Applied multiphase level set function in image segmentation

Tomasz Rymarczyk, Stefan F. Filipowicz, Jan Sikora
Electrical Engineering Institute
04-703 Warszawa, ul. Pożarskiego 28, e-mail: tomasz@rymarczyk.com

The application of the level set function for the image segmentation was presented in this paper. The image segmentation refers to the process of partitioning a digital image into multiple regions. There is typically used to locate objects and boundaries in images. The level set method is a powerful tool for representing moving or stationary interfaces. There was used the idea of the variational formulation for geometric active contours. There was used to minimization problem in image processing to compute piecewise-smooth optimal approximations of the given image. The proposed algorithm has been applied to real pictures with promising results in the image segmentation.

1. Introduction

This paper presents the applications of the level set function for the image segmentation. The level set idea, devised in Osher and Sethian [5], is known to be a powerful and versatile tool to model evolution of interfaces [4, 6, 7, 8].The original idea behind the level set method was a simple one. Given an interface Γ in R^n of dimension one, bounding an open region Ω. It was analyzed and computed its subsequent motion under a velocity field **v**. This velocity can depend on position, time, the geometry of the interface (e.g. its normal or its mean curvature) and the external physical conditions. Level Set Methods is the numerical technique which can follow the evolution of interfaces. These interfaces can develop sharp corners, break apart, and merge together. The variational formulation for geometric active contours forces the level set function to be close to a signed distance function [1, 2]. This idea was used to minimization problem in image processing. For more than two phases were introduced the Mumford-Shah model [3, 9].

2. Level set method

The level set method tracks the motion of an interface by embedding the interface as the zero level set of the signed distance function. The motion of the interface is matched with the zero level set, and the resulting initial value partial differential equation for the evolution of the level set function. The idea is merely to define a smooth function $\phi(x,t)$, that represents the interface as the set where $\phi(x,t)=0$. The motion is analyzed by the convection the ϕ values (levels) with the velocity field. The Hamilton-Jacobi equation of the form [4]:

$$\frac{\partial \phi}{\partial t} + \mathbf{v}\,\nabla\phi = 0 \tag{1}$$

where **v** is the velocity on the interface.

When flat or steep regions complicate the determination of the contour, the reinitialization is necessary. This reinitialization procedure is based by replacing by another function that has the same zero level set but behaves better. This is based on following partial differential equation:

$$\frac{\partial}{\partial t}\phi + S(\phi)(\nabla\phi - 1) = 0 \tag{2}$$

where S(ϕ) is defined as:

$$S(\phi) = \begin{cases} -1 & \text{for } \phi < 0 \\ 0 & \text{for } \phi = 0 \\ 1 & \text{for } \phi > 0 \end{cases} \tag{3}$$

The representation of the level set function is shown on the Fig 1.

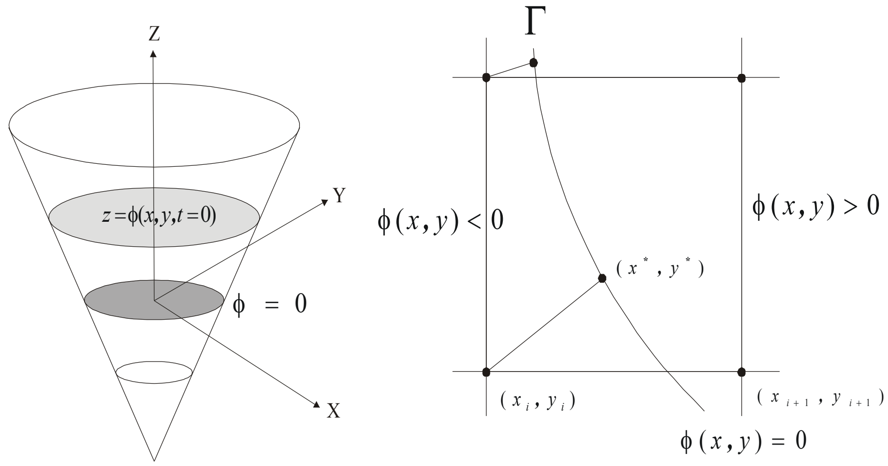

Fig. 1. The representation of the level set function

The numerical algorithm is following (Fig. 2):

- from the level set function (initial) at a time level t, find necessary interface information $\Gamma_0 = \{\phi(x, y) = 0\}$,
- calculate the velocity,
- extend velocity $\left|\phi^k\right| \le \delta$,
- update the level set function,
- reinitialization,
- check convergence.

