rotational system; nonlinear differential system of equation;
Galerkin method; critical states; analytical solutions

Bogumił CHILIŃSKI
Warsaw University of Technology, Faculty of Automotive and Construction Machinery Engineering
Narbutta 84, 02-524 Warsaw, Poland
Corresponding author. E-mail: bogumil.chilinski@gmail.com

ANALYSIS OF DISTURBANCE TORQUE INFLUENCE ON CRITICAL STATE IN ROTATIONAL SYSTEMS

Summary. Currently most of existing means of transport contains different types of rotational systems. In many cases the dynamics of such rotors substantially can influence exploitation of the whole vehicle. Moreover, in order to minimize mass of the whole object modern construction materials are applied. This causes that the dynamic phenomena may be fundamental of exploitation. The paper presents preliminary analysis of disturbance torque influence on critical state in rotational system. The consideration assumed simple physical object in the form of heavy disk embedded on weightless, elastic shaft. The shaft was supported on two bearings. In particular chapters of paper, path leading from proposition of physical model, by solution of it, to qualitative conclusions about considered object and torque disturbances influence of motion of this system, was presented. In introduction, outline of considered problem and potential opportunities of it, were demonstrated. In the next chapter, physical and mathematical model of the analysed object, was described. Next and also the last but one chapter gives a detailed discussion of mathematical model in the form of nonlinear ordinary differential equations proposed earlier. The first part of the chapter presents the possibility to solve such a problem, then it shows the simplifications which are used. Furthermore, the influence of used simplifications on the shape of analysed problem was demonstrated. Additionally, the possibility of equations solution presented in the paper was discussed. Moreover, the series of interesting properties of analysed system of equations has been shown based on founded approximate solutions. The whole paper was summarized with plans for future work and synthetic conclusions concerning the innovative control method of critical states.

ANALIZA WPŁYWU ZABURZENIA MOMENTU SKRĘCAJĄCEGO NA STANY KRYTYCZNE UKŁADÓW WIRUJĄCYCH

Streszczenie. Aktualnie większość istniejących środków transportu zawiera różnego typu układy wirujące. W wielu przypadkach dynamika takich wirników w istotny sposób wpływa na eksploatacje całego pojazdu. Ponadto w celu zminimalizowania masy całego obiektu stosuje się nowoczesne materiały konstrukcyjne. To powoduje, że zjawiska dynamiczne, mogą mieć podstawowe znaczenie eksploatacyjne. Artykuł przedstawia wstępną analizę wpływu zaburzenia momentu skręcającego na stany krytyczne układu wirującego. Do rozważań przyjęto prosty obiekt fizyczny w postaci ciężkiego krążka osadzonego na nieważkim podatnym wale podpartym w dwóch łożyskach.

W poszczególnych rozdziałach artykułu przedstawiono drogę prowadzącą od zaproponowania modelu fizycznego, przez jego rozwiązanie do jakościowych wniosków o rozpatrywanym obiekcie i wpływie zaburzenia momentu skręcającego na jego ruch. We wstępie przedstawiono zarys rozpatrywanego problemu oraz potencjalne możliwości wykorzystania opracowanego zagadnienia. W kolejnym rozdziale zaprezentowano model fizyczny oraz matematyczny dla obiektu będącego podstawą niniejszego artykułu. Następny i zarazem przedostatni rozdział w sposób szczegółowy przedstawia dyskusję zaproponowanego wcześniej modelu matematycznego w postaci układu nieliniowych równań różniczkowych zwyczajnych. Na początku rozdziału przedstawiono rozwiązywalność takiego zagadnienia, następnie opisano zastosowane uproszczenia. W dalszym ciągu zademonstrowano wpływ zastosowanych uproszczeń na kształt analizowanych równań. Dodatkowo podjęto dyskusję o możliwości rozwiązania równań przedstawionych w pracy. Ponadto na podstawie znalezionych przybliżonych rozwiązań wykazano szereg ciekawych własności analizowanego układu równań. Całość została podsumowana celami dalszej pracy oraz syntetycznymi wnioskami, dotyczącymi propozycji innowacyjnej metody sterowania stanami krytycznymi.

1. INTRODUCTION

Almost all means of transport contain rotating elements. In many cases the rotors are long and relatively thin. It means the dynamics of the system may impede the normal exploitation of the technical objects. There are many problems associated with the movement of rotating systems. Each of these problems can significantly affect the exploitation vehicle.

The basic dynamic problem, which exists during the rotor work, is the phenomenon of the movement stability loss. The analysis of the simplest rotor model shows that in certain neighborhood of critical frequency ω_θ there is a sudden increase of bending vibration amplitude. After crossing this frequency (shaft work in supercritical state) bending oscillation decreases. The phenomenon of bending vibration decrease is called the self-centering of shaft [1].

From the design point of view, the shaft self-centering is very valuable because of unavoidable assembly errors, which are a direct reason for stability loss. There is a justified need for the slender shafts to work in supercritical state.

Unfortunately, work with supercritical speed implies a serious exploiting problem, so called problem of transition via critical speed. The simplest shaft model with vibrating mass does not allow to suggest any other technique of passing by via critical state. Thus, it is necessary to analyze a more precise model to find the critical state control technique of the analyzed system.

