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ANALYTICAL AND NUMERICAL SOLVING OF RIGHT TRIANGLES WITH GIVEN A DIFFERENCE OF TWO SIDES LENGTH AND THE ACUTE ANGLE

Abstract

Introduction and aims: The paper shows the analytical models of solving right triangles with appropriate discussion. For right triangles have been discussed six cases taking into account the acute angle and the difference of two sides length in the right triangle. The main aim of this paper is not only to create some analytical algorithms for solving right triangle, but also their implementation in programs *MS-Excel, MathCAD* and *Mathematica.*
Material and methods: Elaboration of six analytical cases of solving right triangles has been made on the basis of the relevant trigonometric properties occurring in a right triangle. In the paper have been used some analytical and numerical methods by using *MS-Excel, MathCAD* and *Mathematica* programs.
Results: As some results have been obtained numerical algorithms in the programs *MS-Excel, MathCAD* and Mathematica for six analytical cases of solving right triangles taking into account the acute angle and the difference of two side length in the right triangle.
Conclusion: Created numerical algorithms of solving the right triangles in the programs *MS-Excel, MathCAD* and *Mathematica* allow for faster significant performance calculations than the traditional way of using logarithms and logarithmic tables.

Keywords: Trigonometry, right triangle, acute angle, difference of two sides length of a triangle, numerical algorithms, *MS-Excel, MathCAD, Mathematica.*

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ANALITYCZNO-NUMERYCZNE ROZWIĄZYWANIE TRÓJKĄTÓW PROSTOKĄTNYCH GDZIE DANA JEST RÓŻNICA DŁUGOŚCI DWÓCH BOKÓW I KĄT OSTRY

Streszczenie

Wstęp i cele: *W pracy pokazano analityczne modele rozwiązywania trójkątów prostokątnych wraz z odpowiednią dyskusją. Dla trójkątów prostokątnych omówiono sześć przypadków z uwzględnieniem kąta ostrego oraz różnicy długości dwóch boków trójkąta. Głównym celem jest pracy jest nie tylko utworzenie algorytmów analitycznych rozwiązywania takich trójkątów lecz również ich implementacja w programach MS-Excel, MathCAD i Mathematica.*
Materiał i metody: *Opracowanie sześciu analitycznych przypadków rozwiązywania trójkątów prostokątnych wykonano opierając się odpowiednich własnościach trygonometrycznych występujących w trójkącie prostokątnym. Zastosowano metodę analityczną i numeryczną wykorzystując programy MS-Excel, MathCAD i Mathematica.*
Wyniki: *Otrzymano algorytmy numeryczne w programach MS-Excel, MathCAD i Mathematica dla sześciu analitycznych przypadków rozwiązywania trójkątów prostokątnych z uwzględnieniem kąta ostrego oraz różnicy długości dwóch boków trójkąta.*
Wniosek: *Utworzone algorytmy numeryczne rozwiązywania trójkątów prostokątnych w programach MS-Excel, MathCAD oraz Mathematica, pozwalają na znaczne szybsze wykonanie obliczeń niż drogą tradycyjną z użyciem logarytmów i tablic logarytmicznych.*

Słowa kluczowe: *Trygonometria, trójkąt prostokątny, kąt ostry, różnica długości dwóch boków trójkąta, algorytmy numeryczne, MS-Excel, MathCAD, Mathematica.*

(Otrzymano: 06.10.2018; Zrecenzowano: 15.10.2018; Zaakceptowano: 22.10.2018)

1. Introduction to trigonometry

1.1. Definitions of trigonometric functions

Let be given the right triangle ABC (angle ACB is the right angle) (Fig. I).

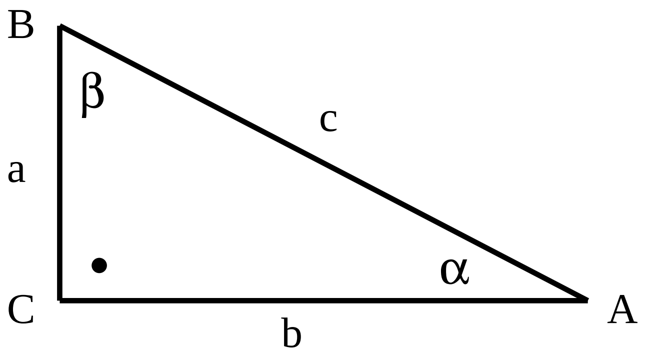

Fig. I. The right triangle with legs a and b and hypotenuse c
Source: Elaboration of the Authors

where a, b, c mean respectively the length of shorter leg, longer leg and hypotenuse. The sides a, b and c satisfy the condition:

$$a^2 + b^2 = c^2 \tag{1}$$

and acute angles α and β satisfying the following condition [4], [6], [7]:

$$\alpha + \beta = \frac{\pi}{2}. \tag{2}$$

Definition 1.

The sine of acute angle is the ratio of the length of leg opposite this angle to the length of hypotenuse, which express the formulae [4], [6], [7]:

$$\sin(\alpha) = \frac{a}{c}, \tag{3}$$

$$\sin(\beta) = \frac{b}{c}. \tag{4}$$

Definition 2.

The cosine of acute angle is the ratio of the length of leg adjacent this angle to the length of hypotenuse, which express the formulae [4], [6], [7]:

$$\cos(\alpha) = \frac{b}{c}, \tag{5}$$

$$\cos(\beta) = \frac{a}{c}. \tag{6}$$

Definition 3.

