFSs.

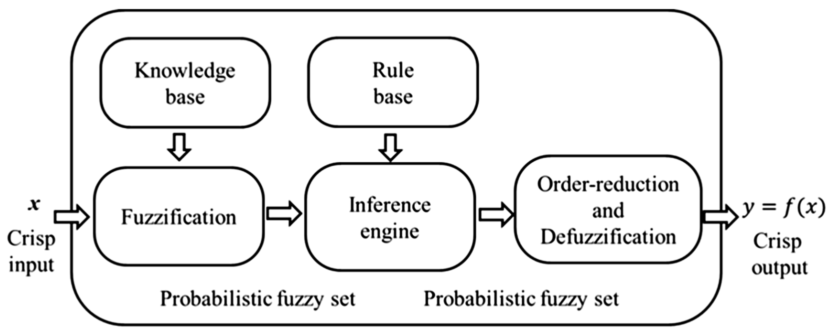

Figure 1: Probabilistic Fuzzy Inference System – structure

2.1. Fuzzification in PFIS

A conventional fuzzy set is a two dimensional MF, in which degree of membership is a crisp value. But PFS is a three dimensional MF, in which degree of membership is a random variable and should be represented with its continuous or discrete PDF. A PFS is denoted as $\widetilde{S}$ and can be expressed mathematically as

$$\widetilde{S} = \{((x,u),\mu_{\widetilde{S}}(x,u)) \quad \forall x \in X, \forall u \in I_x \subseteq [0,1]\} \tag{1}$$

where: x – primary variable ($x \in X$), I_x – primary membership (each value of x has a band of MF values), u – secondary variable ($u \in I_x \subseteq [0,1]$), $\mu_{\widetilde{S}}(x,u)$ – secondary membership, $0 \leqslant \mu_{\widetilde{S}}(x,u) \leqslant 1$ which is presented in Fig. 2 with primary and secondary MFs individually.

The domain of secondary MF is referred as primary MF of x. Hence $\widetilde{S}$ is also expressed as

$$\widetilde{S} = \int_{x \in X} \int_{u \in I_x} \frac{\mu_{\widetilde{S}}(x,u)}{(x,u)}, \quad I_x \subseteq [0,1] \tag{2}$$

where $\iint$ represents union over all admissible x and u. For discrete universes of discourse, $\int$ is replaced by $\sum$.

Gaussian primary membership function with uncertain mean and fixed standard deviation is considered for the design of PFIS, which is presented in Fig. 3 and takes on values as

$$\mu_p^r(x_p) = exp\left[-\frac{1}{2}\left(\frac{x_p - m_p^r}{\sigma_p^r}\right)^2\right] \tag{3}$$

where m_p^r is the mean of p^{th} antecedent probabilistic fuzzy set of r^{th} rule and $m_p^r \in [m_{p_1}^r, m_{p_2}^r]$, σ_p^r is the standard deviation of p^{th} antecedent probabilistic fuzzy set of r^{th} rule, $p = 1,2,\ldots,k$, k is a number of antecedents; $r = 1,2,\ldots,L$ and L is a number of rules.

Uncertainty in the primary membership grades of a Probabilistic Fuzzy MF consists of a bounded region referred as the foot print of uncertainty (FOU) of Probabilistic Fuzzy MF which is the union of all primary membership grades. The FOU may also be described in terms of upper membership function (UMF) and lower membership function (LMF). They are the two conventional MFs that bound the FOU. The UMF is a subset that has the maximum membership grade of FOU and the LMF is a subset has the minimum membership grade of FOU. In Fig. 3 the UMF is represented by the thick solid line and the LMF by the thick dashed line. The footprint of uncertainty is denoted by the shaded region between them. An over bar and under bar are used to represent UMF and LMF. For example, the LMF and UMF of a PFS $\mu_{\widetilde{B}_p^r}(x_p)$ are $\underline{\mu}_{\widetilde{B}_p^r}(x_p)$ and $\overline{\mu}_{\widetilde{B}_p^r}(x_p)$ respectively, so that

$$\mu_{\widetilde{B}_p^r}(x_p) = \int_{b \in \left[\underline{\mu}_{\widetilde{B}_p^r}(x_p), \overline{\mu}_{\widetilde{B}_p^r}(x_p)\right]} \frac{1}{b^r}. \tag{4}$$

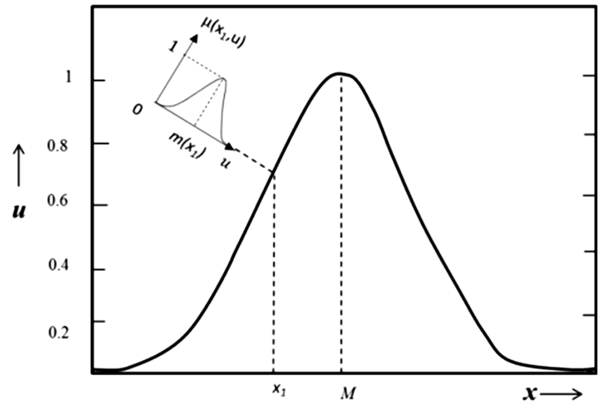

Figure 2: Probabilistic Fuzzy Set (Primary and Secondary MFs shown individually)