Solutions existing in literature [1] and their real constructional application indicates that the method of transition via endangered zone is based on big enough angular acceleration of rotor during the system starting. The only condition is that bending oscillation during the transition process cannot exceed the limiting value resulting from strength or exploiting constraints.

It is obvious that it is not always possible to reach sufficient acceleration so that any rotational system (i.e. with infinite mass moment of inertia) could easily pass by the critical state. It means that there is a certain group of systems for which the work in supercritical state is impossible due to the existence of physical limitation e.g. constructional or energetic.

Thus, the proposition of critical state control technique, different than the acceleration in endangered zone, is justified in the perspective of shaft exploitation. For this purpose, it is possible to use solutions of models available in literature in order to find new methods of critical speed transition.

2. MODELING OF THE SYSTEM

Typical shaft with one rotor can be modeled as the system presented in Fig. 1.

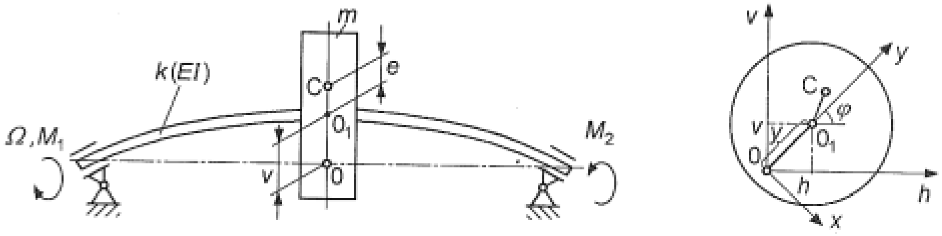

Fig. 1. Shaft model
Rys. 1. Model wału

This model consists of rotor, which has the mass m and the mass moment of inertia I, and the rotor mass centre in eccentric e. The shaft is transversely elastic and torsionally rigid. Transverse elasticity of shaft can be shown as the function $k(\cdot)$, which has Taylor series expansion.

The analyzed system is described as nonlinear second-order differential equations:

$$m \cdot \ddot{h} - m \cdot e \cdot \sin\phi \cdot \ddot{\phi} - m \cdot e \cdot \cos\phi \cdot \dot{\phi}^2 + k(h) \cdot h = 0\,, \tag{1}$$

$$m \cdot \ddot{v} + m \cdot e \cdot \cos\phi \cdot \ddot{\phi} - m \cdot e \cdot \sin\phi \cdot \dot{\phi}^2 + k(v) \cdot v = 0\,, \tag{2}$$

$$(I + m \cdot e^2) \cdot \ddot{\phi} - m \cdot e \cdot \ddot{h} \cdot \sin\phi + m \cdot e \cdot \ddot{v} \cdot \cos\phi = \Delta M(t)\,. \tag{3}$$

3. DISSCUSION OF THE EQUATIONS SYSTEM SOLUTION

Typical shaft with one rotor can be modeled as the system presented in Fig. 1. The nature of a typical shaft work puts some limitations on rotational motion. It is good to present the angular position of shaft described by the function of time $\phi(t)$ in the form of the main motion and its disturbance sum. The use of this form is justified by the simplicity of the formulation of constraints, which were defined earlier. Thus, the map $\phi(t)$ can be written as:

$$\phi(t) = \psi(t) + \theta(t)\,, \tag{4}$$

where: $\psi(t)$ – main motion, $\theta(t)$ – motion disturbance described by the period function.

The quantity described by the function $\psi(t)$ represents global and expected character of motion. In case of shaft work, the angular position constantly increases. It corresponds to the situation when the rotor accelerates (the angle increases with positive acceleration), after that, it reaches steady state of work (the angle increases linearly), to make the rotor stop in the end (the angle increases with negative acceleration). It means that in a typical shaft work the quantity $\psi(t)$ constantly increases with variables acceleration in time – it is a one-way motion (in which the velocity has a constant sign). Therefore, quantity $\psi(t)$ should be described by the weakly monotonic function of time.

The second element of the dependency (4) defined as the function $\theta(t)$ presents the disturbance of angular motion shaft. The reason for motion disturbances is mainly conjugate with other system degrees of freedom or significant torsional shaft elasticity. It is obvious that from the perspective of observing typical shaft exploitation, the impact of the disturbance $\theta(t)$ on the main motion $\psi(t)$ must be "small". Angular motion disturbance is described by upper and lower limited function. Otherwise, there would be the possibility of unlimited function increase $\psi(t)$. Periodic maps, which have Fourier series expansion (i.e. such periodic functions satisfy the conditions which guarantee series convergence – Dirichlet conditions [1 - 3]), are a wide class of the limited function. For this reason, it is assumed that the function $\psi(t)$ belongs to the considered class. In order to formulate the conditions which limit the quantity $\psi(t)$ the following quantities are introduced:

$\theta_0 = \max|\theta(t)|$ – maximum amplitude of angular disturbance,

$\Omega = \max|\dot{\psi}(t)|$ – maximum rotational speed of the basic shaft motion,

$\Omega_0 = \max|\dot{\theta}(t)|$ – maximum velocity amplitude of angular disturbance.

