Scientific Journals
of the Maritime University of Szczecin

Zeszyty Naukowe
Akademii Morskiej w Szczecinie

2016, 47 (119), 85–90
ISSN 1733-8670 (Printed)
ISSN 2392-0378 (Online)
DOI: 10.17402/153

Received: 30.05.2016
Accepted: 17.09.2016
Published: 20.09.2016

Relevance of the relativistic effects in satellite navigation

Eric Kulbiej

Student of the Maritime University of Szczecin,
1–2 Wały Chrobrego St., 70-500 Szczecin, Poland, e-mail: ekulbiej@gmail.com

Key words: GPS, Galileo, GLONASS, BeiDou, COMPASS, IRNSS, satellite navigation, GNSS, relativistic effects, accuracy, clock offset, relativistic error

Abstract

Position determination of Global Navigation Satellite Systems (GNSS) depends on the stability and accuracy of the measured time. However, since satellite vehicles (SVs) travel at velocities significantly larger than the receivers and, more importantly, the electromagnetic impulses propagate through changing gravitational potentials, enormous errors stemming from relativity-based clock offsets would cause a position error of about 11 km to be accumulated after one day. Based on the premise of the constancy of light, two major relativistic effects are described: time dilation and gravitational-frequency shift. Following the individual interests of the author, formulas of both are scrupulously derived from general- and special-relativity theory principles; moreover, in the penultimate section, the equations are used to calculate the author's own numerical values of the studied parameters for various GNSSs and one Land Navigation Satellite System (LNSS).

Introduction

The twentieth century brought the first precise global satellite navigation systems. In 1964, the pioneer TRANSIT was launched, accompanied 10 years later by a nemesis system, Cykada. Both of these were ultimately replaced by their more advanced counterparts and several other satellite systems were installed (Specht, 2007). Today, four major systems can be listed, namely GPS, Galileo, GLONASS, and Chinese BeiDou, though the last is currently under further development (conversion from local BeiDou to global COMPASS). The fact that the Indian Regional Navigation Satellite System (IRNSS) is a local satellite system notwithstanding, it will also be taken into consideration in the final part of this paper. The common factor of all these positioning systems is in terms of precision (Specht, 2003; Januszewski, 2005). Nonetheless, each and every SV's clock is prone to gravitational and motional frequency shifts that are too significant to ignore (Narkiewicz, 2007). If a clock is provided with unsupported time determination due to numerous relativistic effects, then, based on the special and general theory of relativity, the system should be rendered non-operational. In this paper, ways of calculating such effects are undertaken and an example is shown, focusing on the very derivation of error-figuring formulas. This is done in hope that when relativistic effects are fully understood, diminishing them to an insignificant size should be possible.

Principles of satellite systems

The principles of position determination in satellite systems are based on constancy of the speed of electromagnetic signals. It can be accomplished, provided that both the user of the GNSS and the satellite itself have their clocks synchronized in one mutual, underlying, inertial frame (Januszewski, 2010). The signals sent to the receiver are provided with an encrypted message (Specht, 2007). Within the message, the information about the time and position of the constellation is coded. By comparing the time of the SV and receiver, the distance is calculated (Januszewski, 2004). Mathematically it can be stated as follows:

$$d = c\,(t_0 - t_{sv}) \tag{1}$$

Figure 1 illustrates how this is performed in the GPS. In order to determine the position unambiguously, at least four time signals need to be received, so four equations (1) are solved to provide the position (Narkiewicz, 1999; 2007).

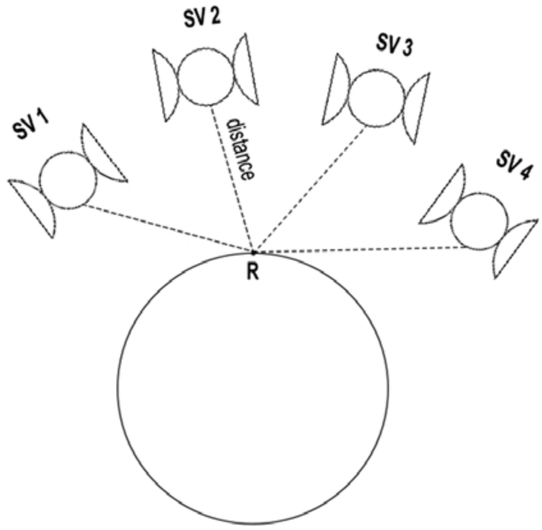

Figure 1. The distance in satellite navigation is calculated as the speed of light multiplied by the time difference

Since the only navigational parameter that is used for calculations is the time, it is convenient to measure the error in seconds (Narkiewicz, 1999; 2007); for instance, an error of 1 nanosecond would cause a position error of about 30 centimeters.