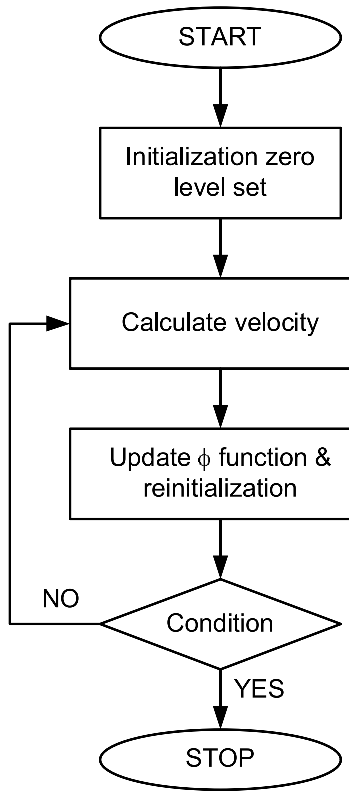

Fig. 2. The scheme of the algorithm – the level set method

Figure 3 present the image segmentation by using the level set numerical algorithm with reinitialization. The zero level set function was defined near edge of the image. It has the yellow colour on the image. The figures show the original image and the topological changes of the shape of the zero level set function after the 100, 200 and 500 iterations.

3. Mumford-shah model

For more than two phases was introduced the multiple level sets idea by Vese and Chan [9]. The algorithm set formulation and algorithm for the general Mumford-Shah minimization problem in image processing, to compute piecewise-smooth optimal approximations of a given image. The proposed model follows and fully generalizes works [3, 9], where there was proposed an active contour model without edges based on a 2-phase segmentation and level sets. The piecewise-constant segmentation of the image allows for more two segments using a new multi-phase level set formulation and partition of the image domain.

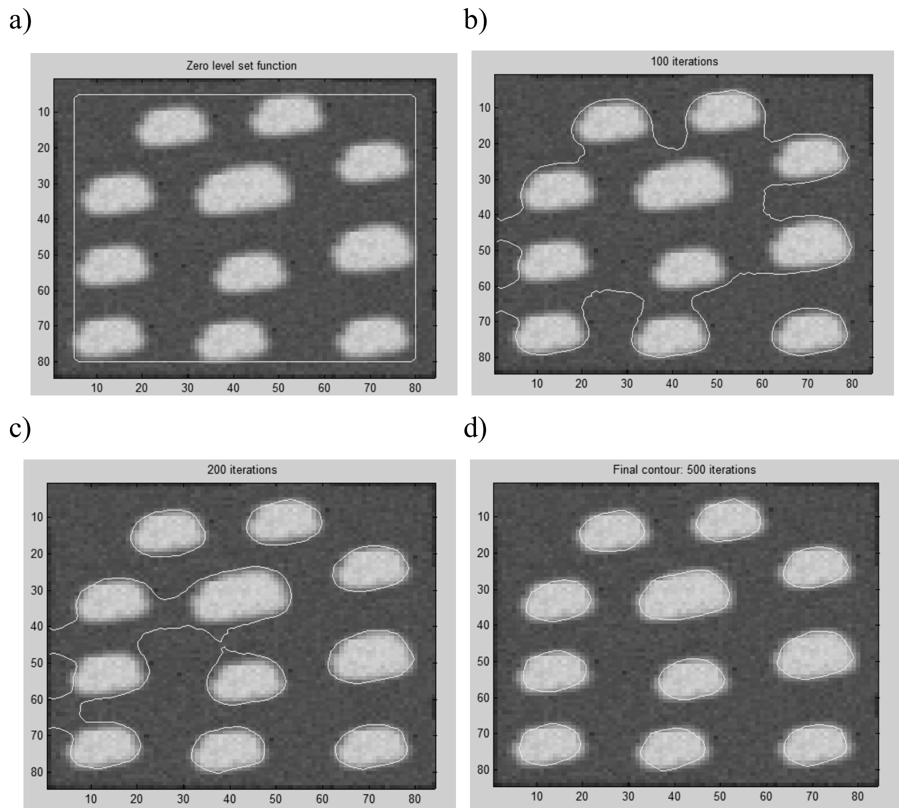

Fig. 3. The image reconstruction by the level set function: a) zero level set, b) after 100 iterations, c) after 200 iterations, d) after 500 iterations

For more than two phases was introduced the multiple level sets idea by Vese and Chan.