The tangent of acute angle is the ratio of the length of leg opposite this angle to the length of leg adjacent to that angle, which express the formulae [4], [6], [7]:

$$\mathrm{tg}(\alpha) = \frac{a}{b}, \tag{7}$$

$$\mathrm{tg}(\beta) = \frac{b}{a}. \tag{8}$$

Definition 4.

The cotangent of acute angle is the ratio of the length of leg adjacent this angle to the length of leg opposite to that angle, which express the formulae [4], [6], [7]:

$$\mathrm{ctg}(\alpha) = \frac{b}{a}, \tag{9}$$

$$\mathrm{ctg}(\beta) = \frac{a}{b}. \tag{10}$$

Theorem 1. *(Basic relationships in the triangle)*

In any length of each side of a triangle is less than the sum of the length of the other side and higher than the absolute value of the difference in length of the sides [10], [11], [13]-[16]:

$$|b-c| < a < b+c, \tag{11}$$

$$|a-c| < b < a+c, \tag{12}$$

$$|a-b| < c < a+b. \tag{13}$$

1.2. Solving right triangles with given the difference of two legs and acute angle of triangle

For the right triangle ABC shown in figure I, there are six cases of solving which are given below (Tab. 1).

Tab. 1. The discussion of solutions for the right triangles

Case:	*Data:*	*Solution:*			
1	c–a=k α, c>a	$\beta=\frac{\pi}{2}-\alpha$	$a=\frac{k\cdot\sin(\alpha)}{1-\sin(\alpha)}$	$b=\frac{k\cdot\cos(\alpha)}{1-\sin(\alpha)}$	$c=\frac{k}{1-\sin(\alpha)}$
2	c–a=k β, c>a	$\alpha=\frac{\pi}{2}-\beta$	$a=\frac{k\cdot\cos(\beta)}{1-\cos(\beta)}$	$b=\frac{k\cdot\sin(\beta)}{1-\cos(\beta)}=k\cdot\mathrm{ctg}\left(\frac{\beta}{2}\right)$	$c=\frac{k}{1-\cos(\beta)}$
3	c–b=p α, c>b	$\beta=\frac{\pi}{2}-\alpha$	$a=\frac{p\cdot\sin(\alpha)}{1-\cos(\alpha)}=p\cdot\mathrm{ctg}\left(\frac{\alpha}{2}\right)$	$b=\frac{p\cdot\cos(\alpha)}{1-\cos(\alpha)}$	$c=\frac{p}{1-\cos(\alpha)}$
4	c–b=p β, c>b	$\alpha=\frac{\pi}{2}-\beta$	$a=\frac{p\cdot\cos(\beta)}{1-\sin(\beta)}$	$b=\frac{p\cdot\sin(\beta)}{1-\sin(\beta)}$	$c=\frac{p}{1-\sin(\beta)}$
5	b–a=q α, b>a	$\beta=\frac{\pi}{2}-\alpha$	$a=\frac{q\cdot\mathrm{tg}(\alpha)}{1-\mathrm{tg}(\alpha)}$	$b=\frac{q}{1-\mathrm{tg}(\alpha)}$	$c=\frac{q\cdot\sec(\alpha)}{1-\mathrm{tg}(\alpha)}$ [1]
6	b–a=q β, b>a	$\alpha=\frac{\pi}{2}-\beta$	$a=\frac{q\cdot\mathrm{ctg}(\beta)}{1-\mathrm{ctg}(\beta)}$	$b=\frac{q}{1-\mathrm{ctg}(\beta)}$	$c=\frac{q\cdot\mathrm{cosec}(\beta)}{1-\mathrm{ctg}(\beta)}$ [2]

Source: Elaborated by the Authors

[1] Note: sec(α) ≡ 1/cos(α).
[2] Note: cosec(α) ≡ 1/sin(α).

2. Solving of right triangles

2.1. Case 1: Difference of two sides length c–a = k and a measure of the angle α.

2.1.1. Theoretical analysis

➢ Let be the right triangle as shown on the figure 1.

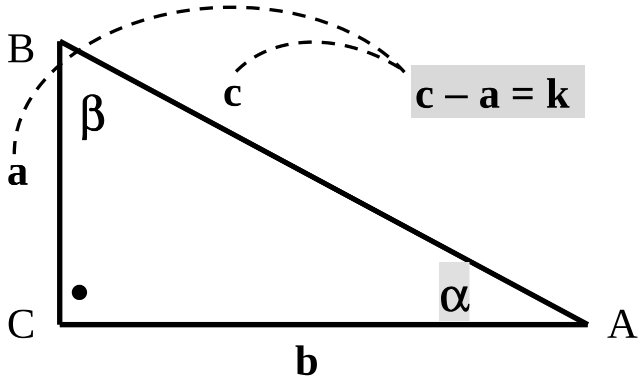

Fig. 1. Right triangle ABC with given a difference of the sides length c–a=k and a measure of the angle α
Source: Elaboration of the Authors