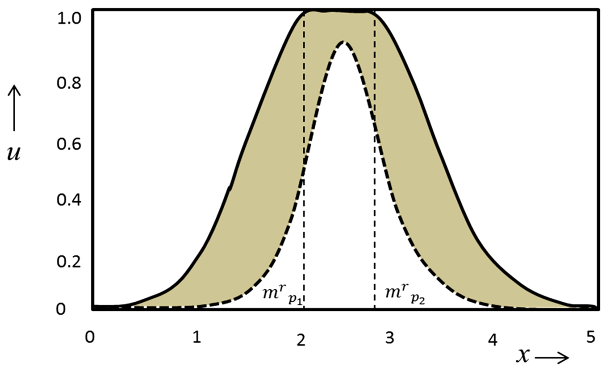

Figure 3: PFS - Primary MFs are Gaussian MFs with Uncertain Mean $(m^r_{p_1},\ m^r_{p_2})$

2.2. Inference in PFIS

The inference engine in PFIS maps the input fuzzy set to output fuzzy set similar to conventional FIS. But the fuzzy sets are PFSs. The inference is based on the set theoretic operations union, intersection and complement on PFSs developed using Zadeh's extension principle [25]. Consider a PFIS having k inputs, $x_1 \in X_1$, $x_2 \in X_2$, $\ldots$, $x_k \in X_k$ and one output $y \in Y$ and has L number of rules. The r^{th} rule is given by

$$R^r \ : \ \text{IF } x_1 \text{ is } \widetilde{S}_1^r \text{ and } x_2 \text{ is } \widetilde{S}_2^r \text{ and } \ldots \text{ and } x_k \text{ is } \widetilde{S}_k^r, \text{ THEN } y \text{ is } \widetilde{C}^r \tag{5}$$

where $\widetilde{S}_p^r$ denotes antecedents and $\widetilde{C}^r$ is the consequent of r^{th} rule.

The equation (5) gives a PFR between the input space $X_1 \times X_2 \times \cdots \times X_k$ and the output space Y of FIS. The MF of this relation is represented as $\mu_{\widetilde{S}_1^r \times \cdots \times \widetilde{S}_k^r \to \widetilde{C}^r}(\mathbf{x}, y)$, where $\widetilde{S}_1^r \times \cdots \times \widetilde{S}_k^r$ represents the Cartesian product of $\widetilde{S}_1^r, \widetilde{S}_2^r, \ldots, \widetilde{S}_k^r$ and $\mathbf{x} =$

$\{x_1, x_2, \ldots, x_k\}$. The extended sup-star composition [12] is used to find the composition of rule R^r and the fuzzy set $\widetilde{\mathbf{X}}'$ to which $\mathbf{x}'$ belongs when an input $\mathbf{x}'$ is applied,

$$\mu_{\widetilde{\mathbf{X}}' \circ \widetilde{S}_1^r \times \cdots \times \widetilde{S}_k^r \to \widetilde{C}^r}(y) = \sqcup_{\mathbf{x} \in \widetilde{\mathbf{X}}'} \left[\mu_{\widetilde{\mathbf{X}}'}(x) \sqcap \mu_{\widetilde{S}_1^r \times \cdots \times \widetilde{S}_k^r \to \widetilde{C}^r}(\mathbf{x}, y) \right] \tag{6}$$

where $\widetilde{\mathbf{X}}'_p$ $(p = 1, 2, \ldots, k)$ are the fuzzy sets describing the inputs.

In singleton fuzzification, the fuzzy set $\widetilde{\mathbf{X}}'$ has a membership grade 1 at $\mathbf{x} = \mathbf{x}'$ and has zero membership grades for all other inputs; hence (6) reduces to

$$\mu_{\widetilde{\mathbf{X}}' \circ \widetilde{S}_1^r \times \cdots \times \widetilde{S}_k^r \to \widetilde{C}^r}(y) = \mu_{\widetilde{S}_1^r \times \cdots \times \widetilde{S}_k^r \to \widetilde{C}^r}(\mathbf{x}', y). \tag{7}$$

We represent $\widetilde{\mathbf{X}}' \circ \widetilde{S}_1^r \times \cdots \times \widetilde{S}_k^r \to \widetilde{C}^r$ as $\widetilde{D}^r$, the output set corresponding to the r^{th} rule and calculated using the meet operation with product or minimum t-norm as given by

$$\mu_{D^r}(y) = \mu_{\widetilde{S}_1^r \times \cdots \times \widetilde{S}_k^r}(\mathbf{x}') \sqcap \mu_{\widetilde{C}^r}(y). \tag{8}$$

Then the MF for a Cartesian product of sets is found out by computing the meet between the MFs of individual sets and hence (8) is given by

$$\begin{aligned} \mu_{D^r}(y) &= \mu_{\widetilde{S}_1^r}(x_1) \sqcap \mu_{\widetilde{S}_2^r}(x_2) \sqcap \cdots \sqcap \mu_{\widetilde{S}_k^r}(x_k) \sqcap \mu_{C^r}(y) = \\ &\quad \mu_{C^r}(y) \sqcap \left[\sqcap_{j=1}^k \mu_{\widetilde{S}_j^r}(x_j) \right]. \end{aligned} \tag{9}$$

In PFIS with singleton fuzzification and meet under minimum or product t-norm, the result of the input and antecedent operations that are contained in the firing set, $\sqcap_{p=1}^k \mu_{\widetilde{H}_p^r}(x'_p) \equiv H^r(\mathbf{x}')$, is a conventional fuzzy set, that is $H^r(\mathbf{x}') = \left[\underline{h}^r(\mathbf{x}'), \bar{h}^r(\mathbf{x}')\right] \equiv \left[\underline{h}^r, \bar{h}^r\right]$, where $\underline{h}^r, \bar{h}^r$ are the left most and right most points which are given by

$$\underline{h}^r = \underline{\mu}_{\widetilde{S}_1^r}(x_1) \star \cdots \star \underline{\mu}_{\widetilde{S}_k^r}(x_k) \quad \text{and} \quad \bar{h}^r = \bar{\mu}_{\widetilde{S}_1^r}(x_1) \star \cdots \star \bar{\mu}_{\widetilde{S}_k^r}(x_k) \tag{10}$$

where x_p $(p = 1, 2, \ldots, k)$ represents the location of the singleton. Fig. 4 presents this result of input and antecedent operations with singleton fuzzification for minimum and product inference with the number of antecedent value for $k = 2$. The firing strength is an conventional fuzzy set, $[\underline{h}^r, \bar{h}^r]$, where $\underline{h}^r = \underline{h}_1^r \star \underline{h}_2^r$ and $\bar{h}^r = \bar{h}_1^r \star \bar{h}_2^r$.

