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Natural frequencies of flexural vibration of a ring with wheel–plate as the Winkler elastic foundation

1 Introduction

The vibration theory of rings with wheel–plates as the elastic foundation has applications in many fields of engineering including the civil engineering, mechanical egineering, and others [1, 5, 8]. The fundamental theory of circular rings vibration is presented in [7]. In the paper [1] theory of curved beam with foundation modeling wheel–plate is used to railway wheels vibration analysis. Free vibration of the Timoshenko beam interacting with Winkler elastic foundation is considered in the paper [2]. Authors of paper [3] employed theory of thin ring to obtain natural frequencies and normal modes of in plane vibration of a circular ring with equi–spaced, identical radial supports. In the paper [4] the exact solution for the free vibration of annular membrane system with Winkler foundation is presented. The majority of previous works in the field present solutions for the free vibration of the thin ring with the Winkler foundation as the wheel–plate [5, 8]. In the present paper the in plane flexural vibration of a compound system consists of circular ring and elastic foundation of a Winkler type, respectively, are analysed. At first the vibration problem of the system is described by partial differential equations. The effect of rotary inertia and shear deformation is included. The solution of the free vibration problem is obtained by the separation of variable method. The other model is formulated by using finite element (FE) method. The achieved results are discussed and compared for these models. This work continues the latest author's research related to vibration of systems with elastic foundation (especially presented in the article [6]). Noticed mistakes and faults in the paper [6] are rectified.

2 Formulation of the problem

Let us consider mechanical model of the system taking into account that it consists of circular ring interacting with massless, linear, Winkler elastic foundation as a wheel–plate. It is assumed that the neutral line of the ring has radius R and an element of the ring, determined by angle θ, displaces in the radial and circumferential direction (see Fig. 1). It is additionally assumed that the ring is perfectly elastic and it has constant cross–sectional area. The small displacements in these directions are denoted as $u(\theta,t)$ and $w(\theta,t)$, respectively. No damping cases are considered. According to the classical theory of vibrating thin rings, the equation of motion for the free in–plane flexural vibrations in terms of the radial deflection $u(\theta,t)$ can be written in the form

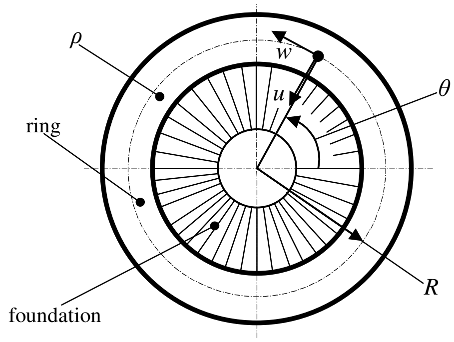

Fig. 1. Model of the system.
Rys. 1. Model układu.

$$\frac{\partial^6 u}{\partial\theta^6}+2\frac{\partial^4 u}{\partial\theta^4}+\left(1+\frac{R^4}{EI_1}k_f\right)\frac{\partial^2 u}{\partial\theta^2}-\frac{R^4}{EI_1}k_p u+\rho A\frac{R^4}{EI_1}\left(\frac{\partial^4 u}{\partial\theta^2\partial^2 t}-\frac{\partial^2 u}{\partial t^2}\right)=0 \quad (1)$$

where E is the Young's modulus of elasticity, I_1 is the area moment of inertia of the ring cross section, ρ is the mass density, A is the area of the ring cross section, k_f and k_p are the radial and tangential stiffness modulus of a elastic foundation, respectively.

Then the Timoshenko's theory is used to include the effects of rotary inertia and shear deformation in the equation of motion of the system under study. The partial differential equation of motion in terms of $u(\theta,t)$, can be expressed in the form

$$\begin{aligned}&\frac{\partial^6 u}{\partial\theta^6}+(2-b_0k_f)\frac{\partial^4 u}{\partial\theta^4}+(1+a_0k_f+b_0k_p)\frac{\partial^2 u}{\partial\theta^2}-a_0k_pu-(c_0+d_0)\frac{\partial^6 u}{\partial\theta^4\partial t^2}+\\&-c_0d_0\frac{\partial^4 u}{\partial t^4}+c_0d_0\frac{\partial^6 u}{\partial\theta^2\partial t^4}+(\rho Aa_0+d_0+h_0k_f-2c_0)\frac{\partial^4 u}{\partial\theta^2\partial t^2}+\\&-(c_0+\rho Aa_0+h_0k_p)\frac{\partial^2 u}{\partial t^2}=0\end{aligned} \quad (2)$$

where

$$a_0=\frac{R^4}{EI_1},\quad b_0=\frac{R^2}{k'AG},\quad c_0=\frac{\rho R^2}{E},\quad d_0=\frac{\rho R^2}{k'G},\quad h_0=\frac{\rho R^4}{k'EAG} \quad (3)$$

and k' is the shear correction factor, G is the modulus of elasticity in shear. The rest of the designations have the same meaning as for in previous case.