The algorithm sets a formulation and models for the general Mumford-Shah minimization problem in image processing, to compute piecewise-smooth optimal approximations of a given image. The problem can be easily generalized to the case where the domain contains more than two materials.

$$F(s,C) = \omega \boldsymbol{L}(C) + \eta \int_{\Omega} (\mathrm{s_o} - \mathrm{s})^2 \mathrm{d}\Omega + \int_{\Omega \setminus C} |\nabla \mathrm{s}|^2 \mathrm{d}\Omega \tag{4}$$

Coefficients c_1 i c_2 are mean values of points in the picture:

$$c_1 = \frac{\int_{\Omega} s_o H(\phi) d\Omega}{\int_{\Omega} H(\phi) d\Omega}, \quad c_2 = \frac{\int_{\Omega} s_o (1 - H(\phi)) d\Omega}{\int_{\Omega} (1 - H(\phi)) d\Omega} \tag{5}$$

The material c is representing following:

$$\mathbf{c} = c_1 H(\phi) + c_2 (1 - H(\phi)) \tag{6}$$

where H is the Heaviside function

$$H = 1 \text{ for } x \geq 0$$
$$H = 0 \text{ for } x < 0$$

The functional can be written as:

$$F(\phi, c_1, c_2) = \omega \int_\Omega |\nabla H(\phi)| d\Omega + \eta_1 \int_\Omega (s_o - c_1)^2 H(\phi) d\Omega + \eta_2 \int_\Omega (s_o - c_2)^2 (1 - H(\phi)) d\Omega \tag{7}$$

The process for minimization of the functional is the following:

$$\frac{\partial \phi}{\partial t} = \delta_\varepsilon(\phi) \left[\omega \nabla \cdot \left(\frac{\nabla \phi}{|\nabla \phi|} \right) - \eta_1 (s_o - c_1)^2 + \eta_2 (s_o - c_2)^2 \right] \tag{8}$$

The numerical algorithm is following (Fig. 5):

- from the level set function (initial) at a time level t, find necessary interface information,
- calculate coefficients c_1, c_2,
- update the level set function,
- reinitialization,
- check convergence.

The level set methods have natural flexibility to create various shapes. Model of the object is shown on the Figure 4. The Figures 6 presents the image segmentation by using the level set numerical algorithm with the Mumford-Shah model. The images show the original image and reconstruction after the 200 iterations. In the example was used the image frame as initial condition for the active contour model. The final contours have the red colour on the original image. They represent the zero value of the level set function. The segmentation gives good results, because the region borders accurately locating the object edges. An increasing numbers of iteration the quantitative results are better.