2.3. Order reduction and defuzzification

Order-reduction is an "extended version" of defuzzification methods of FIS, which is called so, since this operation takes the output probabilistic fuzzy sets (higher order) and results in an conventional fuzzy set (lower order), which is called the "order-reduced set". This set may then be defuzzified to obtain a single crisp number. The order reduced set may be more important in many situations, since it transmits a measure of uncertainties that have fluttered through the PFIS.

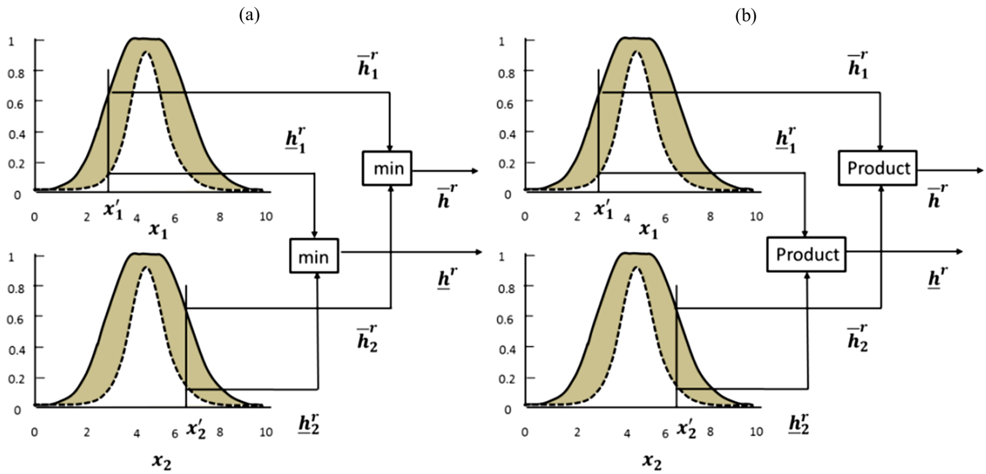

Figure 4: Input and antecedent operations: (a) minimum inference, (b) product inference

The fired output consequent set corresponding to each rule is a PFS and for r^{th} rule which is given by

$$\mu_{\widetilde{D}^r}(y) = \int_{d^r \in \left[\underline{h}^r \star \underline{\mu}_{\widetilde{C}^r}(y),\ \bar{h}^r \star \bar{\mu}_{\widetilde{C}^r}(y)\right]} \frac{1}{d^r} \tag{11}$$

where d^r is the element of output consequent set $\mu_{\widetilde{D}^r}(y)$ corresponding to r^{th} rule and $\underline{\mu}_{\widetilde{C}^r}(y)$, $\bar{\mu}_{\widetilde{C}^r}(y)$ are the lower and upper membership grades of $\mu_{\widetilde{C}^r}(y)$ respectively. All such output sets are combined by the center of sets method of order reduction and represented by

$$Y_{cos}\left(Y^1,\ldots,Y^L,H^1,\ldots,H^L\right) = [y_l, y_r] = \int_{y^1} \cdots \int_{y^L} \int_{h^1} \cdots \int_{h^L} \frac{1}{\frac{\sum_{j=1}^{L} h^j y^j}{\sum_{j=1}^{L} h^j}} \tag{12}$$

where Y is the PFS determined by two end points y_l and y_r, $j = 1,2,\ldots,L$, L is the number of rules, $h^j \in H^j - [\underline{h}^j, \bar{h}^j]$, $y^j \in Y^j = [y_l^j, y_r^j]$ and Y^j – centroid of the consequent PFS. For any value of $y \in Y_{cos}$, y is given by

$$y = \frac{\sum_{j=1}^{L} h^j y^j}{\sum_{j=1}^{L} h^j}. \tag{13}$$

The maximum value of y is y_r and the minimum value of y is y_l, they are given as

$$y_r = \frac{\sum_{j=1}^{L} h_r^j y_r^j}{\sum_{j=1}^{L} h_r^j} \quad \text{and} \quad y_l = \frac{\sum_{j=1}^{L} h_l^j y_l^j}{\sum_{j=1}^{L} h_l^j}. \tag{14}$$

The crisp output from PFIS is obtained by defuzzifying the order – reduced set, i.e., finding the centroid of the order – reduced set. The computation of centroid is equivalent to the computation of a weighted average of the outputs of all conventional FISs embedded in the PFIS, the centroid is the average of y_r, and y_l. Hence the defuzzified output of a PFIS is

$$f(\mathbf{x}) = \frac{y_l + y_r}{2}. \tag{15}$$

The entire chain of computations is summarized as shown in Fig. 5. The firing intervals, which are depend explicitly on the input **x** are computed for all rules. The computations of the centroids are performed for each of the M consequent PFSs and are then stored in memory. Then on order reduction the firing intervals and pre-computed consequent centroids are combined to perform the actual calculations.