➢ *Data in ΔABC:* Difference of the sides length **c–a=k** (c>a**)** and a measure of the angle **α.**
➢ *Unknown in ΔABC:* Length of the sides **a, b**, **c** and a measure of the angle **β**.
➢ *Solution:*
♦ From some properties of acute angles in a right triangle we obtain:

$$\beta = \frac{\pi}{2} - \alpha. \tag{14}$$

♦ From the difference of the triangle sides we have:

$$c - a = k, \tag{15}$$

$$a = c - k. \tag{16}$$

♦ From the formulae (3) and (5) we obtain:

$$a = c \cdot \sin(\alpha), \tag{17}$$

$$b = c \cdot \cos(\alpha). \tag{18}$$

♦ From the formulae (16) and (17) we determine the triangle side c:

$$c - k = c \cdot \sin(\alpha), \tag{19}$$

$$k = c \cdot [1 - \sin(\alpha)], \tag{20}$$

$$c = \frac{k}{1 - \sin(\alpha)}. \tag{21}$$

♦ Substituting (21) into the formulae (17) and (18) we determine the triangle sides a, b:

$$a = \frac{k \cdot \sin(\alpha)}{1 - \sin(\alpha)}, \tag{22}$$

$$b = \frac{k \cdot \cos(\alpha)}{1 - \sin(\alpha)}. \tag{23}$$

➢ *Answer:* $\beta = \frac{\pi}{2} - \alpha, \quad a = \frac{k \cdot \sin(\alpha)}{1 - \sin(\alpha)}, \quad b = \frac{k \cdot \cos(\alpha)}{1 - \sin(\alpha)}, \quad c = \frac{k}{1 - \sin(\alpha)}.$

2.1.2. Numerical algorithms in *MS-Excel, MathCAD* and *Mathematica* programs

For numerical analysis we take a difference of the sides c–a = 13.9690 and the angle α = 31°.

♦ Program *MS-Excel 7.0 [3],[8]*

Algorithm:	*Commentary:*
`A6=`**`13,9690`**	*Given the difference of the sides c–a*
`A7=`**`31`**	*Given the measure of the angle α [°]*
`A8=A7*PI()/180`	*Command for the angle α [rad], $\alpha = 0.541052$*
`A9=90-A7`	*Command for the angle β [°]*
`A10=(A6*SIN(A8))/(1-SIN(A8))`	*Command for calculation of the side* ***a***
`A11=(A6*COS(A8))/(1-SIN(A8))`	*Command for calculation of the side* ***b***
`A12=A6/(1-SIN(A8))`	*Command for calculation of the side* ***c***
`59`	*Result: measure of the angle β [°]*
`14,83532`	*Result: length of the side* ***a***
`24,69012`	*Result: length of the side* ***b***
`28,80432`	*Result: length of the side* ***c***

♦ Program *MathCAD 15 [9],[12]*

Algorithm:	*Commentary:*
k:=**13,9690**	*Given the difference of the sides c–a*
α0:=**31**	*Given the measure of the angle α [°]*
$\alpha := \frac{\alpha 0 \cdot \pi}{180} = 0.541052$	*Command for the angle α [rad] $\alpha = 0.541052$*
$\beta 0 := 90 - \alpha 0 = 59$	*Command for the angle β [°]*
$a := \frac{k \cdot \sin(\alpha)}{1-\sin(\alpha)} = 14.835323$	*Command and result: length of the side* ***a***
$b := \frac{k \cdot \cos(\alpha)}{1-\sin(\alpha)} = 24.690124$	*Command and result: length of the side* ***b***
$a := \frac{k}{1-\sin(\alpha)} = 28.804323$	*Command and result: length of the side* ***c***

♦ Program *Mathematica 7.0 [1],[2],[5]*

Algorithm:	*Commentary:*
`k:=`**`13,969`**	*Given the difference of the sides c–a*
`A0:=`**`31`**	*Given the measure of the angle α [°]*
`A:=A0*N[Pi]/180`	*Command for the angle α [rad] $\alpha = 0.541052$*
`B=90-A0`	*Command for the angle β [°]*
`a=(k*Sin[A])/(1-Sin[A])`	*Command: length of the side* ***a***
`b=(k*Cos[A])/(1-Sin[A])`	*Command: length of the side* ***b***
`c=k/(1-Sin[A])`	*Command: length of the side* ***c***
`59`	*Result: measure of the angle β [°]*
`14.8353`	*Result: length of the side* ***a***
`24.6901`	*Result: length of the side* ***b***
`28.8043`	*Result: length of the side* ***c***

Numerical results: a = 14.835, b = 24.690, c = 28.804, β = 59°.

2.2. Case 2: Difference of two sides length c–a = k and a measure of the angle β

2.2.1. Theoretical analysis

➢ Let be a right triangle as shown on the figure 2.

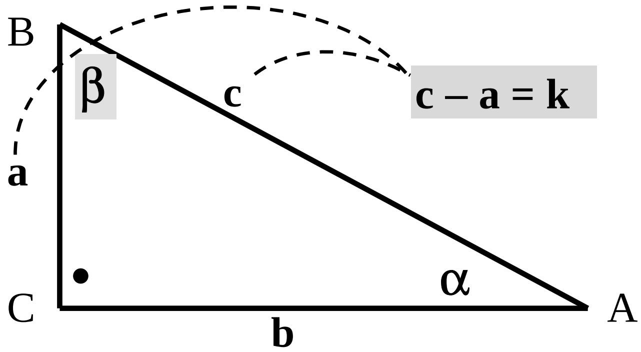

Fig. 2. Right triangle ABC with given a difference of the sides length c–a=k and a measure of the angle β

Source: Elaboration of the Authors

➢ *Data in ΔABC:* Difference of the sides length **c–a=k** (c>a) and a measure of the angle **β**.
➢ *Unknown in ΔABC:* Length of the sides **a, b**, **c** and a measure of the angle **α**.
➢ *Solution:*