Principles of relativity

The special (1905) and general (1915) theory of relativity published by Albert Einstein have fundamentally changed people's understanding of the nature. Contrary to the Newtonian framework of absolute space and time, three postulates of the summed theories of relativity may be stated (Williams, 1968):

(1) The laws of physics have the same form in all inertial and non-inertial reference frames.
(2) The speed of light *c* (299,792.46 km/s) in a vacuum is a constant and does not depend on the motion of the source.
(3) Occurrences due to a gravitational mass are indistinguishable from occurrences due to an inertial mass.

Time dilation

One of the relativity-based errors is caused by the time dilation between the satellites' and the receiver's clocks. That means the moving clocks beat slower than clocks that are stationary (Narkiewicz, 2007). The effect stems from a Lorentz transformation that applies to inertial reference frames (Williams, 1968). Most trivial derivations of the formula for time dilation are as follows: imagine a vehicle moving at velocity *V*. A flashlight is inserted on the floor of the considered vehicle and a single light impulse is emitted towards a mirror set on the very opposite of the torch, onto the celling. The impulse is hence reflected and the time of the operation is recorded. There are two particular observers of the ongoing action: one bound with the frame of the vehicle (inside) and the other outside and stationary. The situation is pictured in the Figure 2.

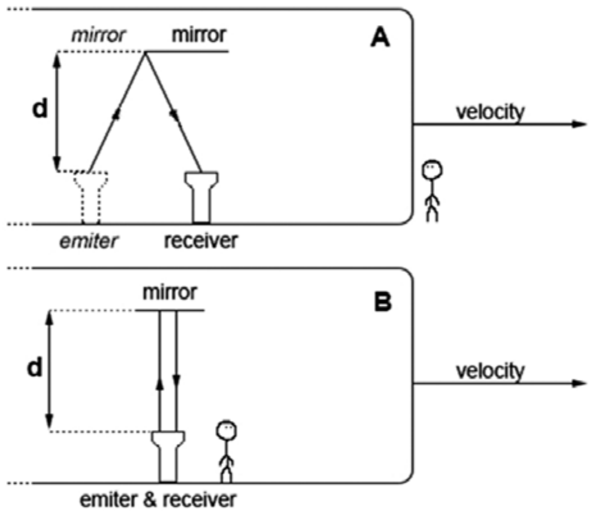

Figure 2. Situation A shows how the route of the flash impulse is seen by a keen observer not bound with the inertial frame of the vehicle. Contrastingly, situation B shows the light pulse as observed within the frame

The point is that although the same event is observed by both people, different routes of the light impulse are perceived. For both situations, a simple equation can be written. For situation A, it is:

$$c = \frac{\sqrt{4d^2 + (vt_B)^2}}{t_B} \tag{2}$$

and for situation B:

$$c = \frac{2v}{t_A} \tag{3}$$

Equations (2) and (3) can be further expressed as follows:

$$t_B = \frac{2d}{c}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{4}$$

$$t_A = \frac{2d}{c} \tag{5}$$

From both (4) and (5), the final formula for time dilation in Lorentz transformation is derived:

$$t_B = t_A \sqrt{1 - \frac{v^2}{c^2}} \tag{6}$$

$$\delta_{\text{time dilation}} = \frac{t_B - t_A}{t_A} \tag{7}$$

This formula states the size of the error based on the time-dilation process. Specifically, it shows that for every piece of time that flows on Earth a change of:

$$\delta_{\text{time dilation}} = \sqrt{1 - \frac{v^2}{c^2}} - 1 \tag{8}$$

occurs on the orbit of the SV. That means that a clock there is slower than a clock on the surface. The newborn error must be naturally compensated in order to maintain the agreement between the clocks and to secure the precise position determination.