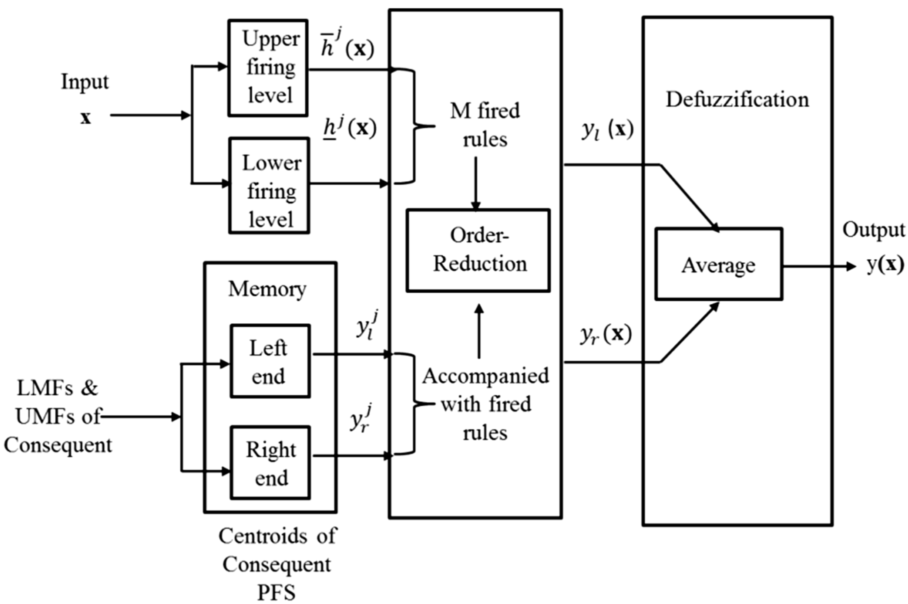

Figure 5: Order-reduction and defuzzification in PFIS

2.4. Systematic design procedure for PFIS

For the given N samples of input - output training pair $(\mathbf{x}, d)$, $\mathbf{x} \in R^k$ the vector input and, $d \in R$ the scalar output, we are interested to design a PFIS with output (15) to minimize the error function as $e = \frac{1}{2}[f(\mathbf{x}) - d]^2$.

i. Set the initial values for all the antecedent and consequent MFs parameters, the epoch training counter $c = 0$ and the training data sample counter $t = 1$.

ii. Apply $k \times 1$ input $\mathbf{x}^{(t)}$ to the PFIS and compute the total firing degree for each rule, i.e., compute $\bar{h}^p$ and $\underline{h}^p$ for $p = 1, 2, \ldots, k$, by using (10), where k is the number of inputs.

iii. Compute y_l and y_r as given by (14).

iv. Compute $f(\mathbf{x}^{(t)}) = (y_l(\mathbf{x}^{(t)}) + y_r(\mathbf{x}^{(t)}))/2$, the defuzzified output.

v. Test each component of $\mathbf{x}^{(t)}$ to find the active branches in $\underline{\mu}_{\tilde{H}_q^r}(x_q)$ and $\bar{\mu}_{\tilde{H}_q^r}(x_q)$, $(q = 1, 2, \ldots, k)$, and represent the active branches as explicit functions of their associated parameters and adjust the parameters of the active branches of the antecedent's MFs and the parameters associated with the consequent using a steepest descent algorithm or any other optimization method for the error function.

vi. Set $t = t + 1$. If $t = N + 1$, go to step vii; otherwise, go to step ii.

vii. Set $c = c + 1$. If $c =$ number of epochs (say E), stop; otherwise set $t = 1$ and repeat from step ii.

3. Mathematical model of an inverted pendulum and cart system

The highly nonlinear and unstable system chosen for the simulation study is the inverted pendulum and cart system which is given in Fig. 6. In this system, the inverted pendulum is mounted on a moving cart; it is a highly nonlinear and open loop unstable system without controller which means the pendulum will just fall over if the cart is not displaced to balance it. The task of the control system is to stabilize the inverted pendulum by driving the cart to which the pendulum is attached, by the force applied to it. Here the pendulum is forced to move in the vertical plane and the force F (in Newtons), which is applied to move the cart horizontally is the control input and the angular position of the pendulum, $Theta$ (in radians) and the horizontal position of the cart, x (in meters) are the outputs. The physical parameters of the considered IPC system are presented in Tab. 1.

The equations governing the IPC system that is the mathematical model is as follows [14, 18]

$$(M+m)\ddot{x} + b\dot{x} + ml\ddot{\Theta}\cos\Theta - ml\dot{\Theta}^2\sin\Theta = F \tag{16}$$

$$\left(I + ml^2\right)\ddot{\Theta} + mgl\sin\Theta = -ml\ddot{x}\cos\Theta. \tag{17}$$

4. Simulation studies

In this section the effectiveness of the proposed PFIS is verified by simulating two cases. A Probabilistic Fuzzy Inference System has been designed for modeling and the

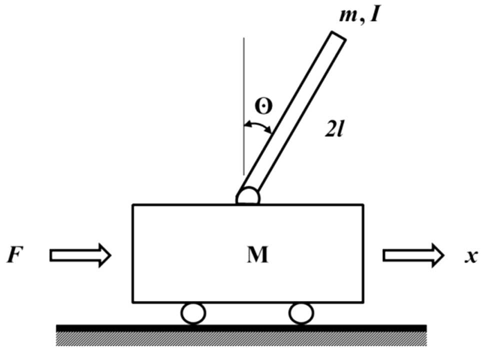

Figure 6: Inverted pendulum and cart system

Table 11: The physical parameters of an IPC system

S.No.	Process parameters	Nomenclature	Normal operating condition
1	Mass of the cart	M	0.5 Kg
2	Mass of the pendulum	m	0.2 Kg
3	Coefficient of friction for cart	b	0.1 N/m/sec
4	Length to pendulum center of mass	l	0.3 m
5	Mass moment of inertia of the pendulum	I	0.006 Kgm^2

control of an inverted pendulum and cart system and its performance is compared with the conventional Fuzzy Inference Systems.