♦ From some properties of acute angles in a right triangle we obtain:

$$\alpha = \frac{\pi}{2} - \beta . \tag{24}$$

♦ From the difference of the triangle sides we have:

$$c - a = k, \tag{25}$$

$$a = c - k. \tag{26}$$

♦ From the formulae (6) and (4) we obtain:

$$a = c \cdot \cos(\beta), \tag{27}$$

$$b = c \cdot \sin(\beta). \tag{28}$$

♦ From the formulae (26) and (27) we determine the triangle side c:

$$c - k = c \cdot \cos(\beta), \tag{29}$$

$$k = c \cdot [1 - \cos(\beta)], \tag{30}$$

$$c = \frac{k}{1 - \cos(\beta)}. \tag{31}$$

♦ Substituting (31) into the formulae (27) and (28) we determine the triangle sides a, b:

$$a = \frac{k \cdot \cos(\beta)}{1 - \cos(\beta)}, \tag{32}$$

$$b = \frac{k \cdot \sin(\beta)}{1 - \cos(\beta)}. \tag{33}$$

➢ *Answer:* $\alpha = \frac{\pi}{2} - \beta, \quad a = \frac{k \cdot \cos(\beta)}{1 - \cos(\beta)}, \quad b = \frac{k \cdot \sin(\beta)}{1 - \cos(\beta)}, \quad c = \frac{k}{1 - \cos(\beta)}.$

2.2.2. Numerical algorithms in *MS-Excel, MathCAD* and *Mathematica* programs

For numerical analysis we take a difference of the sides c–a = 13.9690 and the angle β = 59°.

♦ Program *MS-Excel 7.0 [3],[8]*

Algorithm:	*Commentary:*
`A6=`**`13,9690`**	*Given the difference of the sides c–a*
`A7=`**`59`**	*Given the measure of the angle β [°]*
`A8=A7*PI()/180`	*Command for the angle β [rad], β = 1.029744*
`A9=90-A7`	*Command for the angle α [°]*
`A10=(A6*COS(A8))/(1-COS(A8))`	*Command for calculation of the side **a***
`A11=(A6*SIN(A8))/(1-COS(A8))`	*Command for calculation of the side **b***
`A12=A6/(1-COS(A8))`	*Command for calculation of the side **c***
`31`	*Result: measure of the angle α [°]*
`14,83532`	*Result: length of the side **a***
`24,69012`	*Result: length of the side **b***
`28,80432`	*Result: length of the side **c***

♦ Program *MathCAD 15 [9],[12]*

Algorithm:	*Commentary:*
k:=**13,9690**	*Given the difference of the sides c–a*
β0:=**59**	*Given the measure of the angle β [°]*
$\beta := \frac{\beta 0 \cdot \pi}{180} = 1.029744$	*Command for the angle β [rad] β = 1.029744*
$\alpha 0 := 90 - \beta 0 = 31$	*Command for the angle α [°]*
$a := \frac{k \cdot \cos(\beta)}{1 - \cos(\beta)} = 14.835323$	*Command and result: length of the side **a***
$b := \frac{k \cdot \sin(\beta)}{1 - \cos(\beta)} = 24.690124$	*Command and result: length of the side **b***
$a := \frac{k}{1 - \cos(\beta)} = 28.804323$	*Command and result: length of the side **c***

♦ Program *Mathematica 7.0 [1],[2],[5]*

Algorithm:	*Commentary:*
`k:=`**`13,9690`**	*Given the difference of the sides c–a*
`B0:=`**`59`**	*Given the measure of the angle β [°]*
`B:=A0*N[Pi]/180`	*Command for the angle β [rad] β = 1.029744*
`A=90-B0`	*Command for the angle α [°]*
`a=(k*Cos[A])/(1-Cos[A])`	*Command: length of the side **a***
`b=(k*Sin[A])/(1-Cos[A])`	*Command: length of the side **b***
`c=k/(1-Cos[A])`	*Command: length of the side **c***
`31`	*Result: measure of the angle α [°]*
`14.8353`	*Result: length of the side **a***
`24.6901`	*Result: length of the side **b***
`28.8043`	*Result: length of the side **c***

Numerical results: a = 14.835, b = 24.690, c = 28.804, α = 31°.

2.3. Case 3: Difference of two sides length c–b = p and a measure of the angle α.

2.3.1. Theoretical analysis

➢ Let be a right triangle as shown on the figure 3.

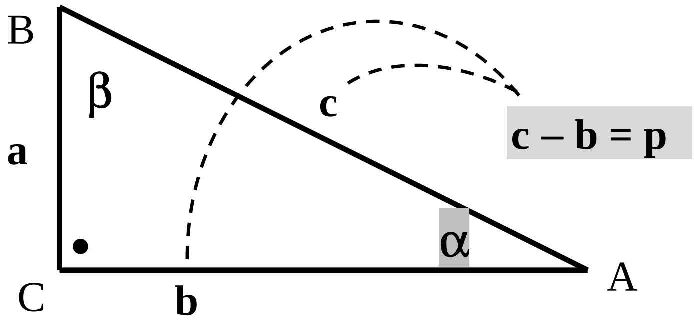

Fig. 3. Right triangle ABC with given difference of the sides length c–b=p and measure of the angle α