To finalize this error analysis, the velocity of the satellite vehicle is to be calculated and inserted into formula (8). From Kepler's Third Law, the period of revolution in Earth's gravitational field can be calculated as (Williams, 1968):

$$T^2 = \frac{4\pi^2 e^3}{G m_E} \tag{9}$$

where $G = 6.6740831 \cdot 10^{-11}$ m^3/kg·s^2 is the gravitational constant, $m_E = 5.9722 \cdot 10^{24}$ kg is the mass of Earth, e is the semi-major axis of concrete SV, and T is its period of revolution. From that, the period of revolution is used to determine the orbital speed of a SV:

$$v = \frac{2\pi e}{T} \tag{10}$$

Substituting $e = 2.65594 \cdot 10^7$ m (for GPS satellites), it is found that T = 43,077 s. Consequently, the velocity is 3,873.95 m/s. Finally, the clock error due to time dilation is calculated as following:

$$\delta_{\text{GPS time dilation}} = \sqrt{1 - \frac{v^2}{c^2}} - 1 = -8.34903 \cdot 10^{-11} \tag{11}$$

To calculate this offset for a particular time, it needs to be integrated:

$$\tau = \int_{\text{time}} \delta_{\text{time dilation}} \, \mathrm{d}t = \int_{\text{time}} \left(\sqrt{1 - \frac{v^2}{c^2}} - 1 \right) \mathrm{d}t \tag{12}$$

For instance, this effect after one day (86,400 seconds) causes a clock offset of about 7.21 µs, which therefore would result in an astonishingly large error of 2,163 meters.

Gravitational frequency shift

Somewhat more complex is the problem of light travel through space-time. The gravitational field is conservative and stems from the pure mass of inducing the object (Williams, 1968). In 1687, Sir Isaak Newton articulated his Law of Gravitation, which is as follows: "Every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them".

Mathematically, this is expressed with:

$$\vec{\mathbf{F}}_{\text{gravitation}} = \frac{G m_1 m_2}{r^3} \vec{\mathbf{r}} \tag{13}$$

where r is the distance between two particular masses m_1 and m_2. More importantly to the studied case, it can be also stated as:

$$\vec{\mathbf{F}}_{\text{gravitation}} = -\nabla \cdot U_{\text{gravitation}} \tag{14}$$

which means the gravitational force is the negative gradient of the gravitational potential energy (Williams, 1968). Further, the potential energy can be calculated as the gravitational potential multiplied by the elementary mass:

$$U_{\text{gravitation}} = \Phi_g \, m \tag{15}$$

Earth may be considered a perfect sphere in regards to an induced gravitational field, to a good approximation. Thus, the gravitational interactions involving such a spherical body can be treated as if all the mass was concentrated at the center of the object. That said, it is easy to derive the formula for gravitational potential energy within Earth's field (by integrating (13) over distance):

$$U_{\text{gravitation}} = -\frac{G M_E m}{r} \tag{16}$$

In order to understand errors in satellite navigation positioning caused by gravitational frequency shift, consider a particle of light (an electromagnetic impulse) that is emitted at point A and received at point B (Figure 3). Upon travelling the described distance, the signal leaves a quasi-potential surface of energy, amounted to:

$$U_A = -\frac{G M_E m}{r_A} \tag{17}$$

and enters a significantly stronger field of:

$$U_B = -\frac{G M_E m}{r_B} \tag{18}$$

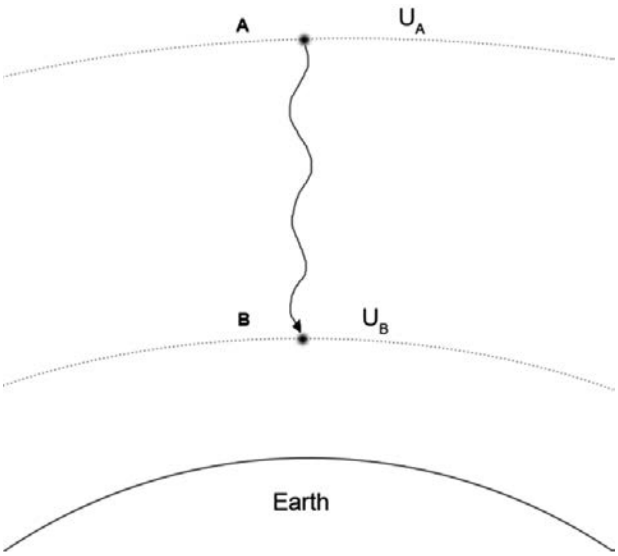

Figure 3. The change of gravitational potential energy as an object approaches the Earth's center of mass