Angular velocity of shaft can be presented as the first time derivative of expression (4):

$$\dot{\phi}(t) = \dot{\psi}(t) + \dot{\theta}(t), \tag{5}$$

and the acceleration as the second time derivative of expression (4):

$$\ddot{\phi}(t) = \ddot{\psi}(t) + \ddot{\theta}(t). \tag{6}$$

During the work of the typical shaft, the amplitude of angular disturbance cannot pass by the limited values. It is natural that maximum deviation of the assumed main motion is not big because it would influence the system as a whole. The limitation can have the following form:

$$\theta_0 \le \theta_{dop} = \frac{\pi}{m_{dop}}, \tag{7}$$

where: θ_{dop} – maximum limited amplitude of the angular disturbance, $m_{dop} = \frac{\pi}{\theta_{dop}}$ – angle multiple θ_{dop} in π angle.

The second constraint of motion disturbance is the ratio of disturbance velocity amplitude to the main motion velocity. It results from the fact, that the work of mechanical system with big changes in rotational speed is not allowed in most cases. This limitation has the form of the following condition:

$$\Omega_\theta \le \kappa_{dop} \cdot \Omega. \tag{8}$$

where: κ_{dop} – maximum allowed quotient of angular disturbance amplitude of the shaft.

In conclusion, it can be written that the inequality (7) shows a small assumption of angles, whereas inequality (8) represents the condition of small disturbance of angular velocity.

In the specific case, while the main motion has the constant velocity Ω, and the disturbance is defined by the function of $\theta_0 \cdot \sin(\omega_\theta \cdot t)$ type, (7) and (8) conditions can be presented in a different form. The first condition can be written in the form of amplitude of angular velocity disturbance and the second one as the limitation of the disturbance angle amplitude. These conditions have the following form:

$$\theta_0 \cdot \omega_\theta \le \theta_{dop} \cdot \omega_\theta = \frac{\pi}{m_{dop}} \cdot \omega_\theta, \tag{9}$$

$$\frac{\Omega_\theta}{\omega_\theta} \le \frac{\kappa_{dop} \cdot \Omega}{\omega_\theta}, \tag{10}$$

which leads to the following formulation of (7) and (8) conditions:

$$\frac{\Omega_\theta}{\omega_\theta} \le \omega_{dop} \cdot \omega_\theta = \frac{\pi}{m_{dop}} \cdot \omega_\theta, \tag{11}$$

$$\theta_0 \le \frac{\kappa_{dop} \cdot \Omega}{\omega_\theta}. \tag{12}$$

On the basis of the (7) and (12) (or (8) and (11)) conditions, it is possible to find the constraints which have to be satisfied by the amplitude of vibration acceleration. These constraints are defined by the following dependencies:

$$\theta_0 \cdot \omega_\theta^{\ 2} \le \frac{\pi}{12} \cdot \omega_\theta^{\ 2}, \tag{13}$$

$$\Omega_\theta \cdot \omega_\theta \le 0.1 \cdot \Omega \cdot \omega_\theta. \tag{14}$$

Assuming that $\theta_0 \cdot \omega_\theta^{\ 2} = \Omega_\theta \cdot \omega_\theta = \varepsilon_\theta$ limiting conditions for accelerations have the following form:

$$\varepsilon_\theta \le \frac{\pi}{12} \cdot \omega_\theta^{\ 2}, \tag{15}$$

$$\varepsilon_\theta \le 0.1 \cdot \Omega \cdot \omega_\theta. \tag{16}$$

Unfortunately, for each pair of sets defined by inequalities (7) - (12) and (15) - (16) there is no inclusion relation. Therefore, the disturbance frequency range ω_θ is divided into two areas: the area in which the limitation of angular disturbance value is significant, and the area in which the limitation of disturbance velocity value is significant. The point dividing the whole area of frequency into the previously defined ranges can be marked by the comparison of limitations (7) and (12) (or (8) and (11)). The solution of such an equation has always the following form:

$$\omega_{\theta gr} \theta_0 \le \frac{\kappa_{dop}}{\theta_{dop}} \cdot \Omega = \frac{\kappa_{dop} \cdot m_{dop}}{\pi} \cdot \Omega. \tag{17}$$

The formulated assumptions should be used for transformation and simplification of the differential equation system, which is defined by formulae (1) - (3) describing the motion of analyzed system, in order to find a strict solution. Such approach is the result of the fact that closed and analytical solutions of this system (1) - (3) do not exist.