3 Free vibration analysis

The objective of this section is the determination of the analytical solution of free vibration of the system under study. The solution of Eqs. (1) and (2) is assumed in the form

$$u(\theta,t)=U(\theta)e^{i\omega t} \tag{4}$$

where ω is the natural frequency of vibration and $i=\sqrt{-1}$ is the imaginary unit. After substitution into the Eq. (1), it becomes

$$\frac{d^6U}{d\theta^6}+2\frac{d^4U}{d\theta^4}+\left(1+\frac{R^4}{EI_1}k_f\right)\frac{d^2U}{d\theta^2}-\frac{R^4}{EI_1}k_pU-\rho A\frac{R^4}{EI_1}\omega^2\left(\frac{d^2U}{d\theta^2}-U\right)=0 \tag{5}$$

Substituting solutions (4) into Eq. (2) gives the following expression

$$\begin{aligned}&\frac{d^6U}{d\theta^6}+\left(2-b_0k_f\right)\frac{d^4U}{d\theta^4}+\left(1+a_0k_f+b_0k_p\right)\frac{d^2U}{d\theta^2}-a_0k_pU+\\&+\left(\left(c_0+d_0\right)\frac{d^4U}{d\theta^4}+\left(2c_0-d_0-\rho Aa_0-h_0k_f\right)\frac{d^2U}{d\theta^2}+\left(c_0+\rho Aa_0+h_0k_p\right)U\right)\omega^2+\\&+c_0d_0\left(\frac{d^2U}{d\theta^2}-U\right)\omega^4=0\end{aligned} \tag{6}$$

The general solution of Eqs. (5) and (6) is assumed as

$$U(\theta)=D_1\sin(n\theta+\varphi),\qquad n=2,3,\ldots \tag{7}$$

where D_1 and φ are constants. When Eq. (7) is substituted into Eq. (5), it yelds the natural frequencies of vibration as (thin ring theory)

$$\omega_n^2=\frac{EI_1\left(n^6-2n^4+n^2\right)+R^4k_fn^2+R^4k_p}{\rho AR^4\left(n^2+1\right)} \tag{8}$$

Substituting Eq. (7) into Eq. (6) gives the frequency equation (thick ring theory)

$$\begin{aligned}&-n^6+\left(2-b_0k_f\right)n^4-\left(1+a_0k_f+b_0k_p\right)n^2-a_0k_p+\left(\left(c_0+d_0\right)n^4+\right.\\&\left.+\left(\rho Aa_0+d_0+h_0k_f-2c_0\right)n^2+\left(c_0+\rho Aa_0+h_0k_p\right)\right)\omega_n^2-c_0d_0\left(n^2+1\right)\omega_n^4=0,\ n=2,3,\ldots\end{aligned} \tag{9}$$

Equation (9) is a quadratic in ω_n^2 and hence two frequency values are associated with each value of n. The smaller of the two ω_n values corresponds to the flexural mode, and the higher value corresponds to the thickness–shear mode.

For both cases the mode shapes of the ring can be expressed as follows

$$u_n(\theta,t)=D_1\sin(n\theta+\varphi)e^{i\omega_nt},\qquad n=2,3,\ldots \tag{10}$$

where D_1 and φ may be determined from the initial conditions of the ring system.

4 The finite element technique

In the next instance the discrete model of the sytsem under consideration is formulated using finite element method (ANSYS program). The elaborated FE model is treated as an approximation of the analytical solution given by Eqs. (6) and (9), respectively. The natural frequencies and corresponding mode shapes of the system may be obtained by using the block Lanczos method. The principal problem of this section is elaborated the FE model of the elastic layer. The proposed FE model is prepared as follows. Ring is modelled as the solid body with taking into account the structural geometry of the rim. The foundation is modelled as the massless solid body with allowing for the structural geometry of the system. The ten node tetrahedral element (solid187) with three degrees of freedom in each node is employed to realize the system. The elaborated model is displayed in Fig. 2 and it contains 41740 solid elements.