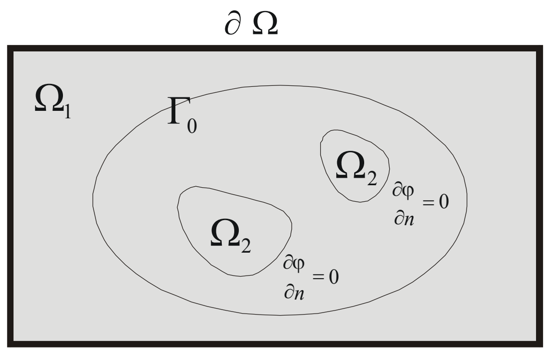

Fig. 4. Model of the object

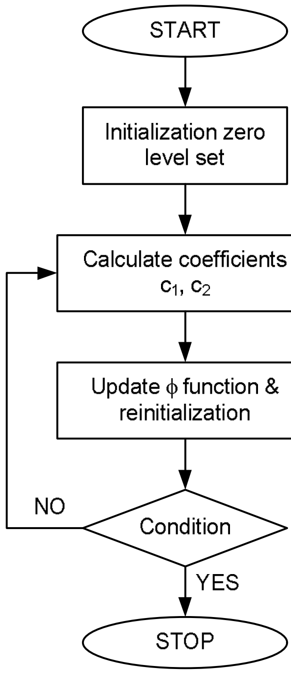

Fig. 5. The scheme of the algorithm – the level set method

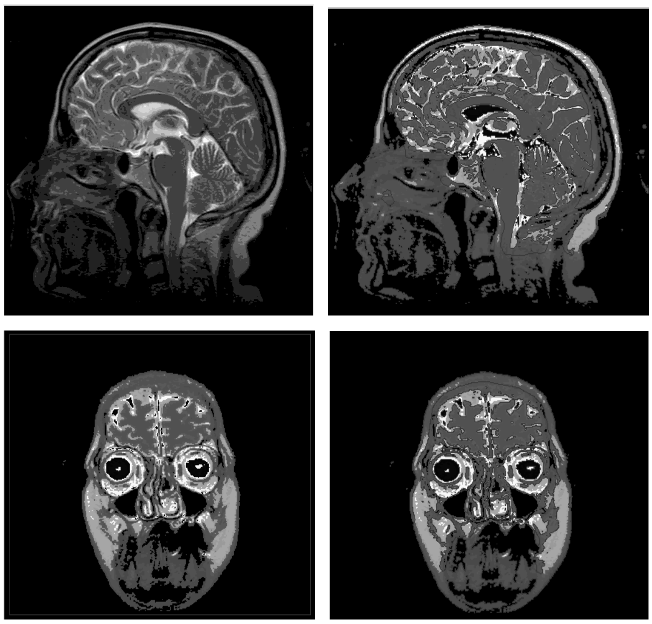

Fig. 6. Image segmentation - the original images and the reconstructions after the 200 iterations

4. Variational level set method

The formulation of the variational level set method consists of an internal energy term that penalizes the deviation of the level set function and an external energy term that drives the motion of the zero level set toward the desired image features. When flat or steep regions complicate the determination of the contour, the reinitialization is necessary. This reinitialization procedure is based by replacing by another function that has the same zero level set but behaves better. Variational formulation for geometric active contours that forces the level set function to be close to a signed distance function, and therefore completely eliminates the need of the costly reinitialization procedure.

The resulting evolution of the level set function is the gradient flow that minimizes the overall energy functional [2]:

$$\boldsymbol{P}(\phi) = \int_{\Omega} \frac{1}{2}(|\nabla\phi| - 1)^2 \, \mathrm{d}x\mathrm{d}y \tag{9}$$

An external energy for a function $\phi(x, y)$ is defined as below:

$$\boldsymbol{E}(\phi) = \mu \boldsymbol{P}(\phi) + \boldsymbol{E}_m(\phi) \tag{10}$$

where: $\boldsymbol{P}(\phi)$ – internal energy, $\boldsymbol{E}_m(\phi)$ – external energy.

Denoting by $\frac{\partial \mathbf{E}}{\partial \phi}$ the Gateaux derivative of the functional **E** receiving the following evolution equation:

$$\frac{\partial \phi}{\partial t} = -\frac{\partial \mathbf{E}}{\partial \phi} \tag{11}$$

In the image segmentation active contours are dynamic curves that moves towards the object boundaries. Denoting letter I as an image, and g be the edge indicator function defined by:

$$g = \frac{1}{1 + |\nabla G_\sigma * I|^2} \tag{12}$$

where G_σ is the Gaussian kernel with standard deviation σ.

Total energy functional is defined by:

$$\boldsymbol{E}_{g,\lambda,v}(\phi) = \lambda \boldsymbol{L}_g(\phi) + \omega \boldsymbol{A}_g(\phi) \tag{13}$$

where λ and ω are constant, $\boldsymbol{L}_g(\phi)$ and $\boldsymbol{A}_g(\phi)$ are defined by:

$$\boldsymbol{L}_g(\phi) = \int_{\Omega} g\delta(\phi)|\nabla\phi| \mathrm{d}x\mathrm{d}y \tag{14}$$

$$\boldsymbol{A}_g(\phi) = \int_{\Omega} g\mathrm{H}(-\phi)\mathrm{d}x\mathrm{d}y \tag{15}$$

The functional ***E*** can be written as:

$$\frac{\partial \boldsymbol{E}}{\partial \phi} = -\mu\left[\Delta\phi - \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\right] - \lambda\delta(\phi)\operatorname{div}\left(g\frac{\nabla\phi}{|\nabla\phi|}\right) - \omega g\delta(\phi) \tag{16}$$

where Δ is Laplace operator.