When the angle $\phi(t)$ is put into the system of equation (1) - (3) in the form (4) the following equations are obtained:

$$m \cdot \ddot{h} - m \cdot e \cdot \sin(\psi(t) + \theta(t)) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t)) - m \cdot e \cdot \cos(\psi(t) + \theta(t)) \cdot (\dot{\psi}(t) + \dot{\theta}(t))^2 + k(h) \cdot h = 0 \tag{18}$$

$$m \cdot \ddot{v} + m \cdot e \cdot \cos(\psi(t) + \theta(t)) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t)) - m \cdot e \cdot \sin(\psi(t) + \theta(t)) \cdot (\dot{\psi}(t) + \dot{\theta}(t))^2 + k(v) \cdot v = 0 \tag{19}$$

$$(I + m \cdot e^2) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t)) - m \cdot e \cdot \ddot{h} \cdot \sin(\psi(t) + \theta(t)) + m \cdot e \cdot \ddot{v} \cdot \cos(\psi(t) + \theta(t)) = \Delta M(t) \tag{20}$$

When the trigonometric identity for the sum of angles of sinus and cosines is used to divide unknown functions $\psi(t)$ and $\theta(t)$ the following formula is obtained:

$$\begin{aligned} &m \cdot \ddot{h} - m \cdot e \cdot \sin\psi(t) \cdot \cos\theta(t) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t)) - m \cdot e \cdot \cos\psi(t) \cdot \sin\theta(t) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t)) + \\ &- m \cdot e \cdot \cos\psi(t) \cdot \cos\theta(t) \cdot (\dot{\psi}(t) + \dot{\theta}(t))^2 + m \cdot e \cdot \sin\psi(t) \cdot \sin\theta(t) \cdot (\dot{\psi}(t) + \dot{\theta}(t))^2 + k(h) \cdot h = 0 \end{aligned} \tag{21}$$

$$\begin{aligned} &m \cdot \ddot{v} + m \cdot e \cdot \cos\psi(t) \cdot \cos\theta(t) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t)) - m \cdot e \cdot \sin\psi(t) \cdot \sin\theta(t) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t)) + \\ &- m \cdot e \cdot \sin\psi(t) \cdot \cos\theta(t) \cdot (\dot{\psi}(t) + \dot{\theta}(t))^2 - m \cdot e \cdot \cos\psi(t) \cdot \sin\theta(t) \cdot (\dot{\psi}(t) + \dot{\theta}(t))^2 + k(h) \cdot h = 0 \end{aligned} \tag{22}$$

$$\begin{aligned} &(I + m \cdot e^2) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t)) - m \cdot e \cdot \ddot{h} \cdot \sin\psi(t) \cdot \cos\theta(t) - m \cdot e \cdot \ddot{h} \cdot \cos\psi(t) \cdot \sin\theta(t) + \\ &+ m \cdot e \cdot \ddot{v} \cdot \cos\psi(t) \cdot \cos\theta(t) - m \cdot e \cdot \ddot{v} \cdot \sin\psi(t) \cdot \sin\theta(t) = \Delta M(t) \end{aligned} \tag{23}$$

In connection with assumptions of small angles (assumption (7)), the following assumptions can be written with satisfying approximation:

$$\sin\theta(t) \cong \theta(t), \tag{24}$$

$$\cos\theta(t) \cong 1. \tag{25}$$

The approximation precision (24) and (25) is the bigger, the smaller the angle value $\theta(t)$ is. Maximum error is marked by the biggest acceptable disturbance amplitude θ_{dop}. In engineering

calculations the range of arguments, for which dependencies (24) and (25) make sense is $\left|\theta(t)\right| \le \frac{\pi}{12}$.
It means that the analyzed quantity θ_{dop} must be smaller than:

$$\theta_{dop} \le \frac{\pi}{12}. \tag{26}$$

For these approximations the equations have the following form:

$$\begin{aligned} & m\cdot\ddot{h} - m\cdot e\cdot\sin\psi(t)\cdot(\ddot{\psi}(t)+\ddot{\theta}(t)) - m\cdot e\cdot\cos\psi(t)\cdot\theta(t)\cdot(\ddot{\psi}(t)+\ddot{\theta}(t)) + \\ & - m\cdot e\cdot\cos\psi(t)\cdot(\dot{\psi}(t)+\dot{\theta}(t))^2 + m\cdot e\cdot\sin\psi(t)\cdot\theta(t)\cdot(\dot{\psi}(t)+\dot{\theta}(t))^2 + k(h)\cdot h = 0 \end{aligned}, \tag{27}$$

$$\begin{aligned} & m\cdot\ddot{v} + m\cdot e\cdot\cos\psi(t)\cdot(\ddot{\psi}(t)+\ddot{\theta}(t)) - m\cdot e\cdot\sin\psi(t)\cdot\theta(t)\cdot(\ddot{\psi}(t)+\ddot{\theta}(t)) + \\ & - m\cdot e\cdot\sin\psi(t)\cdot(\dot{\psi}(t)+\dot{\theta}(t))^2 - m\cdot e\cdot\cos\psi(t)\cdot\theta(t)\cdot(\dot{\psi}(t)+\dot{\theta}(t))^2 + k(h)\cdot h = 0 \end{aligned}, \tag{28}$$

$$\begin{aligned} & (I + m\cdot e^2)\cdot(\ddot{\psi}(t)+\ddot{\theta}(t)) - m\cdot e\cdot\ddot{h}\cdot\sin\psi(t) - m\cdot e\cdot\ddot{h}\cdot\cos\psi(t)\cdot\theta(t) + \\ & + m\cdot e\cdot\ddot{v}\cdot\cos\psi(t) - m\cdot e\cdot\ddot{v}\cdot\sin\psi(t)\cdot\theta(t) = \Delta M(t) \end{aligned}. \tag{29}$$