The electromagnetic wave (photon) itself has an energy proportional to its frequency (Williams, 1968), by:

$$E = h\,v \tag{19}$$

and, on the other hand, its energy is described by Einstein's best known equation:

$$E = m\,c^2 \tag{20}$$

Since there is conservation of energy, the overall energy must be stated as a constant function of distance from the mass inducting the gravitational field. That means that the overall energy of a signal at point A must precisely equal the energy at point B:

$$E_A = E_B \tag{21}$$

Yet there is a significant change in the gravitational potential energy:

$$\Delta U = U_A - U_B \tag{22}$$

$$\Delta U = \Delta \Phi_g\, m \tag{23}$$

Having that formulated, it is now possible to compare the energy of the impulse from positions A and B. In order to maintain the same amount of energy, the frequency of the electromagnetic wave changes, thereby compensating for the deficit of gravitational potential energy. By comparing the energy at point A and B, the exact gravitational frequency shift can be derived:

$$h\,v_A = h\,v_B + \Delta \Phi_g\, m \tag{24}$$

Putting:

$$m = -\frac{h\,v_A}{c^2} \tag{25}$$

in (24) for m will result in:

$$h\,v_A = h\,v_B - \Delta \Phi_g \cdot \frac{h\,v_A}{c^2} \tag{26}$$

and eventually:

$$\frac{v_B}{v_A} - 1 = \frac{\Delta \Phi_g}{c^2} \tag{27}$$

$$\frac{\Delta v}{v} = \frac{\Delta \Phi_g}{c^2} \tag{28}$$

Hence, with the change of the gravitational field, the received frequency of the light signal changes according to (28). This phenomenon is recognized within the general theory of relativity and is referred to as gravitational frequency shift or gravitational redshift.

This model was labeled as one with good approximation but for the purposes of this study a deeper insight is needed, as far as Earth's gravitational field is concerned. Equation (16) shows, for an ideally-spherical body, the distribution of gravitational potential as a function of only distance from the center of the mass. Adding to this the fact that Earth is a geoid of a complex structure, the equation would be:

$$\Phi(r,\theta) = -G\frac{M_E}{r}\left[1 - J\left(\frac{R_E}{r}\right)^2 P_2(\cos\theta)\right] \tag{29}$$

where: r is the distance from the center of the Earth's mass, θ is the polar angle measured downward from the axis of rotational symmetry, J is earth's quadrupole moment coefficient, and P_2 is the Legendre polynomial of degree 2, which stands for:

$$P_2(\cos\theta) = \frac{1}{2}\left(3\cos^2\theta - 1\right) \tag{30}$$

On the whole, the time rate is lower in a more intense gravity field; hence on the surface of Earth, a clock will run slower than on the orbit of the SV. Since the formula has been derived, the real error of this kind may be calculated. To calculate it, only the height (altitude) of the satellite is needed, as it is the only variable contributing to the change of gravitational field. Using $H = 20{,}162$ km, $\theta = 90°–55°$ (as for GPS SV), $J = 1.08263 \cdot 10^{-3}$, and $R_E = 6378.14$ km, the error is found to be:

$$\delta_{\text{GPS redshift}} = \frac{\Phi_g(R_E + H) - \Phi_g(R_E)}{c^2} \tag{31}$$

$$\delta_{\text{GPS redshift}} = 5.2835 \cdot 10^{-10} \tag{32}$$

Similarly to (12), in order to gain a numerical value of a one-day offset, (32) needs to be integrated over time:

$$\tau = \int_{\text{time}} \delta_{\text{GPS redshift}}\, \mathrm{d}t = = \int_{\text{time}} \left(\frac{\Phi_g(R_E + H) - \Phi_g(R_E)}{c^2} \right) \mathrm{d}t \quad (33)$$

Comparing this effect to the time dilation after one day (86,400 seconds), it causes a clock offset of about 45.65 , which therefore would result in an even larger error of 13,685.53 meters.