The process for minimization of the functional ***E*** is the following:

$$\frac{\partial \phi}{\partial t} = \mu\left[\Delta\phi - \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\right] + \lambda\delta(\phi)\operatorname{div}\left(g\frac{\nabla\phi}{|\nabla\phi|}\right) + \omega g\delta(\phi) \tag{17}$$

The numerical algorithm is following (Fig. 7):

- from the level set function (initial) at a time level t, find necessary interface information,
- calculate H & δ,
- update the level set function,
- reinitialization,
- check convergence.

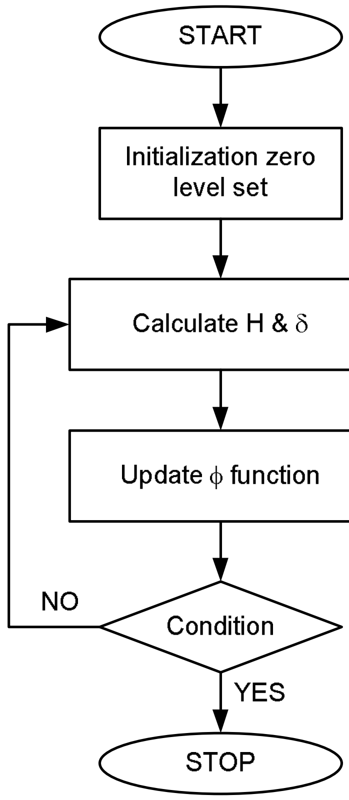

Fig. 7. The scheme of the algorithm – the level set method

Fig. 8. The image segmentation - the original image and reconstruction after the 200 iterations

Figure 8 presents the image segmentation in the following iterative process. The algorithm of the image reconstruction consists the variational level set method. The zero level set function was defined the near edge of the image (6 pixels). The final contours have the red colour on the original images. These contours represent the zero value of the level set function. The segmentation gives good results, because the region borders accurately locating the object edges. The process reconstruction is finished after the 200 iterations.

5. Variational level set method with Mumford-Shah model

An energy for a function $\phi(x, y)$ is defined as below:

$$\boldsymbol{E}(\phi) = \mu\, \boldsymbol{P}(\phi) + \boldsymbol{E}_{\varepsilon}(\phi, c_1, c_2) \tag{18}$$

The functional can be written as:

$$F(u, C) = \mu\, \boldsymbol{P}(\phi) + \omega\, \boldsymbol{L}(C) + \eta \int_{\Omega} (u_o - u)^2 d\Omega + \int_{\Omega \setminus C} |\nabla u|^2 d\Omega \tag{19}$$

The formulation of the variational level set method with Mumford-Shah model is following:

$$F(\phi, c_1, c_2) = \int_\Omega \frac{1}{2}(|\nabla\phi| - 1)^2 \, dxdy + \omega \int_\Omega |\nabla H(\phi)| d\Omega + \lambda_1 \int_\Omega (u_o - c_1)^2 H(\phi) d\Omega + \lambda_2 \int_\Omega (u_o - c_2)^2 (1 - H(\phi)) d\Omega \tag{20}$$

The process for minimization of the functional E is the following:

$$\frac{\partial\phi}{\partial t} = \mu\left[\Delta\phi - \mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\right] + \delta_\varepsilon(\phi)\left[\omega\nabla \cdot \left(\frac{\nabla\phi}{|\nabla\phi|}\right) - \eta_1 (u_o - c_1)^2 + \eta_2 (u_o - c_2)^2\right] \tag{21}$$

The numerical algorithm (Fig. 9):

- from the level set function (initial) at a time level t, find necessary interface information,
- calculate coefficients c_1, c_2,
- update the level set function,
- reinitialization,
- check convergence.

Figure 10 presents the roentgen images segmentation in the iterative process. The algorithm of the image reconstruction consists variational level set method and the Mumford-Shah function.