4.1. Modeling of an IPC system using PFIS

An IPC system is simulated using the nonlinear first principle model presented in section 3 in all simulation runs. A collection of N input-output data training pairs $(x^{(1)}\ y^{(1)})$, $(x^{(2)}\ y^{(2)})$, …, $(x^{(N)}\ y^{(N)})$, where $\mathbf{x}$ is the vector input and y is the scalar output. Simulation is based on 500 samples, the first 250 data samples are for training and the remaining is for testing. Four antecedents are used for estimating the model of an IPC system; the force applied to the cart $F(k)$, $F(k-1)$, cart position $x(k-1)$ and pendulum angle $\Theta(k-1)$. Three fuzzy sets are used for each antecedent, so the number of rules is $3^4 = 81$.

In order to prove the performance of the proposed probabilistic fuzzy modeling, first conventional fuzzy model is obtained [7, 20, and 22]. For the design of a fuzzy model of an IPC system Gaussian MFs, product implication and t-norm, and weighted average are assumed.

To design the probabilistic fuzzy modeling for our process, Gaussian primary MFs with constant standard deviation and uncertain mean for the antecedents, product implication and t-norm, and centroid defuzzification are assumed. The initial locations of the antecedents MFs were based on the mean m_t and standard deviation σ_t of the first 250 samples. The Root Mean Square Error (RMSE) for both models (FIS and PFIS) is computed as defined by

$$RMSE = \left(\frac{1}{N} \sum_{k=1}^{N} (y(k) - \hat{y}(k))^2 \right)^{\frac{1}{2}} \tag{18}$$

where $N = 250$ is the number of samples, $y(k)$ is the desired output and $\hat{y}(k)$ is the estimated output of a modeled system. Data is corrupted by random noise (Gaussian noise) with various SNR and in each case RMSE is calculated which is presented in Tab. 2. The simulation comparison of a FIS and PFIS for the system is shown in Fig. 7. From the simulation results it can be concluded that the RMSE values of PFIS have been found to be considerably less than that of FIS.

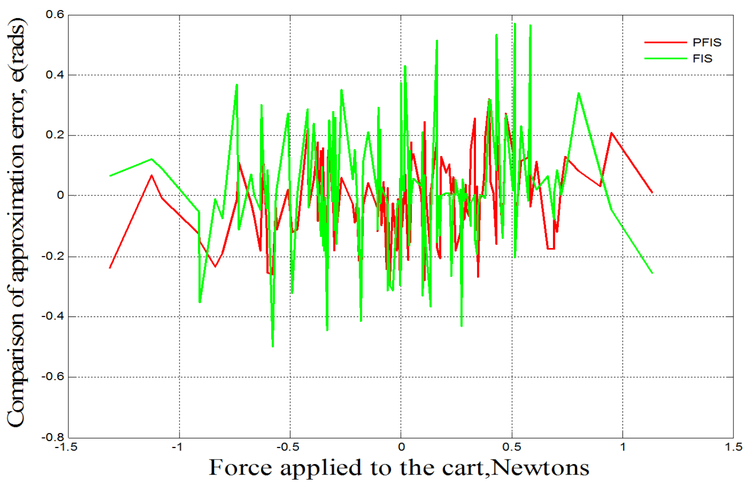

Figure 7: Comparison of approximation error $e = y - \hat{y}$

4.2. Design of Probabilistic Fuzzy Logic Controller for an IPC system

The objective of the control system is to keep the pendulum in the upright position by means of force, F for any initial position of the pendulum without concern to the position and velocity of the cart. At the stable equilibrium point of the system, any small perturbation to the pendulum position from its vertical equilibrium position will cause control to return the pendulum to the equilibrium position. To prove the performance of

Table 12: TRMSE values of FIS and PFIS for noise with various SNR values

Noise with various Signal to noise ratio (SNR) dB	RMSE	
	FIS	PFIS
-8.0000	0.9412	0.6828
2.3975	0.2042	0.1660
3.4888	0.2152	0.1532
3.8000	0.2213	0.1618
4.9501	0.1919	0.1435
7.1686	0.1735	0.1182
11.9399	0.1311	0.0963

the proposed Probabilistic Fuzzy Logic Controller over other controllers, PID and Fuzzy Logic Controllers are designed for an IPC system. The Internal Model Control (IMC) based PID tuning procedure is used for the design of PID controller. The parameters of PID controller are the controller gain K_c, integral time T_i and derivative time T_d and they have been determined using the internal model control (IMC) tuning rules. The conventional PI Fuzzy Logic Controller is designed using Gaussian membership functions for both, antecedents and the consequents. Seven MFs are used for each of the two inputs, error (e) and change in error (Δe), and also for the output, hence forty nine rules. The fuzzy inference system (FIS) has been used with minimum implication, maximum aggregation and centroid defuzzification [11, 17 and 21].