The assumption of small velocity disturbance (formula (8)) allows to write an expression $(\dot{\psi}(t)+\dot{\theta}(t))^2$ in the following form:

$$\begin{aligned} & (\dot{\psi}(t)+\dot{\theta}(t))^2 = \dot{\psi}(t)^2 + 2\cdot\dot{\psi}(t)\cdot\dot{\theta}(t) + \dot{\theta}(t)^2 \le \dot{\psi}(t)^2 + 2\cdot\dot{\psi}(t)\cdot\max\left|\dot{\theta}(t)\right| + \max\left|\dot{\theta}(t)\right|^2 = \\ & = \dot{\psi}(t)^2 + 2\cdot\dot{\psi}(t)\cdot\Omega_\theta + {\Omega_\theta}^2 \le \max\left|\dot{\psi}(t)\right|^2 + 2\cdot\max\left|\dot{\psi}(t)\right|\cdot\Omega_\theta + {\Omega_\theta}^2 = \Omega^2 + 2\cdot\Omega\cdot\Omega_\theta + {\Omega_\theta}^2 \end{aligned} \tag{30}$$

In extreme case, when the value of the amplitude of angular velocity disturbance is $\Omega_\theta = \kappa_{dop}\cdot\Omega$ the following formula is obtained:

$$(\dot{\psi}(t)+\dot{\theta}(t))^2 \le \Omega^2 + 2\cdot\Omega\cdot\kappa_{dop}\cdot\Omega + {\kappa_{dop}}^2\cdot\Omega^2 = (1 + 2\cdot\kappa_{dop} + {\kappa_{dop}}^2)\cdot\Omega^2. \tag{31}$$

In engineering practice the error can be acceptable if it does not exceed 10%. The quantity of this error defines the maximum value of κ_{dop} because it has to satisfy the following condition:

$$1 + 2\cdot\kappa_{dop} + {\kappa_{dop}}^2 \le 0.1. \tag{32}$$

Additionally, it is worth mentioning that $(1 + 2\cdot\kappa_{dop} + {\kappa_{dop}}^2)\cdot\Omega^2$ is the limiting value, which is rarely taken by expression $(\dot{\psi}(t)+\dot{\theta}(t))^2$. Thus, with the satisfying approximation (if condition (32) is satisfied) it can be assumed that:

$$(\dot{\psi}(t)+\dot{\theta}(t))^2 = \dot{\psi}(t)^2. \tag{33}$$

That is why, the simplified equations system has the following form:

$$\begin{aligned} & m\cdot\ddot{h} - m\cdot e\cdot\sin\psi(t)\cdot(\ddot{\psi}(t)+\ddot{\theta}(t)) - m\cdot e\cdot\cos\psi(t)\cdot\theta(t)\cdot(\ddot{\psi}(t)+\ddot{\theta}(t)) + \\ & - m\cdot e\cdot\cos\psi(t)\cdot\dot{\psi}(t)^2 + m\cdot e\cdot\sin\psi(t)\cdot\theta(t)\cdot\dot{\psi}(t)^2 + k(h)\cdot h = 0 \end{aligned}, \tag{34}$$

$$\begin{aligned} & m\cdot\ddot{v} + m\cdot e\cdot\cos\psi(t)\cdot(\ddot{\psi}(t)+\ddot{\theta}(t)) - m\cdot e\cdot\sin\psi(t)\cdot\theta(t)\cdot(\ddot{\psi}(t)+\ddot{\theta}(t)) + \\ & - m\cdot e\cdot\sin\psi(t)\cdot\dot{\psi}(t)^2 - m\cdot e\cdot\cos\psi(t)\cdot\theta(t)\cdot\dot{\psi}(t)^2 + k(h)\cdot h = 0 \end{aligned}, \tag{35}$$

$$\begin{aligned} & (I + m\cdot e^2)\cdot(\ddot{\psi}(t)+\ddot{\theta}(t)) - m\cdot e\cdot\ddot{h}\cdot\sin\psi(t) - m\cdot e\cdot\ddot{h}\cdot\cos\psi(t)\cdot\theta(t) + \\ & + m\cdot e\cdot\ddot{v}\cdot\cos\psi(t) - m\cdot e\cdot\ddot{v}\cdot\sin\psi(t)\cdot\theta(t) = \Delta M(t) \end{aligned}. \tag{36}$$