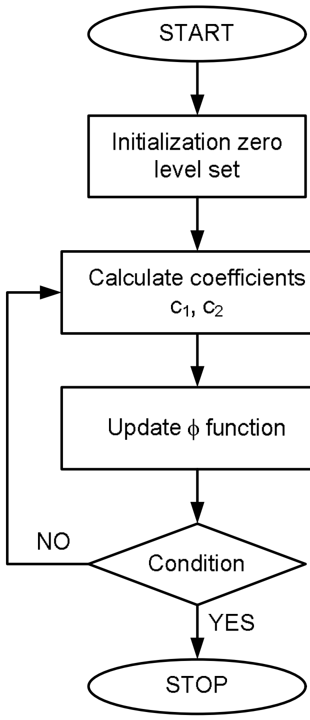

Fig. 9. The scheme of the algorithm – the level set method

a)
b)
c)
d)

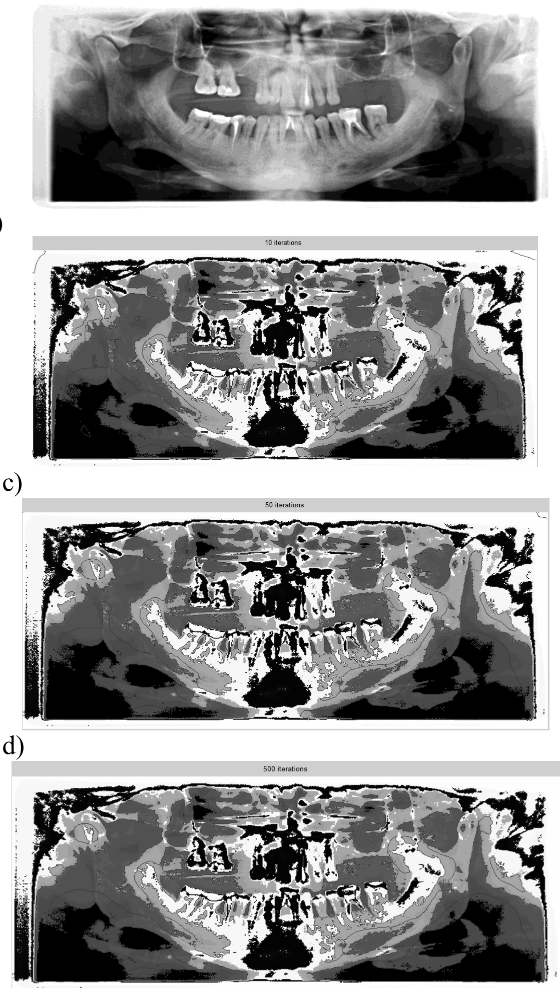

Fig. 10. The image segmentation: a) the oryginal image, b) after 10 iterations, c) after 50 iterations, d) after 500 iterations

6. Conclusion

The applications of the level set function for image segmentation was presented in this paper. The level set idea is known to be a powerful and versatile tool to model evolution of interfaces. The piecewise-constant segmentation of the image allows for more two segments. Variational formulation for geometric active contours that forces the level set function to be close to a signed distance function. The proposed algorithms have been used to real pictures with promising results in the roentgen images segmentation.

References

[1] Rymarczyk T., Filipowicz S.F., Sikora J., Tymburski M.: Variational Level Set Methods in the Roentgen Images Segmentation. COMPUMAG 2009, Florianópolis, Brasil, November 22-26, 2009.

[2] Li C., Xu C., Gui C., and M. D. Fox., "Level set evolution without re-initialization: A new variational formulation", In IEEE Conference on Computer Vision and Pattern Recognition (CVPR), volume 1, pages 430–436, 2005.

[3] Mumford D., Shah J.: Optimal approximation by piecewise smooth functions and associated variational problems. Comm. Pure Appl. Math., (42):577–685, 1989.

[4] Osher S., Fedkiw R.: Level Set Methods and Dynamic Implicit Surfaces. Springer, New York 2003.

[5] Osher S., Sethian J.A.: Fronts Propagating with Curvature Dependent Speed: Algorithms Based on Hamilton-Jacobi Formulations. J. Comput. Phys. 79, 12-49, 1988.

[6] Osher, S., Fedkiw, R.: Level Set Methods: An Overview and Some Recent Results. J. Comput. Phys. 169, 463-502, 2001.

[7] Osher S., Santosa F.: Level set methods for optimization problems involving geometry and constraints. Frequencies of a two-density inhomogeneous drum. Journal of Computational Physics, 171, pp. 272-288, 2001.

[8] Sethian J.A.: Level Set Methods and Fast Marching Methods. Cambridge Univeristy Press 1999.

[9] Vese L. Chan T.: A new multiphase level set framework for image segmentation via the Mumford and Shah model. CAM Report 01-25, UCLA Math. Dept., 2001.