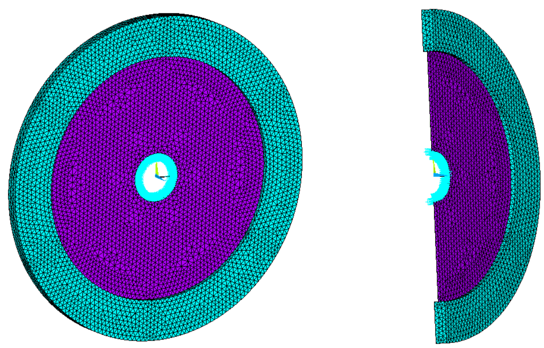

Fig. 2. Finite element model.
Rys. 2. Model MES układu.

The difference between the analytical and the FE solutions is defined by [4]

$$\varepsilon_n = \left(\omega_n^f - \omega_n^c\right)/\,\omega_n^c \cdot 100\,\% \qquad (11)$$

where ω_n^f and ω_n^c are the natural frequencies of the FE and analytical models, respectively. Eq. (11) is the so–called frequency error.

5 Numerical computations

Numerical solutions for free vibration analysis of the circular ring with elastic foundation models suggested earlier, are computed. For all results presented here, only the first seven natural frequencies and mode shapes are discussed. Table 1 demonstrates the parameters characterizing the system under study (the same as in the paper [6]).

Table 1. Parameters characterizing the system under study.
Tabela 1. Dane techniczne rozważanego układu.

d_{01} [m]	s_0 [m]	R [m]	d_{if} [m]	I_1 [m^4]	A [m^2]	ρ [kg/m^3]	E [Pa]	ν	**k'**
0.025	0.008	0.0875	0.03	$1.0417\cdot10^{-8}$	$2\cdot10^{-4}$	$7.83\cdot10^{3}$	$2.08\cdot10^{11}$	0.3	5/6

In Table 1, d_{01} and s_0 are the depth and width of the ring, respectively; ν is the Poisson ratio; d_{if} is the inner diameter of the foundation area. To compare the achieved results with the ref. [6], in the first instance, the computation for the both analytical models of the system with different values of the k_f and k_p, respectively, are executed. For the analytical approach the natural frequencies are determined from numerical solution of Eqs. (8) and (9), respectively. In the presented paper an analytical solution coming from the Timoshenko's theory is considered as more satisfactory solution. The results of the calculation of the natural frequencies are displayed in Table 2. In each case the best compatibility between analytical solutions is obtained for the first natural frequency. For each case of k_f and k_p the difference between the results of the thick and thin ring model, respectively, grows in parallel with the increase of the natural frequencies number. Worth pointing out is the fact that for higher values of k_f the less frequency error for all frequencies is noticed (see last row of the Table 2).

Table 2. Results of computation related to analytical models.

Tabela 2. Wyniki obliczeń otrzymane na podstawie rozwiązania analitycznego.

k_f [N/m^2]	k_p [N/m^2] \ n	2	3	4	5	6	7	8
natural frequencies of the system under study ω_n [Hz] (the Timoshenko's theory)								
0	0	1982	5296	9483	14243	19373	24739	30254
$5 \cdot 10^5$	$4 \cdot 10^4$	1983	5297	9483	14243	19373	24740	30254
$6 \cdot 10^7$	$6 \cdot 10^6$	2172	5376	9529	14274	19396	24757	30268
$9.82 \cdot 10^9$	$6 \cdot 10^6$	11387	12907	15227	18589	22763	27476	32534
natural frequencies of the system under study ω_n [Hz] (the thin ring solution)								
0	0	2074	5868	11252	18197	26695	36742	48337
$5 \cdot 10^5$	$4 \cdot 10^4$	2076	5869	11252	18197	26695	36742	48337
$6 \cdot 10^7$	$6 \cdot 10^6$	2258	5943	11293	18223	26712	36755	48347
$9.82 \cdot 10^9$	$6 \cdot 10^6$	11463	13319	16617	21997	29447	38802	49928
frequency error ε_n [%]								
0	0	4.64	10.8	18.65	27.76	37.8	48.52	59.77
$5 \cdot 10^5$	$4 \cdot 10^4$	4.69	10.8	18.65	27.76	37.8	48.51	59.77
$6 \cdot 10^7$	$6 \cdot 10^6$	3.96	10.55	18.51	27.67	37.72	48.46	59.73
$9.82 \cdot 10^9$	$6 \cdot 10^6$	0.67	3.19	9.13	18.33	29.36	41.22	53.46