Source: Elaboration of the Authors

➢ *Data in ΔABC:* Difference of the sides length **c–b=p** (c>b) and a measure of the angle **α**.
➢ *Unknown in ΔABC:* Length of the sides **a, b**, **c** and a measure of the angle **β**.
➢ *Solution:*
♦ From some properties of acute angles in a right triangle we obtain:

$$\beta = \frac{\pi}{2} - \alpha\,. \tag{34}$$

♦ From the difference of the triangle sides we have:

$$c - b = p, \tag{35}$$

$$b = c - p. \tag{36}$$

♦ From the formulae (3) and (5) we obtain:

$$a = c \cdot \sin(\alpha), \tag{37}$$

$$b = c \cdot \cos(\alpha). \tag{38}$$

♦ From the formulae (36) and (38) we determine the triangle side c:

$$c - p = c \cdot \cos(\alpha), \tag{39}$$

$$p = c \cdot [1 - \cos(\alpha)], \tag{40}$$

$$c = \frac{p}{1 - \cos(\alpha)}. \tag{41}$$

♦ Substituting (41) into the formulae (37) and (38) we determine the triangle sides a, b:

$$a = \frac{p \cdot \sin(\alpha)}{1 - \cos(\alpha)} = \frac{p \cdot 2\sin(\frac{\alpha}{2})\cos(\frac{\alpha}{2})}{2\sin^2(\frac{\alpha}{2})} = \frac{p \cdot \cos(\frac{\alpha}{2})}{\sin(\frac{\alpha}{2})} \equiv p \cdot \text{ctg}\left(\frac{\alpha}{2}\right), \tag{42}$$

$$b = \frac{p \cdot \cos(\alpha)}{1 - \cos(\alpha)}. \tag{43}$$

➢ *Answer:* $\beta = 90° - \alpha, \quad a = \frac{p \cdot \sin(\alpha)}{1 - \cos(\alpha)} = p \cdot \text{ctg}\left(\frac{\alpha}{2}\right), \quad b = \frac{p \cdot \cos(\alpha)}{1 - \cos(\alpha)}, \quad c = \frac{p}{1 - \cos(\alpha)}.$

2.3.2. Numerical algorithms in *MS-Excel, MathCAD* and *Mathematica* programs

For numerical analysis we take a difference of the sides c–b = 4.1142 and the angle $\alpha = 31°$.

♦ Program *MS-Excel 7.0 [3],[8]*

Algorithm:	*Commentary:*
`A6=`**`4,1142`**	*Given the difference of the sides c–b*
`A7=`**`31`**	*Given the measure of the angle α [°]*
`A8=A7*PI()/180`	*Command for the angle α [rad], $\alpha = 0.541052$*
`A9=90-A7`	*Command for the angle β [°]*
`A10=(A6*SIN(A8))/(1-COS(A8))`	*Command for calculation of the side **a***
`A11=(A6*COS(A8))/(1-COS(A8))`	*Command for calculation of the side **b***
`A12=A6/(1-COS(A8))`	*Command for calculation of the side **c***
`59`	*Result: measure of the angle β [°]*
`14,83533`	*Result: length of the side **a***
`24,69013`	*Result: length of the side **b***
`28,80433`	*Result: length of the side **c***

♦ Program *MathCAD 15 [9],[12]*

Algorithm:	*Commentary:*
p:=**4,1142**	*Given the difference of the sides c–b*
α0:=**31**	*Given the measure of the angle α [°]*
$\alpha := \frac{\alpha 0 \cdot \pi}{180} = 0.541052$	*Command for the angle α [rad] $\alpha = 0.541052$*
$\beta 0 := 90 - \alpha 0 = 59$	*Command for the angle β [°]*
$a := \frac{p \cdot \sin(\alpha)}{1-\cos(\alpha)} = 14.835326$	*Command and result: length of the side **a***
$b := \frac{p \cdot \cos(\alpha)}{1-\cos(\alpha)} = 24.690129$	*Command and result: length of the side **b***
$a := \frac{p}{1-\cos(\alpha)} = 28.804329$	*Command and result: length of the side **c***

♦ Program *Mathematica 7.0 [1],[2],[5]*

Algorithm:	*Commentary:*
`p:=`**`4,1142`**	*Given the difference of the sides c–b*
`A0:=`**`31`**	*Given the measure of the angle α [°]*
`A:=A0*N[Pi]/180`	*Command for the angle α [rad] $\alpha = 0.541052$*
`B=90-A0`	*Command for the angle β [°]*
`a=(p*Sin[A])/(1-Cos[A])`	*Command: length of the side **a***
`b=(p*Cos[A])/(1-Cos[A])`	*Command: length of the side **b***
`c=p/(1-Cos[A])`	*Command: length of the side **c***
`59`	*Result: measure of the angle β [°]*
`14,835`	*Result: length of the side **a***
`24,690`	*Result: length of the side **b***
`28,804`	*Result: length of the side **c***

Numerical results: a = 14.835, b = 24.690, c = 28.804, $\beta = 59°$.

2.4. Case 4: Difference of two sides length c–b = p and a measure of the angle β

2.4.1. Theoretical analysis

- Let be a triangle as shown on the figure 4.

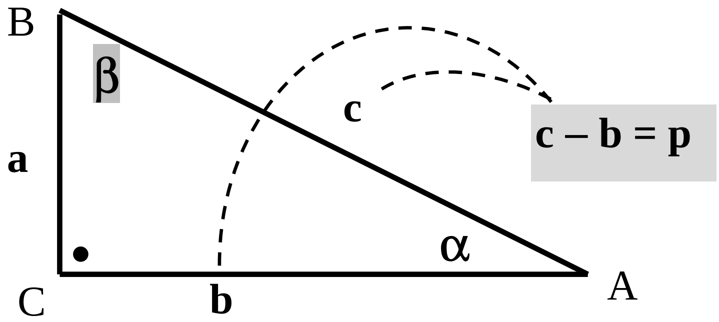

Fig. 4. Right triangle ABC with given a difference of the sides length c–b=p and a measure of the angle β

Source: Elaboration of the Authors

- *Data in ΔABC:* Difference of the sides length **c – b = p** (c>b) and a measure of the angle **β**.
- *Unknown in ΔABC:* Length of the sides **a, b**, **c** and a measure of the angle **α**.
- *Solution:*