After grouping the similar terms and factoring out, the following dependencies are obtained:

$$\begin{aligned} & m\cdot\ddot{h} - m\cdot e\cdot\sin\psi(t)\cdot(\ddot{\psi}(t)+\ddot{\theta}(t)) - m\cdot e\cdot\cos\psi(t)\cdot\dot{\psi}(t)^2 + \\ & + k(h)\cdot h - [m\cdot e\cdot\cos\psi(t)\cdot(\ddot{\psi}(t)+\ddot{\theta}(t)) - m\cdot e\cdot\sin\psi(t)\cdot\dot{\psi}(t)^2]\cdot\theta(t) = 0 \end{aligned}, \tag{37}$$

$$m \cdot \ddot{v} + m \cdot e \cdot \cos\psi(t) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t)) - m \cdot e \cdot \sin\psi(t) \cdot \dot{\psi}(t)^2 + \\ + k(v) \cdot v - [m \cdot e \cdot \sin\psi(t) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t)) + m \cdot e \cdot \cos\psi(t) \cdot \dot{\psi}(t)^2] \cdot \theta(t) = 0, \tag{38}$$

$$(I + m \cdot e^2) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t)) - m \cdot e \cdot \ddot{h} \cdot \sin\psi(t) + m \cdot e \cdot \ddot{v} \cdot \cos\psi(t) + \\ - [m \cdot e \cdot \ddot{h} \cdot \cos\psi(t) + m \cdot e \cdot \ddot{v} \cdot \sin\psi(t)] \cdot \theta(t) = \Delta M(t). \tag{39}$$

It is worth mentioning that in formulae (37) and (39) there are terms, which are proportional to disturbance angle $\theta(t)$. These elements are much smaller than the rest of them, because the maximum disturbance value θ_0 does not exceed $\frac{\pi}{12} \cong 0.2$ (see formula (26)). Thus, small terms can be eliminated because they are much smaller than the rest of the elements of analyzed dependencies. After this elimination the following formula is obtained:

$$m \cdot \ddot{h} - m \cdot e \cdot \sin\psi(t) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t)) - m \cdot e \cdot \cos\psi(t) \cdot \dot{\psi}(t)^2 + k(h) \cdot h = 0, \tag{40}$$

$$m \cdot \ddot{v} + m \cdot e \cdot \cos\psi(t) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t)) - m \cdot e \cdot \sin\psi(t) \cdot \dot{\psi}(t)^2 + k(v) \cdot v = 0, \tag{41}$$

$$(I + m \cdot e^2) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t)) - m \cdot e \cdot \ddot{h} \cdot \sin\psi(t) + m \cdot e \cdot \ddot{v} \cdot \cos\psi(t) = \Delta M(t), \tag{42}$$

The solution of system of equations (40) - (42) by the means of analytical methods is not possible. The approximate or numerical solution can be found [3,5,6]. In the case of vibrating systems, in which the nonlinearities are weak, the satisfying results can be obtained when the Galerkin method [1,2] is used. This method is based on the proposition of approximate solution, which belongs to the functional space. It is assumed that this space is the subspace of precise solutions space of the analyzed problem. Finding the approximate solution results in the projection of analyzed equation into the elements of expected solution, in the sense of previously defined scalar product.

Predicted approximate solution has the following form:

$$h = A_h \cdot \cos(\Omega \cdot t), \tag{43}$$

$$v = A_v \cdot \sin(\Omega \cdot t). \tag{44}$$

If the predicted solution is put into the system of equations the following result is obtained:

$$-A_h \cdot \Omega^2 \cdot \cos(\Omega \cdot t) + \frac{k(A_h \cdot \cos(\Omega \cdot t))}{m} \cdot A_h \cdot \cos(\Omega \cdot t) = \\ = e \cdot [\cos\psi(t) \cdot \dot{\psi}(t)^2 + \sin\psi(t) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t))], \tag{45}$$

$$-A_v \cdot \Omega^2 \cdot \sin(\Omega \cdot t) + \frac{k(A_v \cdot \sin(\Omega \cdot t))}{m} \cdot A_v \cdot \sin(\Omega \cdot t) = \\ = e \cdot [\sin\psi(t) \cdot \dot{\psi}(t)^2 - \cos\psi(t) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t))], \tag{46}$$

The scalar product for the functional space of solutions (43) and (45) is defined as:

$$\int_0^T f(t) \cdot g(t) \cdot dt, \tag{47}$$

The result of formulae projection (45) and (46) is:

$$\int_0^T A_h \cdot \cos(\Omega \cdot t) + \left[\frac{k(A_h \cdot \cos(\Omega \cdot t))}{m} - \cdot\Omega^2\right] \cdot \cos^2(\Omega \cdot t) \cdot dt = \\ = \int_0^T e \cdot [\cos\psi(t) \cdot \dot{\psi}(t)^2 + \sin\psi(t) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t))] \cdot \cos(\Omega \cdot t) \cdot dt, \tag{48}$$

$$\int_0^T A_v \cdot \sin(\Omega \cdot t) + \left[\frac{k(A_v \cdot \sin(\Omega \cdot t))}{m} - \cdot\Omega^2\right] \cdot \sin^2(\Omega \cdot t) \cdot dt = \\ = \int_0^T e \cdot [\sin\psi(t) \cdot \dot{\psi}(t)^2 - \cos\psi(t) \cdot (\ddot{\psi}(t) + \ddot{\theta}(t))] \cdot \sin(\Omega \cdot t) \cdot dt. \tag{49}$$