Table 3 displays the results obtained for the FE model case. For this instance the FE results are compared with both analytical solutions. More satisfied compatibility between the thick ring solution and the FE representation is observed. In the case when the foundation is omitted (i.e. $k_f = k_p = 0$), for both cases the frequency error grows in parallel with the increase of the number of the natural frequencies. In the case when the foundation is taken into consideration, for both cases the worst fit is achieved for the first natural frequency. The best compatibility is observed between the thick ring solution and the FE computation for the natural frequency ω_3. The smallest distinction between the thin ring solution and the FE representation is noticed for frequencies ω_4, ω_5 and ω_6, respectively. These results are even better compared to the results from the thick ring solution (see 7th and last row of the Table 3).

Table 3. Results of computation related to the FE model.
Tabela 3. Wyniki obliczeń otrzymane z modelu MES układu.

k_f [N/m^2]	n k_p [N/m^2]	2	3	4	5	6	7	8
natural frequencies of the system under study ω_n [Hz] (the FE model)								
0	0	2001	5358	9611	14458	19692	25175	30813
$9.82 \cdot 10^9$	$6 \cdot 10^6$	8692	12916	17221	21801	26665	31749	36984
frequency error ε_n [%] (related to the thick ring solution)								
0	0	0.96	1.17	1.35	1.51	1.65	1.76	1.85
$9.82 \cdot 10^9$	$6 \cdot 10^6$	-23.67	0.07	13.1	17.28	17.14	15.55	13.68
frequency error ε_n [%] (related to the thin ring solution)								
0	0	-3.52	-8.69	-14.58	-20.55	-26.23	-31.48	-36.25
$9.82 \cdot 10^9$	$6 \cdot 10^6$	-24.17	-3.03	3.63	-0.89	-9.45	-18.18	-25.93

Presented results are better in comparison with the ref. [6] but still not satifactory. To large difference between the obtained results are observable. In the Figs. 3–6 the mode shapes of vibration corresponding to the presented pairs of the natural frequencies obtained from the FE model are shown.

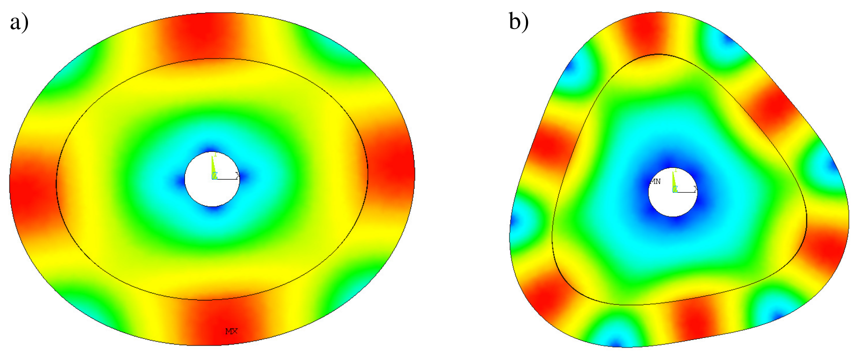

Fig. 3. Mode shapes related to following frequencies; a) ω_2, b) ω_3 .
Rys. 3. Postacie drgań własnych odpowiadające częstościom; a) ω_2, b) ω_3 .

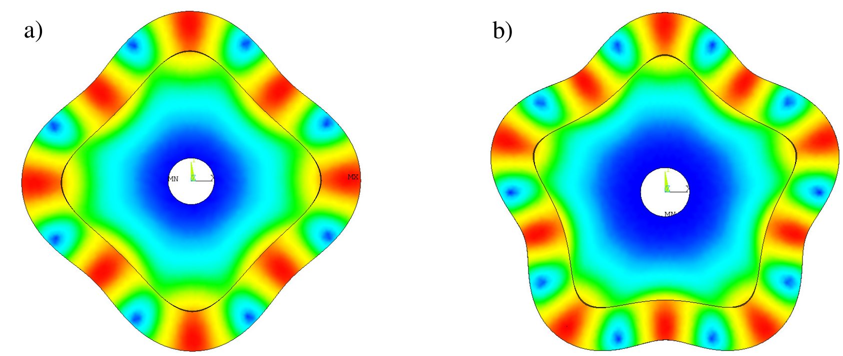

Fig. 4. Mode shapes related to following frequencies; a) ω_4, b) ω_5 .
Rys. 4. Postacie drgań własnych odpowiadające częstościom; a) ω_4, b) ω_5 .