The Probabilistic Fuzzy Logic Controller is similar to the conventional Fuzzy Logic Controller; the major structural difference is that the defuzzifier block of conventional fuzzy controller is actually replaced by the output processing block, which consists of an order-reduction followed by defuzzification. In the simulation, the inverse dynamic controller of an IPC system is modeled by a PFIS with the th fuzzy rule as follows

$$\text{Rule } r\ :\ \text{ IF } x_1 \text{ is } \widetilde{S}_1^r \text{ and } x_2 \text{ is } \widetilde{S}_2^r \text{ and } \ldots \text{ and } x_k \text{ is } \widetilde{S}_k^r,\ \text{ THEN } y \text{ is } \widetilde{C}^r.$$

Gaussian primary membership functions with constant standard deviation and uncertain mean for antecedent and consequent MFs, minimum implication, maximum aggregation and centroid defuzzification are assumed. Seven MFs are chosen for each of the two antecedents, error (e) and change in error (Δe) and also for the controller output, hence forty nine rules.

Fine tuning of the parameters of PID controller is more complicated for the unstable process. Tuning of the FLC parameters is simple but it is not cable of dealing uncertainty effectively. In spite of complications in implementation of PFLC it produces better corrective action for random uncertainties. The tool used for simulation of both modeling and controller is Matlab (7.14.0.739) R2012a.

4.2.1. Servo performance

The pendulum in an IPC is bumped with an impulse force, F. The dynamic responses of the pendulum angle for a balancing problem of an inverted pendulum with (i) PID controller (ii) conventional Fuzzy Logic Controller (FLC) and (iii) proposed Probabilistic Fuzzy Logic Controller (PFLC) have shown in Fig. 8(a) and 8(c). It can be inferred from the response that all the controllers are able to stabilize the pendulum. But the PFLC balances the pendulum a little bit faster with small overshoot than the other controllers. The corresponding variations in the controller outputs are presented in Fig. 8(b) and Fig. 8(d). The ISE, IAE and ITAE values of PID, FLC and PFLC are reported in Tab. 3, which are defined mathematically as follows

$$\text{Integral of the square error, ISE} = \int_0^\infty e^2(t)dt \tag{19}$$

$$\text{Integral of the absolute value of the error, IAE} = \int_0^\infty |e(t)|dt \tag{20}$$

$$\text{Integral of time-weighted absolute error, ITAE} = \int_0^\infty t|e(t)|dt \tag{21}$$

where $e(t)$ is the difference between the set point and the process variable, which represents the deviation of the response from the desired set point.

It can be inferred that from Tab. 3 that the ISE, IAE and ITAE values of PFLC are found to be considerably less than PID and FLC. However the performance of the proposed controller is found to be better than conventional PID and FLC, as there is less overshoot and balances the pendulum a little bit faster

Table 13: ISE, IAE and ITAE values of PID, FLC and PFLC for set point tracking

Controllers	ISE	IAE	ITAE
PID	7.999e-11	6.718e-6	2.756e-6
FLC	2.915e-13	7.483e-7	8.115e-7
PFLC	9.225e-14	4.269e-7	4.177e-7

4.2.2. Servo-regulatory performance

The disturbance rejection capability that is the robustness of the proposed controller has been demonstrated by simulation studies and also compared with PID and FLC. A step change of small magnitude is given in the force applied to the cart and this simulation results are shown in Fig. 9(a) and Fig. 9(c). The corresponding controller outputs

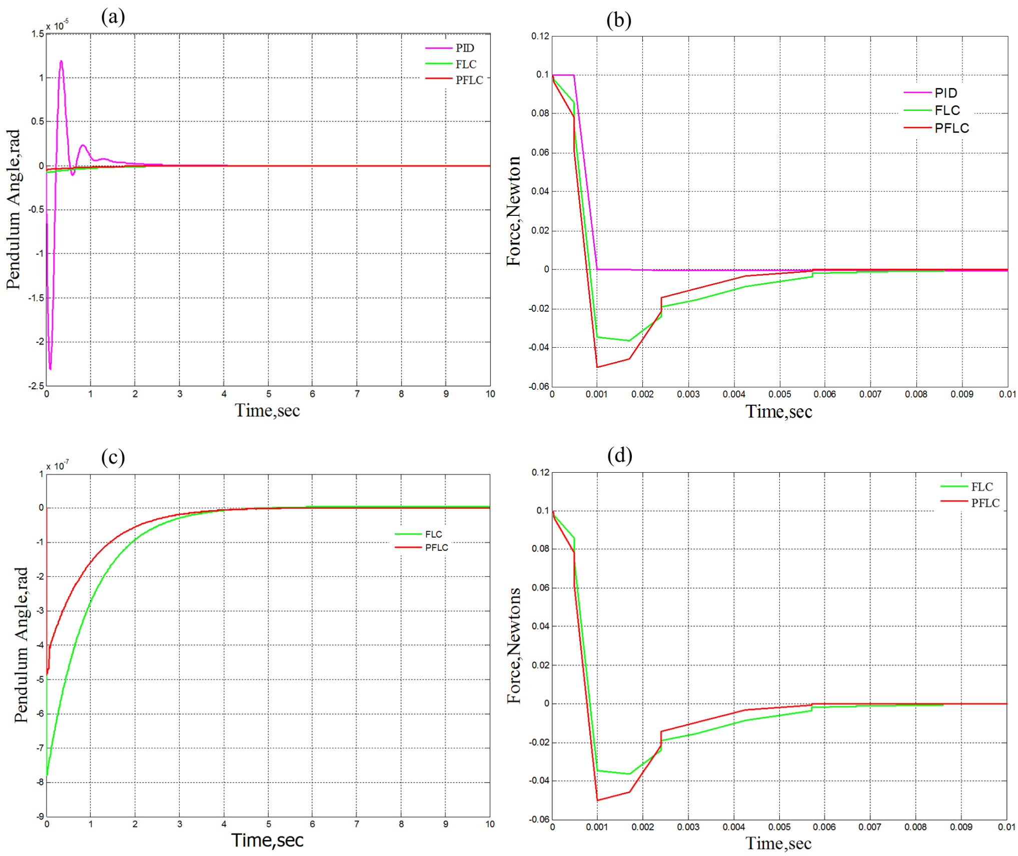

Figure 8: Dynamic Responses of the Pendulum angle with: (a) PID, FLC and PFLC – process output, (b) PID, FLC and PFLC – controllers output, (c) FLC and PFLC – process output, (d) FLC and PFLC – controllers output

are presented in Fig. 9(b) and Fig. 9(d). The ISE, IAE and ITAE values of PID, FLC and PFLC are reported in Tab. 4.