♦ From some properties of acute angles in a right triangle we obtain:

$$\alpha = \frac{\pi}{2} - \beta. \tag{44}$$

♦ From the difference of the triangle sides we have:

$$c - b = p, \tag{45}$$

$$b = c - p. \tag{46}$$

♦ From the formulae (6) and (4) we obtain:

$$a = c \cdot \cos(\beta), \tag{47}$$

$$b = c \cdot \sin(\beta). \tag{48}$$

♦ From the formulae (46) and (48) we determine the triangle side c:

$$c - p = c \cdot \sin(\beta), \tag{49}$$

$$p = c \cdot [1 - \sin(\beta)], \tag{50}$$

$$c = \frac{p}{1 - \sin(\beta)}. \tag{51}$$

♦ Substituting (51) into the formulae (47) and (48) we determine the triangle sides a, b:

$$a = \frac{p \cdot \cos(\beta)}{1 - \sin(\beta)}, \tag{52}$$

$$b = \frac{p \cdot \sin(\beta)}{1 - \sin(\beta)}. \tag{53}$$

- *Answer:* $\alpha = \frac{\pi}{2} - \beta, \quad a = \frac{p \cdot \cos(\beta)}{1 - \sin(\beta)}, \quad b = \frac{p \cdot \sin(\beta)}{1 - \sin(\beta)}, \quad c = \frac{p}{1 - \sin(\beta)}.$

2.4.2. Numerical algorithms in *MS-Excel, MathCAD* and *Mathematica* programs

For numerical analysis we take a difference of the sides c–b = 4.1142 and the angle β = 59°.

♦ Program *MS-Excel 7.0 [3],[8]*

Algorithm:	*Commentary:*
`A6=`**`4,1142`**	*Given the difference of the sides c–b*
`A7=`**`59`**	*Given the measure of the angle β [°]*
`A8=A7*PI()/180`	*Command for the angle β [rad], β = 1.029744*
`A9=90-A7`	*Command for the angle α [°]*
`A10=(A6*COS(A8))/(1-SIN(A8))`	*Command for calculation of the side **a***
`A11=(A6*SIN(A8))/(1-SIN(A8))`	*Command for calculation of the side **b***
`A12=A6/(1-SIN(A8))`	*Command for calculation of the side **c***
`31`	*Result: measure of the angle α [°]*
`14,83533`	*Result: length of the side **a***
`24,69013`	*Result: length of the side **b***
`28,80433`	*Result: length of the side **c***

♦ Program *MathCAD 15 [9],[12]*

Algorithm:	*Commentary:*
p:=**4,1142**	*Given the difference of the sides c–b*
β0:=**59**	*Given the measure of the angle β [°]*
$\beta := \frac{\beta 0 \cdot \pi}{180} = 1.029744$	*Command for the angle β [rad] β = 1.029744*
$\alpha 0 := 90 - \beta 0 = 31$	*Command for the angle α [°]*
$a := \frac{p \cdot \cos(\beta)}{1-\sin(\beta)} = 14.835326$	*Command and result: length of the side **a***
$b := \frac{p \cdot \sin(\beta)}{1-\sin(\beta)} = 24.690129$	*Command and result: length of the side **b***
$a := \frac{p}{1-\sin(\beta)} = 28.804329$	*Command and result: length of the side **c***

♦ Program *Mathematica 7.0 [1],[2],[5]*

Algorithm:	*Commentary:*
`p:=`**`4,1142`**	*Given the difference of the sides c–b*
`A0:=`**`59`**	*Given the measure of the angle β [°]*
`A:=A0*N[π]/180`	*Command for the angle β [rad] β = 1.029744*
`B=90-A0`	*Command for the angle α [°]*
`a=(p*Cos[A])/(1-Sin[A])`	*Command: length of the side **a***
`b=(p*Sin[A])/(1-Sin[A])`	*Command: length of the side **b***
`c=p/(1-Sin[A])`	*Command: length of the side **c***
`31`	*Result: measure of the angle α [°]*
`14,8353`	*Result: length of the side **a***
`24,6901`	*Result: length of the side **b***
`28,8043`	*Result: length of the side **c***

Numerical results: a = 14.835, b = 24.690, c = 28.804, α = 31°.

2.5. Case 5: Difference of two sides length b–a = q and a measure of the angle α.

2.5.1. Theoretical analysis

➢ Let be a triangle as shown on the figure 5.

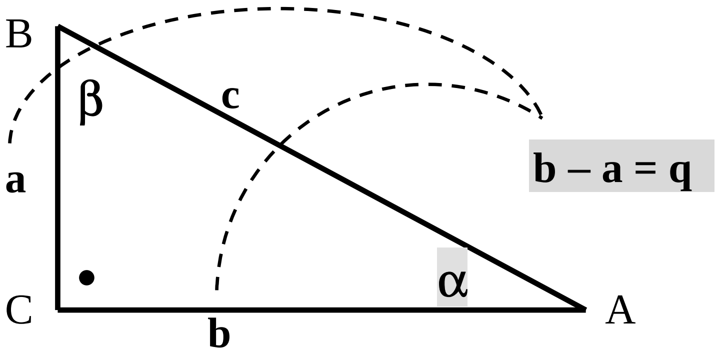

Fig. 5. Right triangle ABC with given a difference of the sides length b–a=q and a measure of the angle α
Source: Elaboration of the Authors