Overall offset and other effects

It is important to emphasize that, interestingly, the relative-error effects derived from time dilation and gravitational frequency shift have opposite signs and are diminishing themselves (Narkiewicz, 1999; 2007). That is to say that a detailed calculation may prove that at the height of about 6,000 km, the effects would counteract each other (Narkiewicz, 2007). That fact notwithstanding, the overall relativistic effect in a satellite navigational system can be calculated from an integral, which is a combination of (12) and (33):

$$\tau_o = \int_{\text{time}} \left(\delta_{\text{redshift}} + \delta_{\text{time dilation}} \right) \mathrm{d}t = = \int_{\text{time}} \left(\frac{\Phi_g(R_E + H) - \Phi_g(R_E)}{c^2} + \sqrt{1 - \frac{v^2}{c^2}} - 1 \right) \mathrm{d}t \quad (34)$$

For GPS satellites, the overall error is calculated as 38.44 µs/day or $\Delta f/f = 4.4486 \cdot 10^{-10}$. Since this clock offset tends to decrease the quality of service and is omnipresent, GPS frequency is modified in a way that the frequency of satellite vehicles of Global Positioning System on Earth would be measured as:

$$10.23\ \text{MHz} - 10.23\ \text{MHz} \cdot 4.4486 \cdot 10^{-10} = = 10.2299999954491\ \text{MHz} \quad (35)$$

Thanks to this effect, a receiver on Earth is capable of receiving a standard frequency of 10.23 MHz. The value of the change calculated in (35) is nearly equal to the one stated by Narkiewicz (2007). It is possible to calculate that daily clocks on SVs are slowed by $4.4486 \cdot 10^{-10} \cdot 86{,}400$ s = 38.436 µs. The greatest part of the relativity-based error is thus removed and compensated, albeit other types of errors are still present, nonetheless of lesser significance. These are Sagnac effects and the periodic tidal effect of the Moon and Sun. But then again, they are removed by way of signal processing in the receiver using following equations:

$$t = t_{SV} - \Delta t_{SV} \quad (36)$$

$$\Delta t_{SV} = A_0 + A_1 (t - t_0) + A_2 (t - t_0)^2 \quad (37)$$

where: A_0, A_1, and A_2 are coefficients of the provided polynomial and are directly transferred via the GPS message, and t_0 is the referent time. Ultimately, there is the eccentricity-effect correction, which is mathematically specified with the formula:

$$\Delta t_{SV} = Fe\sqrt{a} \cdot \sin E \quad (38)$$

where: F, e, and E are parameters of the satellite and are also taken from the GPS message, while F is a system constant = $4.442807 \cdot 10^{-10}\ \text{s} \cdot \text{m}^{-0.5}$.

Table 1. Satellite parameters and corresponding to them relativistic effects for major GNSSs (Dana, 1995; FindTheData, 2016; GSC, 2016; IAC, 2016; ILRS, 2016; N2YO, 2016; PosiTim, 2010; SatelliteCoverage, 2016; Spaceflight Insider, 2015)

Factor	GPS	GLONASS	Galileo	Beidou Compass	IRNSS
Satellite's name / number	GPS IIF	13	Galileo-101	COMPASS-M3	IRNSS-1A
Orbit	MEO	MEO	MEO	MEO	GEO
Semi-Major axis [km]	26559.4	25508	29599.8	21528	42164
Altitude [km]	20183.5	19132	23014.5	21527.5	35786
Perigee [km]	19652	18622	23013	21460	35707
Apogee [km]	20715	19642	23016	21595	35884.7
Eccentricity	0.002	0.00085	0.001	0.0025	0.0002
Inclination [rad]	0.95	1.13	0.97	0.95	0.51
Period of revolution [s]	43077.6	40543.9	50688	46403.4	86172
Average velocity [m/s]	3873.95	3669.598826	3952.981413	3779.324488	3074.624434
Relativistic effect	GPS	GLONASS	Galileo	Beidou Compass	IRNSS
Frequency change of time dilation $\Delta f/f$ [s/s]	–8.34903E–11	–7.49145E–11	–8.69317E–11	–7.94615E–11	–5.25912E–11
Time dilation[µs per day]	–7.21	–6.47	–7.51	–6.86	–4.54
Gravitational frequency change $\Delta f/f$ [s/s]	5.28355E–10	5.21454E–10	5.44268E–10	5.36396E–10	5.90633E–10
Gravitational redshift [µs per day]	45.65	45.05	47.02	46.34	51.03
Total relativistic offset [µs per day]	38.44	38.58	39.51	39.48	46.49
Total relativistic offset [km per day]	11.52	11.56	11.84	11.83	13.94