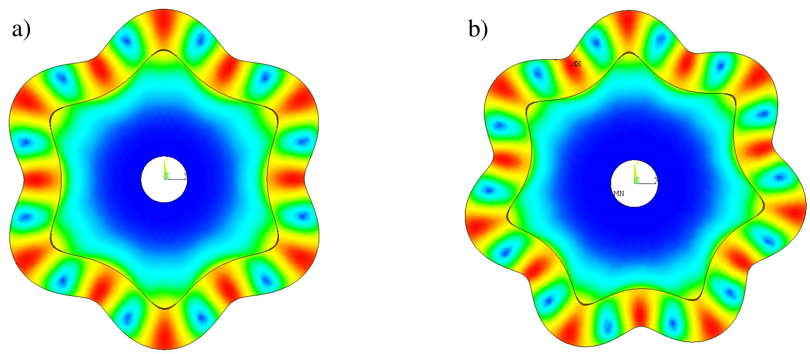

Fig. 5. Mode shapes related to following frequencies; a) ω_6, b) ω_7 .

Rys. 5. Postacie drgań własnych odpowiadające częstościom; a) ω_6, b) ω_7 .

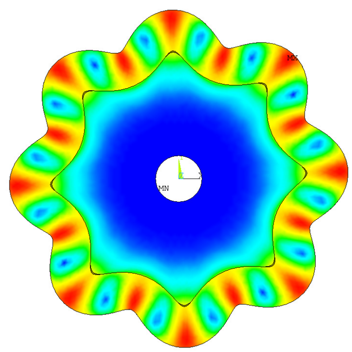

Fig. 6. Mode shape related to frequency ω_8.

Rys. 6. Postać drgań własnych odpowiadająca częstości ω_8.

6 Conclusions

The paper deals with the free in–plane flexural vibration of a rings with wheel–plate as the elastic foundation of the Winkler type. The effect of rotary inertia and shear deformation is comprised. The separation of variables method is employed to solve the eigenvalue problem. The analytical solution of the system under study connected with the thick ring theory is treated as the most satisfactory solution compared to the FE representation which is treated as an approximation of the exact solution. The FE model of the system under consideration is elaborated. The numerical solution results showed that further investigations refered to the rings interacting with foundation are required.

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Summary

In this work the in plane flexural vibration of a circular ring with wheel–plate as a foundation of the Winkler type is studied on the basis of the analytical method and numerical simulation. To begin with the free vibration of the system is described by partial differential equations. The effect of rotary inertia and shear deformation is taken into account. The general solution of the analyzed problem is derived by the separation of variable method. Then the solution by using finite element method is received. The obtained results of calculation are discussed and compared for these solutions. FE models are formulated by using ANSYS software.

Keywords: circular ring, Timoshenko's theory, Winkler foundation, in plane vibration

Częstości drgań własnych giętnych w płaszczyźnie pierścienia koła o tarczy modelowanej podłożem sprężystym typu Winklera

Streszczenie

W pracy analizowane są drgania własne giętne pierścienia kołowego współpracującego z tarczą modelowaną warstwą sprężystą typu Winklera. Prezentowane modele matematyczne układu opracowano na podstawie klasycznej teorii drgań giętnych pierścieni oraz metodę elementów skończonych. W modelu ścisłym uwzględniono wpływ bezwładności obrotowej i odkształcenia postaciowego. Analityczne rozwiązanie drgań własnych układu otrzymano stosując metodę rozdzielenia zmiennych. Otrzymane rezultaty (częstości własne i odpowiadające im formy własne) porównano z rezultatami otrzymanymi z metody elementów skończonych. Obliczenia MES wykonano w programie ANSYS.

Słowa kluczowe: pierścień kołowy, teoria Timoshenki, podłoże typu Winklera, drgania giętne w płaszczyźnie