The following observations can be made from this part of the simulation study. From Fig. 9(a) and Fig. 9(c) it is concluded that the PFLC outperforms than PID and FLC by all means and similarly it is driving the cart quickly to the desired position in order to balance a pendulum. From Fig. 9(c) it is proved that the proposed controller is able to quickly discard the disturbance and balance the pendulum. The values of ISE, IAE and ITAE for the proposed controller are found to be considerably less from the Tab. 4.

4.2.3. Performance of FLC and PFLC in the presence of noise measurement

Gaussian noise has been added to the true value of the process variable, i.e. pendulum angle. The performances of the FLC and proposed PFLC controllers in the presence

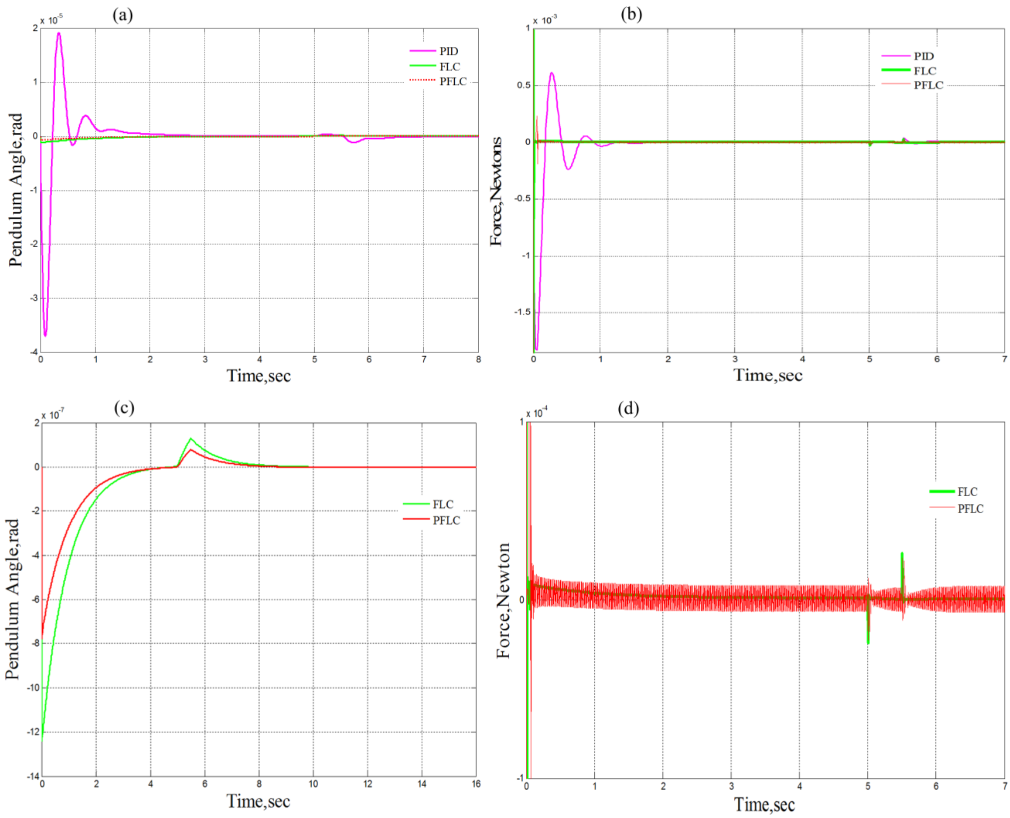

Figure 9: Regulatory responses of an IPC System with: (a) PID, FLC and PFLC – process output, (b) PID, FLC and PFLC – controllers output, (c) FLC and PFLC – process output, (d) FLC and PFLC – controllers output

of measurement noise of various levels and their corresponding controller outputs are shown in the Fig. 10, Fig. 11 and Fig. 12. The mean and the standard deviation of the true value of the measured variable (pendulum angle) for noise with various values of the standard deviations are reported in Tab. 5.

The following observations can be made from this part of the simulation study. From Fig. 10(a) it seems that the FLC performs satisfactorily if the noise level is small in magnitude. From Fig. 11(a) it is proved that the PFLC outperforms irrespective of magnitude of noise level. From Fig. 12(a) it is proved that the PFLC produces robust control action than FLC in the presence of measurement noise. The standard deviation of the controlled variable has been found to be very less in the case of PFLC than FLC from Tab. 5.