➢ *Data in ΔABC:* Difference of the sides length **b–a=q** (b>a) and a measure of the angle **α**.
➢ *Unknown in ΔABC:* Length of the sides **a, b**, **c** and a measure of the angle **β**.
➢ *Solution:*
♦ From some properties of acute angles in a right triangle we obtain:

$$\beta = \frac{\pi}{2} - \alpha . \tag{54}$$

♦ From the difference of the triangle sides we have:

$$b - a = q, \tag{55}$$

$$a = b - q. \tag{56}$$

♦ From the formulae (7) i (9) we obtain:

$$a = b \cdot tg(\alpha), \tag{57}$$

$$b = c \cdot \cos(\alpha). \tag{58}$$

♦ From the formula (58) we determine the triangle side c:

$$c = \frac{b}{\cos(\alpha)}. \tag{59}$$

♦ From the formula (56) and (57) we determine the triangle side b:

$$b - q = b \cdot tg(\alpha), \tag{60}$$

$$q = b \cdot [1 - tg(\alpha)], \tag{61}$$

$$b = \frac{q}{1 - tg(\alpha)}. \tag{62}$$

♦ From the formulae (57), (59) and (62) we finally determine the triangle side a and c:

$$a = \frac{q \cdot tg(\alpha)}{1 - tg(\alpha)}, \tag{63}$$

$$c = \frac{q}{\cos(\alpha)[1 - tg(\alpha)]} \equiv \frac{q \cdot \sec(\alpha)}{1 - tg(\alpha)}. \tag{64}$$

➢ *Answer:* $\beta = \frac{\pi}{2} - \alpha$, $a = \frac{q \cdot tg(\alpha)}{1 - tg(\alpha)}$, $b = \frac{q}{1 - tg(\alpha)}$, $c = \frac{q \cdot \sec(\alpha)}{1 - tg(\alpha)}$.

2.5.2. Numerical algorithms in *MS-Excel, MathCAD* and *Mathematica* programs

For numerical analysis we take a difference of the sides b–a = 9.8548 and the angle $\alpha = 31°$.

♦ Program *MS-Excel 7.0 [3],[8]*

Algorithm:	*Commentary:*
`A6=`**`9,8548`**	*Given the difference of the sides b–a*
`A7=`**`31`**	*Given the measure of the angle α [°]*
`A8=A7*PI()/180`	*Command for the angle α [rad], $\alpha = 0.541052$*
`A9=90-A7`	*Command for the angle β [°]*
`A10=(A6*TAN(A8))/(1-TAN(A8))`	*Command for calculation of the side **a***
`A11=A6/(1-TAN(A8))`	*Command for calculation of the side **b***
`A12=A6/(COS(A8)*(1-TAN(A8)))`	*Command for calculation of the side **c***
`59`	*Result: measure of the angle β [°]*
`14,83532`	*Result: length of the side **a***
`24,69012`	*Result: length of the side **b***
`28,80432`	*Result: length of the side **c***

♦ Program *MathCAD 15 [9],[12]*

Algorithm:	*Commentary:*
q:=**`9,8548`**	*Given the difference of the sides b–a*
α0:=**`31`**	*Given the measure of the angle α [°]*
$\alpha := \frac{\alpha 0 \cdot \pi}{180} = 0.541052$	*Command for the angle α [rad] $\alpha = 0.541052$*
$\beta 0 := 90 - \alpha 0 = 59$	*Command for the angle β [°]*
$a := \frac{q \cdot \tan(\alpha)}{1 - \tan(\alpha)} = 14.835322$	*Command and result: length of the side **a***
$b := \frac{q}{1 - \tan(\alpha)} = 24.690122$	*Command and result: length of the side **b***
$a := \frac{q \cdot \sec(\alpha)}{1 - \tan(\alpha)} = 28.804321$	*Command and result: length of the side **c***

♦ Program *Mathematica 7.0 [1],[2],[5]*

Algorithm:	*Commentary:*
`q:=`**`9,8548`**	*Given the difference of the sides b–a*
`A0:=`**`31`**	*Given the measure of the angle α [°]*
`A:=A0*N[Pi]/180`	*Command for the angle α [rad] $\alpha = 0.541052$*
`B=90-A0`	*Command for the angle β [°]*
`a=(q*Tan[A])/(1-Tan[A])`	*Command: length of the side **a***
`b=q/(1-Tan[A])`	*Command: length of the side **b***
`c=(q*Sec[A])/(1-Tan[A])`	*Command: length of the side **c***
`59`	*Result: measure of the angle β [°]*
`14,8353`	*Result: length of the side **a***
`24,6901`	*Result: length of the side **b***
`28,8043`	*Result: length of the side **c***

Numerical results: a = 14.835, b = 24.690, c = 28.803, $\beta = 59°$.

2.6. Case 6: Difference of two sides length b–a = q and a measure of the angle β
2.6.1. Theoretical analysis

- Let be a triangle as shown on the figure 6.