The transformation of equation (48) and (49) results in approximate amplitude-frequency characteristics:

$$A_h = \frac{\int_0^T e\cdot[\cos\psi(t)\cdot\dot{\psi}(t)^2 + \sin\psi(t)\cdot(\ddot{\psi}(t)+\ddot{\theta}(t))]\cdot\cos(\Omega\cdot t)\cdot dt}{\int_0^T\left[\frac{k(A_h\cdot\cos(\Omega\cdot t))}{m} - \cdot\Omega^2\right]\cdot\cos^2(\Omega\cdot t)\cdot dt}, \tag{50}$$

$$A_v = \frac{\int_0^T e\cdot[\sin\psi(t)\cdot\dot{\psi}(t)^2 - \cos\psi(t)\cdot(\ddot{\psi}(t)+\ddot{\theta}(t))]\cdot\sin(\Omega\cdot t)\cdot dt}{\int_0^T\left[\frac{k(A_v\cdot\sin(\Omega\cdot t))}{m} - \cdot\Omega^2\right]\cdot\sin^2(\Omega\cdot t)\cdot dt}, \tag{51}$$

which after regrouping and arranging the terms have the following form:

$$A_h = \frac{\int_0^T e\cdot\cos\psi(t)\cdot\dot{\psi}(t)^2\cdot\cos(\Omega\cdot t)\cdot dt + \int_0^T e\cdot\sin\psi(t)\cdot\ddot{\theta}(t)\cdot\cos(\Omega\cdot t)\cdot dt}{\int_0^T\left[\frac{k(A_h\cdot\cos(\Omega\cdot t))}{m} - \cdot\Omega^2\right]\cdot\cos^2(\Omega\cdot t)\cdot dt} + \\ + \frac{\int_0^T e\cdot\sin\psi(t)\cdot\ddot{\psi}(t)\cdot\cos(\Omega\cdot t)\cdot dt}{\int_0^T\left[\frac{k(A_h\cdot\cos(\Omega\cdot t))}{m} - \cdot\Omega^2\right]\cdot\cos^2(\Omega\cdot t)\cdot dt}, \tag{52}$$

$$A_v = \frac{\int_0^T e\cdot\sin\psi(t)\cdot\dot{\psi}(t)^2\cdot\sin(\Omega\cdot t)\cdot dt + \int_0^T e\cdot\cos\psi(t)\cdot\ddot{\theta}(t)\cdot\sin(\Omega\cdot t)\cdot dt}{\int_0^T\left[\frac{k(A_v\cdot\sin(\Omega\cdot t))}{m} - \cdot\Omega^2\right]\cdot\sin^2(\Omega\cdot t)\cdot dt} + \\ + \frac{\int_0^T e\cdot\cos\psi(t)\cdot\ddot{\psi}(t)\cdot\sin(\Omega\cdot t)\cdot dt}{\int_0^T\left[\frac{k(A_v\cdot\sin(\Omega\cdot t))}{m} - \cdot\Omega^2\right]\cdot\sin^2(\Omega\cdot t)\cdot dt}, \tag{53}$$

The analysis of expressions (52) and (53) is difficult in general case. The main problem is the unknown form of functions $\psi(t)$ and $\theta(t)$. Only certain estimates concerning the properties which result from general form of formulae (52) i (53) can be proposed.

The form of the expression indicates that critical states occur when the value Ω is the zero of the denominator of dependencies (52) and (53) i.e. expressions:

$$\int_0^T\left[\frac{k(A_h\cdot\cos(\Omega\cdot t))}{m} - \cdot\Omega^2\right]\cdot\cos^2(\Omega\cdot t)\cdot dt, \tag{54}$$

and

$$\int_0^T\left[\frac{k(A_v\cdot\sin(\Omega\cdot t))}{m} - \cdot\Omega^2\right]\cdot\sin^2(\Omega\cdot t)\cdot dt. \tag{55}$$

In the numerator of approximate amplitude-frequency characteristics the following sum occurs:

$$\int_0^T e\cdot\cos\psi(t)\cdot\dot{\psi}(t)^2\cdot\cos(\Omega\cdot t)\cdot dt + \int_0^T e\cdot\sin\psi(t)\cdot\ddot{\theta}(t)\cdot\cos(\Omega\cdot t)\cdot dt + \\ + \int_0^T e\cdot\sin\psi(t)\cdot\ddot{\psi}(t)\cdot\cos(\Omega\cdot t)\cdot dt, \tag{56}$$

and

$$\int_0^T e\cdot\sin\psi(t)\cdot\dot{\psi}(t)^2\cdot\sin(\Omega\cdot t)\cdot dt+\int_0^T e\cdot\cos\psi(t)\cdot\ddot{\theta}(t)\cdot\sin(\Omega\cdot t)\cdot dt+ \\ +\int_0^T e\cdot\cos\psi(t)\cdot\ddot{\psi}(t)\cdot\sin(\Omega\cdot t)\cdot dt \quad , \tag{57}$$

this sum is the reason of the resonance. If this sum was zero, the critical state could not occur (the limit of type $\frac{0}{0}$ would have to be found). The first term of this sum is connected with the influence of centrifugal reaction force on the resonance:

$$\int_0^T \cos\psi(t)\cdot\dot{\psi}(t)^2\cdot\cos(\Omega\cdot t)\cdot dt, \tag{58}$$

and

$$\int_0^T \sin\psi(t)\cdot\dot{\psi}(t)^2\cdot\sin(\Omega\cdot t)\cdot dt. \tag{59}$$