Numerical data

Using formulas derived in previous sections of this paper, the specific, relativistic clock offsets were calculated by the author. It is also important to note that each error and its value is strictly individual and depends on the characteristics of a particular SV's orbit. In the Table 1 most basic and significant parameters of different satellite systems are collected, as well as time-dilation error and gravitational-frequency shift error are calculated for those GNSSs using data published by authorized sources for particular SVs. All constants used in calculations are those used and stated previously; that also applies to the formulas themselves.

Conclusions

In order to maintain precise positioning, the relativistic effects need to be taken into consideration and thoroughly calculated. All of the currently-used, global-navigation satellite systems base their working schema on the concept of clock synchronization to receivers within Earth's inertial reference frame. However, since their value stems from each orbit's constellation numerical parameters, relativistic errors of different GNSS have comparable values. Nonetheless, clocks' offsets are too significant to be ignored and ways of reducing them are presented, thus making diminishment of the error forced by the very forces of gravitation possible.

References

1. Dana, P.H. (1995) *GPS Satellite almanac data samples*. [Online] Available from: http://www.colorado.edu/geography/gcraft/notes/gps/almanacs.html [Accessed: May 03, 2016]
2. FindTheData (2016) *Compass M3 – Medium Earth Orbit Satellite*. [Online] Available from: http://www.satellites.findthedata.com/l/134/Compass-M3 [Accessed: May 03, 2016]
3. GSC (2016) *Orbital and Technical Parameters*. [Online] Available from: http://www.gsc-europa.eu/system-status/orbital-and-technical-parameters [Accessed: May 03, 2016]
4. IAC (2016) *Evaluation of characteristics*. [Online] Available from: https://www.glonass-iac.ru/en/GLONASS/technical_characteristics.php [Accessed: May 03, 2016]
5. ILRS (2016) *Current Missions*. [Online] Available from: http://ilrs.gsfc.nasa.gov/missions/satellite_missions/current_missions/index.html [Accessed: May 03, 2016]
6. Januszewski, J. (2004) *System GPS i inne systemy satelitarne w nawigacji morskiej*. Gdynia: Fundacja Rozwoju Wyższej Szkoły Morskiej w Gdyni.
7. Januszewski, J. (2005) GPS Vertical Accuracy for Different Constellations. *Scientific Journals of the Maritime University of Szczecin* 6 (78). pp. 181–190.
8. Januszewski, J. (2010) *Systemy satelitarne GPS, Galileo i inne*. Warszawa: Wydawnictwo Naukowe PWN.
9. N2YO (2016) *COSMOS 2514* (*GLONASS*). [Online] Available from: http://www.n2yo.com/satellite/?s=41330 [Accessed: May 03, 2016]
10. Narkiewicz, J. (1999) *Podstawy układów nawigacyjnych*. Warszawa: Wydawnictwa Komunikacji i Łączności.
11. Narkiewicz, J. (2007) *GPS i inne satelitarne systemy referencyjne*. Warszawa: Wydawnictwa Komunikacji i Łączności.
12. PosiTim (2010) *Global Navigation Satellite Systems* (*GNSS*). [Online] Available from: http://www.positim.com/navsys_overview.html [Accessed: May 03, 2016]
13. SatelliteCoverage (2016) *Space station* (*ISS*). [Online] Available from: https://satellitecoverage.net/satellite-tracking/ [Accessed: May 03, 2016]
14. Spaceflight Insider (2015) *China successfully launches BDS M3-S satellite for its homegrown BeiDou navigation system*. [Online] Available from: http://www.spaceflightinsider.com/space-flight-news/china-successfully-launches-bds-m3-s-satellite-homegrown-beidou-navigation-system/ [Accessed: May 03, 2016]
15. Specht, C. (2003) Availability, Reliability and Continuity Model of Differential GPS Transmission. Polish Academy of Sciences. *Annual of Navigation* 5.
16. Specht, C. (2007) *System GPS*. Pelplin: Bernardinum.
17. Williams, J.E. (1968) *Modern Physics*. New York: Holt, Rinehart and Winston, Inc.