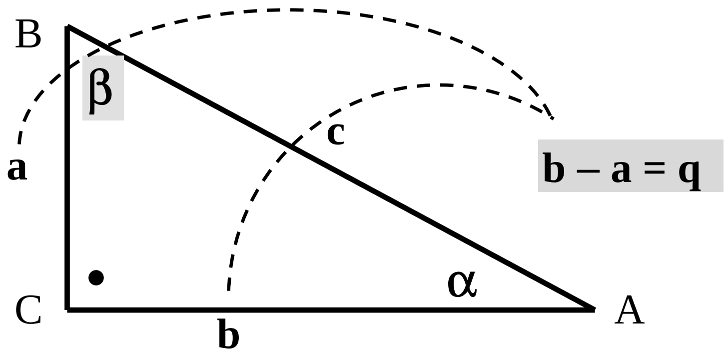

Fig. 6. Right triangle ABC with given a difference of the sides length b–a=q and a measure of the angle β
Source: Elaboration of the Authors

- *Data in ΔABC:* Difference of the sides length **b–a=q** (b>a) and a measure of the angle **β**.
- *Unknown in ΔABC:* Length of the sides **a, b**, **c** and a measure of the angle **α**.
- *Solution:*

♦ From some properties of acute angles in a right triangle we obtain:

$$\alpha = \frac{\pi}{2} - \beta \,. \tag{65}$$

♦ From the difference of the triangle sides we have:

$$b - a = q, \tag{66}$$

$$a = b - q. \tag{67}$$

♦ From the formulae (10) i (4) we obtain:

$$a = b \cdot \mathrm{ctg}(\beta), \tag{68}$$

$$b = c \cdot \sin(\beta). \tag{69}$$

♦ From the formula (69) we determine the triangle side c:

$$c = \frac{b}{\sin(\beta)}. \tag{70}$$

♦ From the formula (67) and (68) we determine the triangle side b:

$$b - q = b \cdot \mathrm{ctg}(\beta), \tag{71}$$

$$q = b \cdot [1 - \mathrm{ctg}(\beta)], \tag{72}$$

$$b = \frac{q}{1 - \mathrm{ctg}(\beta)}. \tag{73}$$

♦ From the formulae (68), (70) and (73) we finally determine the triangle side a and c:

$$a = \frac{q \cdot \mathrm{ctg}(\beta)}{1 - \mathrm{ctg}(\beta)}, \tag{74}$$

$$c = \frac{q}{\sin(\beta)[1 - \mathrm{ctg}(\beta)]} \equiv \frac{q \cdot \mathrm{cosec}(\beta)}{1 - \mathrm{ctg}(\beta)}. \tag{75}$$

- *Answer:* $\alpha = \frac{\pi}{2} - \beta$, $a = \frac{q \cdot \mathrm{ctg}(\beta)}{1 - \mathrm{ctg}(\beta)}$, $b = \frac{q}{1 - \mathrm{ctg}(\beta)}$, $c = \frac{q \cdot \mathrm{cosec}(\beta)}{1 - \mathrm{ctg}(\beta)}$.

2.6.2. Numerical algorithms in *MS-Excel, MathCAD* and *Mathematica* programs

For numerical analysis we take a difference of the sides b–a = 9.8548 and the angle β = 59°.

♦ Program *MS-Excel 7.0 [3],[8]*

Algorithm:	*Commentary:*
`A6=`**`9,8548`**	*Given the difference of the sides b–a*
`A7=`**`59`**	*Given the measure of the angle β [°]*
`A8=A7*PI()/180`	*Command for the angle β [rad], β = 1,030*
`A9=90-A7`	*Command for the angle α [°]*
`A10=(A6/TAN(A8))/(1-(1/TAN(A8)))`	*Command for calculation of the side **a***
`A11=A6/(1-(1/TAN(A8)))`	*Command for calculation of the side **b***
`A12=A6/((SIN(A8))*(1-TAN(A8))))`	*Command for calculation of the side **c***
`31`	*Result: measure of the angle β [°]*
`14,83532`	*Result: length of the side **a***
`24,69012`	*Result: length of the side **b***
`28,80432`	*Result: length of the side **c***

♦ Program *MathCAD 15 [9],[12]*

Algorithm:	*Commentary:*
q:=**9,8548**	*Given the difference of the sides b–a*
β0:=**59**	*Given the measure of the angle β [°]*
$\beta := \frac{\beta 0 \cdot \pi}{180} = 1.029744$	*Command for the angle β [rad]*
$\alpha 0 := 90 - \beta 0 = 31$	*Command for the angle α [°]*
$a := \frac{q \cdot \cot(\beta)}{1-\cot(\beta)} = 14.835322$	*Command and result: length of the side **a***
$b := \frac{q}{1-\cot(\beta)} = 24.690321$	*Command and result: length of the side **b***
$a := \frac{q}{\sin(\beta)\cdot(1-\cot(\beta))} = 28.804425$	*Command and result: length of the side **c***

♦ Program *Mathematica 7.0 [1],[2],[5]*

Algorithm:	*Commentary:*
`q:=`**`9,8548`**	*Given the difference of the sides b–a*
`A0:=`**`59`**	*Given the measure of the angle β [°]*
`A:=A0*N[Pi]/180`	*Command for the angle β [rad] β = 1.02974*
`B=90-A0`	*Command for the angle α [°]*
`a=(q*Cot[A])/(1-Cot[A])`	*Command: length of the side **a***
`b=q/(1-Cot[A])`	*Command: length of the side **b***
`c=q/((Sin[A])*(1-Cot[A]))`	*Command: length of the side **c***
`31`	*Result: measure of the angle α [°]*
`14,8353`	*Result: length of the side **a***
`24,6901`	*Result: length of the side **b***
`28,8043`	*Result: length of the side **c***

Numerical results: a = 14.835, b = 24.690, c = 28.804, α = 31°.

3. Conclusions

- Discussed analytical models of solving right triangles allow to perform the other theoretical considerations in each of the six basic cases (i.e. for difference of sides), taking into account the definition of trigonometric functions and the corresponding properties and formulae.
- Created numerical algorithms of solving the right triangles in *MS-Excel, MathCAD,* and *Mathematica* programs allow for significant execution of faster calculations than the traditional way of using logarithms and logarithmic tables.

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