The second term of this sum can be interpreted as the influence of angular shaft acceleration on the critical state:

$$\int_0^T \sin\psi(t)\cdot\ddot{\psi}(t)\cdot\cos(\Omega\cdot t)\cdot dt, \tag{60}$$

and

$$\int_0^T \cos\psi(t)\cdot\ddot{\psi}(t)\cdot\sin(\Omega\cdot t)\cdot dt. \tag{61}$$

The third term of this sum presents the influence of the angular velocity disturbances on the critical state:

$$\int_0^T \sin\psi(t)\cdot\ddot{\theta}(t)\cdot\cos(\Omega\cdot t)\cdot dt, \tag{62}$$

and

$$\int_0^T \cos\psi(t)\cdot\ddot{\theta}(t)\cdot\sin(\Omega\cdot t)\cdot dt. \tag{63}$$

If functions $\psi(t)$ and $\theta(t)$ are chosen in such a way that integrals (60) - (63) do not disappear, there is a possibility of the influence on the critical state of the system, because the numerator depends on the value of angular velocity disturbances of the shaft (dependencies (52) and (53)). It is difficult to evaluate the weight of the influence of particular elements. This evaluation requires the analysis of specific cases (concrete functions defining main motion and angular shaft disturbances should be assumed). In most cases, definite integrals exist for elementary functions. Thus, it is not difficult to find the concrete solutions. Besides, there is always the possibility to use numerical calculation of the value of analyzed integral. It means that the evaluation of torsional vibrations influence on vibrating shafts critical states is possible. Moreover, it is possible to evaluate the use of this influence to control the dynamics of the analyzed system.

4. PRACTICAL APPLICATION

The solution presented in the paper can be used to replace typical steel shafts by shafts made from modern construction materials. For example, drive shaft from Iveco van has length of 540 mm and diameter of 75 mm. Currently used drive shaft can be replaced by much thinner object made of carbon composite. Replacement would have completely different dynamic properties. The technique presented in the paper, provides a way to avoid some of the problems with the dynamics of the new rotor. Only the existence of opportunities to influence the bending vibrations was presented in the paper. This qualitative result is the basis for further research on this problem. Works on obtaining quantitative results are pending. Quantitative results will be presented in future articles. A similar approach can be used for other dynamic systems, such as a ball screw feed drive systems [7].

5. CONCLUSIONS

Moreover, it is possible to evaluate the use of this influence to control the dynamics of the analyzed system. The analysis carried out in this article indicates that there is a connection between torsional and transverse vibrations in the system defined by equations (1) - (3). It means that the application of the torque disturbance in the analyzed system (the reason for the occurrence of torsional vibrations) will influence the character of transverse shaft vibrations. Thus, it is possible to control the rotor vibrations with the help of the quantity disturbance $\Delta M(t)$ from equation (3). The evaluation of the range of transverse vibrations control, i.e. the influence of torsional oscillations on the shaft critical states requires:

- a thorough discussion of integrals (60) - (63),
- the application of a more precise method of the analyzed equations system solution,
- a precise research of the workstation.

Bibliography

1. Nayfeh, A.H. & Mook, D.T. *Nonlinear Oscillations*. Birkach: Willey-Vch. 2008.
2. Pilipchuk, V.N. *Nonlinear Dynamics: Between Linear and Impact Limits*. Berlin: Springer. 2010.
3. Batko, W. & Dąbrowski, Z. & Kiciński, J. *Nonlinear Effects in Technical Diagnostics*. Radom: ITE-PIB, 2008.
4. Minorsky, N. *Drgania nieliniowe*. Warszawa: Państwowe Wydawnictwo Naukowe. 1967. [In Polish: Minorsky, N. *Nonlinear oscillations*. Warsaw: PWN. 1967].
5. Klekot, G. *Zastosowanie miar propagacji energii wibroakustycznej do monitorowania stanu obiektów oraz jako narzędzie w zarządzaniu hałasem.* Radom: ITE, 2012. [In Polish: Klekot, G. *Application of vibro-acoustic energy propagation measures to monitor status of the object and as a tool in the manage of noise*].
6. Pakowski, R. & Pankiewicz, J. Symulacja dynamiki układu wirującego jako pomoc przy projektowaniu wałów maszynowych. *Materiały XIX Sympozjonu Podstaw Konstrukcji Maszyn*. Świnoujście − Zielona Góra. 1999. [In Polish: Pakowski, R. & Pankiewicz, J. Rotating system dynamics simulation as an aid in the design of machine shafts. *Materials of XIX Symposium of Machine Design Fundamentals.* Świnoujście − Zielona Góra. 1999].
7. Dong, L. & Tang W.C. Hybrid modelling and analysis of structural dynamic of a ball screw feed drive system. *Mechanika*. 2013. Vol. 19 (3). P. 316-323.

Received 14.03.2012; accepted in revised form 04.10.2013