Table 14: ISE, IAE and ITAE values of PID, FLC and PFLC in the presence of load change

Controllers	ISE	IAE	ITAE
PID	2.052e-10	1.156e-5	9.345e-6
FLC	7.500e-13	1.315e-6	2.004e-6
PFLC	2.725e-13	7.904e-7	1.121e-6

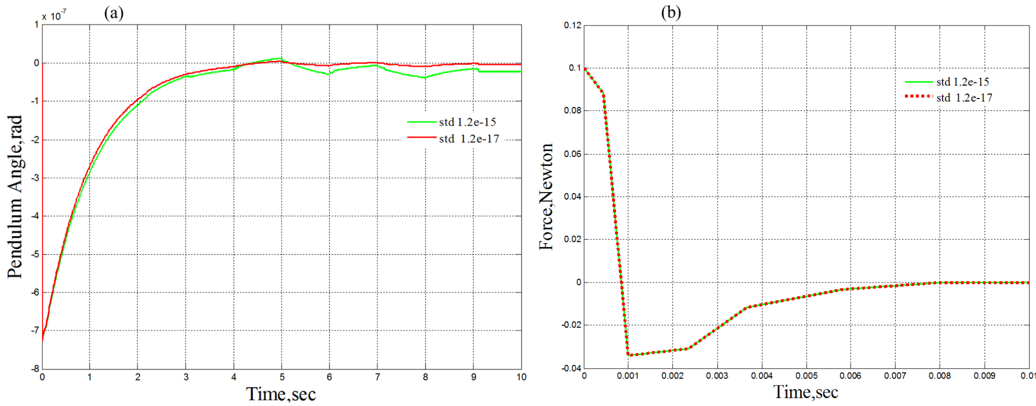

Figure 10: Performance of FLC in the presence of measurement noise of various levels: (a) process output, (b) controllers output

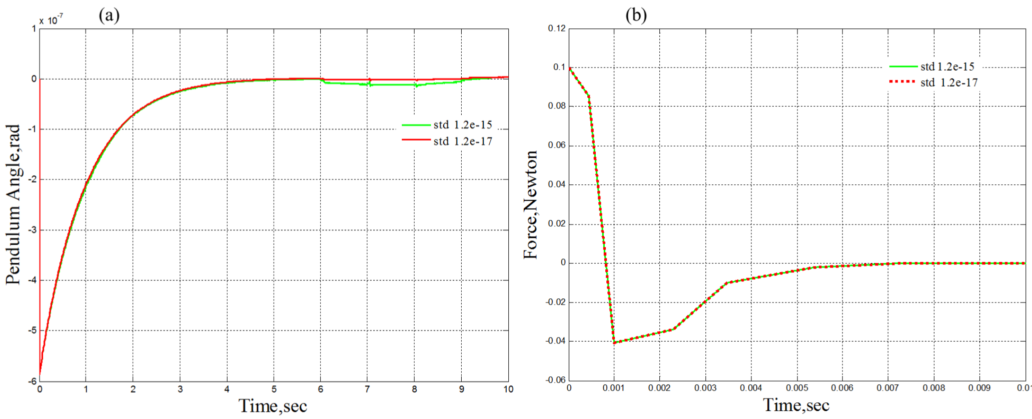

Figure 11: Performance of PFLC in the presence of measurement noise of various levels: (a) process output, (b) controllers output

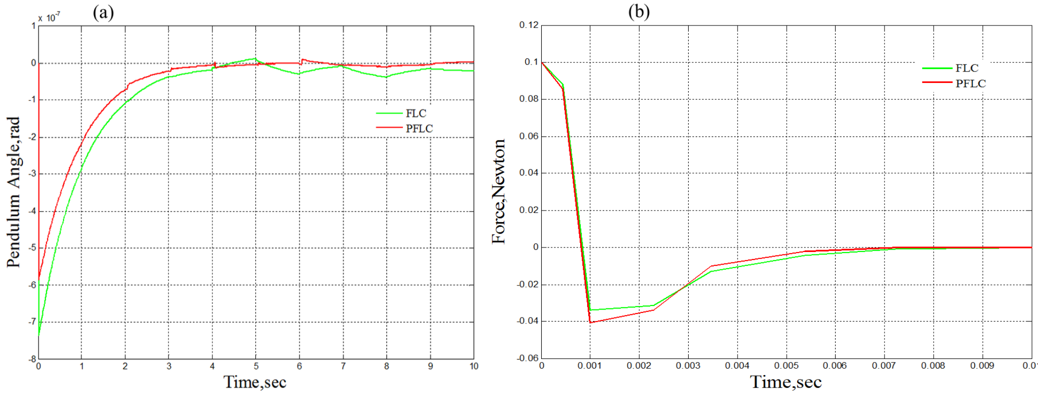

Figure 12: Performance of FLC and PFLC in the Presence of Measurement Noise: (a) process output, (b) controllers output

Table 15: Mean and std of the true value of the process variable (pendulum angle) for various values of std of noise signal

Standard Deviation (std) of noise signal	FLC		PFLC	
	Mean	std	Mean	std
1.2e-15	-8.4903e-8	1.4769e-7	-5.7312e-8	1.1842e-7
1.2e-17	-7.2948e-8	1.4848e-7	-5.6131e-8	1.1838e-7

5. Conclusions

In this paper to handle random uncertainties in the modeling and control problems, a Probabilistic Fuzzy Inference system is introduced. Also straightforward procedure for designing PFIS is provided and has been applied for modeling and control of an IPC system. From the widespread simulation studies, it can be concluded that the performance of the proposed model is better than the conventional fuzzy model. Similarly the proposed controller has successfully driven the cart of an IPC to the desired position as well as stabilizing the pole of the pendulum in the upright position with minimum ISE, IAE and ITAE and also has good disturbance rejection capability and robustness properties. The performance of the proposed controller has been compared with conventional PID and FLC, its performance is proven to be better than the others by all means. It has also been proved that the proposed controller captures the stochastic uncertainties effectively and outperforms particularly in the presence of measurement noises. It is trusted that the PFIS will be capable for many engineering applications like pattern recognition, decision making, system identification, robust control where stochastic uncertainties could be expected.

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